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Reflexivity, Transitivity, Symmetry, and Anti-Symmetry of the Intersection Convolution of Relations

ABSTRACT. We give some sufficient conditions in order that the intersection convolution $F * G$ of two relations F and G on a groupoid X be reflexive, transitive, symmetric, and anti-symmetric.

Here, $F * G$ is a relation on X such that

$$\big(F * G \big)(x) = \bigcap \big\{ F(u) + G(v) : \qquad x = u + v \, , \quad F(u) \neq \emptyset \, , \quad G(v) \neq \emptyset \big\}$$

for all $x \in X$.

KEY WORDS. Groupoids, binary relations, intersection convolution, reflexivity, transitivity, symmetry, and anti-symmetry

1 A few basic facts on relations and groupoids

A subset F of a product set $X \times Y$ is called a relation on X to Y. If in particular $F \subset X^2$, then we may simply say that F is a relation on X. Thus, a relation F on X to Y is also a relation on $X \cup Y$.

If F is a relation on X to Y, then for any $x \in X$ and $A \subset X$ the sets $F(x) = \{y \in X : (x, y) \in F\}$ and $F[A] = \bigcup_{a \in A} F(a)$ are called the images of x and A under F, respectively.

Moreover, the sets $D_F = \{x \in X : \ F(x) \neq \emptyset\}$ and $R_F = F[D_F]$ are called the domain and range of F, respectively. If in particular $D_F = X$, then we say that F is a relation of X to Y, or that F is a total relation on X to Y.

Now, a relation F on X is called

(1) reflexive if $x \in F(x)$ for all $x \in D_F$;

(2) symmetric if $y \in F(x)$ implies $x \in F(y)$;

(3) transitive if $y \in F(x)$ and $z \in F(y)$ implies $z \in F(x)$;

(4) anti-symmetric if $y \in F(x)$ and $x \in F(y)$ implies $x = y$;

In particular, a relation f on X to Y is called a function if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$.

If X is a set and $+$ is a function of X^2 to X, then the function $+$ is called an operation in X and the ordered pair $X(+) = (X, +)$ is called a groupoid even if X is void.

In this case, we may simply write $x + y$ in place of $+(x, y)$ for any $x, y \in X$. Moreover, we may also simply write X in place of $X(+)$ whenever the operation $+$ is clearly understood.

In the practical applications, instead of groupoids, it is usually sufficient to consider only semigroups. However, several definitions and theorems on semigroups can be naturally extended to groupoids.

For instance, if X is a groupoid, then for any $A, B \subset X$, we may naturally write $A + B = \{a + b : \ a \in A, \ b \in B\}$. Moreover, we may also write $x + A = \{x\} + A$ and $A + x = A + \{x\}$ for any $x \in X$.

Note that if in particular X is a group, then we may also naturally write $-A = \{-a : \ a \in A\}$ and $A - B = A + (-B)$ for any $A, B \subset X$. Though, the family $\mathcal{P}(X)$ of all subsets of X is only a semigroup with zero.

2 The intersection convolution of relations

Definition 2.1 *If X is a groupoid, then for any $x \in X$ and $A, B \subset X$, we define*

$$\Gamma(x, A, B) = \big\{ (u, v) \in A \times B : \quad x = u + v \big\} .$$

Remark 2.2 Now, in particular, we may simply write

$$\Gamma(x) = \Gamma(x, X, X) .$$

Thus, Γ is just the inverse relation of the operation $+$ in X. Moreover, we have

$$\Gamma\big(x, A, B\big) = \Gamma(x) \cap (A \times B) .$$

Definition 2.3 *If F and G are relations on one groupoid X to another Y, then we define a relation $F * G$ on X to Y such that*

$$\big(F * G\big)(x) = \bigcap \big\{ F(u) + G(v) : \quad (u, v) \in \Gamma(x, D_F, D_G) \big\}$$

for all $x \in X$. The relation $F * G$ will be called the intersection convolution of the relations F and G.

Remark 2.4 If in particular F and G are relations of X to Y, then we may simply write

$$\big(F * G\big)(x) = \bigcap_{x=u+v} \big(F(u) + G(v)\big) = \bigcap \big\{F(u) + G(v) : \quad (u, v) \in \Gamma(x)\big\} .$$

A particular case of Definition 2.3 was already considered in [2]. But, the following theorem has only been proved in [3].

Theorem 2.5 *If F and G are relations on a group X to a groupoid Y, then for any $x \in X$ we have*

$$(F * G)(x) = \bigcap \big\{F(x - v) + G(v) : \quad v \in (-D_F + x) \cap D_G\big\} =$$
$$= \bigcap \big\{F(u) + G(-u + x) : \quad u \in D_F \cap (x - D_G)\big\} .$$

Hence, by using that $-X + x = X$ and $x - X = X$ for all $x \in X$, we can immediately get

Corollary 2.6 *If F and G are relations on a group X to a groupoid Y, then for any $x \in X$ we have*

(1) $\big(F * G\big)(x) = \bigcap_{v \in D_G} \big(F(x - v) + G(v)\big)$ *whenever F is total;*

(2) $\big(F * G\big)(x) = \bigcap_{u \in D_F} \big(F(u) + G(-u + x)\big)$ *whenever G is total.*

Hence, it is clear that in particular we also have

Corollary 2.7 *If F and G are relations of a group X to a groupoid Y, then for any $x \in X$ we have*

$$(F * G)(x) = \bigcap_{v \in X} \big(F(x - v) + G(v)\big) = \bigcap_{u \in X} \big(F(u) + G(-u + x)\big) .$$

3 Reflexivity and transitivity of the intersection convolution

Theorem 3.1 *If F and G are reflexive relations on a groupoid X, then $F * G$ is a reflexive relation of X.*

Proof: If $x \in X$, then by the reflexivity of F and G, for any $(u, v) \in \Gamma(x, D_F, D_G)$, we have

$$x = u + v \in F(u) + G(v) .$$

Therefore, by the corresponding definitions,

$$x \in \bigcap \big\{F(u) + G(v) : \quad (u, v) \in \Gamma(x, D_F, D_G)\big\} = \big(F * G\big)(x) ,$$

and thus the required assertion is also true.

Theorem 3.2 *If F and G are transitive relations on a groupoid X such that $R_F \subset D_F$ and $R_G \subset D_G$, then $F * G$ is also a transitive relation on X.*

Proof: If $x \in X$,

$$y \in (F * G)(x) \qquad \text{and} \qquad z \in (F * G)(y)\,,$$

then by Definition 2.3

$$y \in \bigcap \big\{ F(u) + G(v) : \quad (u, v) \in \Gamma(x, D_F, D_G) \big\}$$

and

$$z \in \bigcap \big\{ F(s) + G(t) : \quad (s, t) \in \Gamma(y, D_F, D_G) \big\}\,.$$

Thus, for any $(u, v) \in \Gamma(x, D_F, D_G)$, we have $y \in F(u) + G(v)$. Therefore, there exist $s \in F(u)$ and $t \in G(v)$ such that $y = s + t$. Hence, by using the transitivity of F and G, we can infer that

$$F(s) \subset F\big[F(u)\big] \subset F(u) \qquad \text{and} \qquad G(t) \subset G\big[G(v)\big] \subset G(v).$$

Moreover, by using that

$$s \in F(u) \subset R_F \subset D_F \qquad \text{and} \qquad t \in G(v) \subset R_G \subset D_G,$$

we can also see that $(s, t) \in \Gamma(y, D_F, D_G)$. Hence, since $z \in \big(F * G\big)(y)$, it is clear that

$$z \in F(s) + G(t) \subset F(u) + G(v)\,.$$

Therefore,

$$z \in \bigcap \big\{ F(u) + G(v) : \quad (u, v) \in \Gamma(x, D_F, D_G) \big\} = (F * G)(x)\,,$$

and thus the required assertion is also true.

Now, as an immediate consequence of Theorem 3.1 and 3.2, we can also state

Corollary 3.3 *If F and G are preorder relations on a groupoid X such that $R_F \subset D_F$ and $R_G \subset D_G$, then $F * G$ is a preorder relation of X.*

4 Symmetry and anti-symmetry of the intersection convolution

Unfortunately, concerning the symmetry and anti-symmetry of the intersection convolution, we can only prove the following less satisfactory theorems.

Theorem 4.1 *If F is a symmetric relation of a group X and g is a symmetric function on X, then $F * g$ is a symmetric relation on X.*

Proof: If $x \in X$ and $y \in (F * g)(x)$, then by Corollary 2.6

$$y \in \bigcap_{v \in D_g} \big(F(x-v) + g(v) \big) .$$

Therefore, for any $v \in D_g$, we have

$$y \in F(x-v) + g(v) , \qquad \text{and thus} \qquad y - g(v) \in F(x-v) .$$

Hence, by the symmetry of F, it follows that

$$x - v \in F\big(y - g(v)\big) , \qquad \text{and thus} \qquad x \in F\big(y - g(v)\big) + v .$$

Moreover, by using our assumptions on g, we can see that

$$v = g\big(g(v)\big) .$$

Therefore,

$$x \in F\big(y - g(v)\big) + g\big(g(v)\big) .$$

Hence, it is clear that

$$x \in \bigcap_{v \in D_g} \Big(F\big(y - g(v)\big) + g\big(g(v)\big) \Big) = \bigcap_{t \in R_g} \big(F(y-t) + g(t) \big) .$$

Moreover, by using the symmetry of g, we can see that $R_g = D_g$. Therefore,

$$x \in \bigcap_{t \in D_g} \big(F(y-t) + g(t) \big) = (F * g)(y) ,$$

and thus the required assertion is also true.

By using the second statement of Corollary 2.6, we can quite similarly prove the following

Theorem 4.2 *If f is a symmetric function on a group X and G is a symmetric relation of X, then $f * G$ is a symmetric relation on X.*

Moreover, by using the corresponding definitions, we can easily prove the following

Theorem 4.3 *If F and G are relations on a groupoid X such that $D_{F*G} \subset D_F + D_G$ and there exists an anti-symmetric relation H on X such that*

$$F(u) + G(v) \subset H(u+v)$$

for all $u \in D_F$ and $v \in D_G$, then $F * G$ is an anti-symmetric relation on X.

Proof: Assume that $x, y \in X$ such that

$$y \in (F * G)(x) \qquad \text{and} \qquad x \in (F * G)(y)\,.$$

Then, by Definition 2.3 and the hypotheses of the theorem, we have

$$\begin{aligned} y \in \bigcap \big\{ F(u) + G(v) : \quad (u, v) \in \Gamma(x, D_F, D_G) \big\} \subset \\ \subset \bigcap \big\{ H(u + v) : \quad (u, v) \in \Gamma(x, D_F, D_G) \big\} = \\ = \bigcap \big\{ H(x) : \quad (u, v) \in \Gamma(x, D_F, D_G) \big\} = H(x)\,, \end{aligned}$$

and quite similarly $x \in H(y)$. Hence, by the assumed anti-symmetry of H, it follows that $x = y$. Therefore, the required assertion is also true.

Remark 4.4 If F, G and H are relations on one groupoid X to another Y, then by using the global sum

$$F \oplus G = \big\{ (u + s, v + t) : \quad (u, v) \in F,\ (s, t) \in G \big\}\,,$$

investigated in [1], we can easily see that the following assertions are equivalent:

(1) $F \oplus G \subset H$;

(2) $F(u) + G(v) \subset H(u + v)$ for all $u \in D_F$ and $v \in D_G$.

Therefore, the anti-symmetry of the global sum of two relations should also be investigated.

5 Two illuminating examples

Example 5.1 If in particular

$$F = \big\{ (0, 1), (1, 0) \big\} \cup \big(\mathbb{R} \setminus \{0\} \big)^2\,,$$

then F is a symmetric relation of $\mathbb{R}$ and the relation $F * F$ is not symmetric.

From the definition of F, it is clear that F is symmetric. Moreover, we can also at once see that

$$\begin{aligned} &F(0) = \{1\}\,, \qquad F(1) = \mathbb{R}\,, \\ &F(x) = \mathbb{R} \setminus \{0\} \quad \text{for all} \quad x \in \mathbb{R} \setminus \{0, 1\}\,. \end{aligned}$$

Hence, it is clear that

$$\begin{aligned} &F(1 - 0) + F(0) = \mathbb{R} + \{1\} = \mathbb{R}\,, \\ &F(1 - 1) + F(1) = \{1\} + \mathbb{R} = \mathbb{R}\,, \\ &F(1 - v) + F(v) = \big(\mathbb{R} \setminus \{0\} \big) + \big(\mathbb{R} \setminus \{0\} \big) = \mathbb{R} \quad \text{for all} \quad v \in \mathbb{R} \setminus \{0, 1\}\,. \end{aligned}$$

Therefore,

$$(F * F)(1) = \bigcap_{v \in \mathbb{R}} \big(F(1-v) + F(v) \big) = \mathbb{R} \,.$$

Moreover, we can quite similarly see that

$$\begin{aligned}
&F(0-0) + F(0) = \{1\} + \{1\} = \{2\} \,, \\
&F(0-1) + F(1) = \big(\mathbb{R} \setminus \{0\} \big) + \mathbb{R} = \mathbb{R} \,, \\
&F\big(0-(-1)\big) + F(-1) = \mathbb{R} + \big(\mathbb{R} \setminus \{0\} \big) = \mathbb{R} \,, \\
&F(0-v) + F(v) = \big(\mathbb{R} \setminus \{0\} \big) + \big(\mathbb{R} \setminus \{0\} \big) = \mathbb{R} \quad \text{for all} \quad v \in \mathbb{R} \setminus \{-1, 0, 1\} \,.
\end{aligned}$$

Therefore,

$$(F * F)(0) = \bigcap_{v \in \mathbb{R}} \big(F(0-v) + F(v) \big) = \{2\} \,.$$

Now, since

$$0 \in \mathbb{R} = (F * F)(1) \,, \qquad \text{but} \qquad 1 \notin \{2\} = (F * F)(0) \,,$$

it is clear that $F * F$ is not symmetric.

Example 5.2 If in particular F is the usual order relation on $\mathbb{R}$, that is,

$$F = \big\{ (x, y) \in \mathbb{R}^2 : \quad x \leq y \big\} \,,$$

then F and F^{-1} are linear order relations of $\mathbb{R}$ such that the relation $F * F^{-1}$ is not anti-symmetric.

To check this, it is convenient to note that

$$F(x) = x + [0, +\infty[\qquad \text{and} \qquad F^{-1}(y) = y + \,]-\infty, 0]$$

for all $x, y \in \mathbb{R}$. Namely,

$$\begin{aligned}
x \in F^{-1}(y) \iff y \in F(x) \iff (x, y) \in F \iff \\
\iff x \leq y \iff x \in \,]-\infty, y] \iff x \in y + \,]-\infty, 0] \,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\big(F * F^{-1} \big)(x) &= \bigcap_{v \in \mathbb{R}} \big(F(x-v) + F^{-1}(v) \big) = \\
&= \bigcap_{v \in \mathbb{R}} \big(x - v + [0, +\infty[+ v +]-\infty, 0] \big) = \\
&= \bigcap_{v \in \mathbb{R}} \big(x - v + v + [0, +\infty[+]-\infty, 0] \big) = \\
&= \bigcap_{v \in \mathbb{R}} \big(x + \mathbb{R} \big) = \bigcap_{v \in \mathbb{R}} \mathbb{R} = \mathbb{R}
\end{aligned}$$

for all $x \in \mathbb{R}$, and thus $F * F^{-1} = \mathbb{R}^2$.

Remark 5.3 In view of this example and Corollary 3.3, it would be interesting to construct two equivalence relations F and G of $\mathbb{R}$ such that the relation $F * G$ be non-symmetric.

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