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Generalized Noor iterations with errors for asymptotically nonexpansive mappings

ABSTRACT. In the present paper, we define and study a new three-step iterative schemes with errors. Several strong convergence theorems of this scheme are established for asymptotically nonexpansive mappings. Our results extend and improve the recent ones announced by Osilike and Aniagbosor, Cho et.al, Liu and Kang, Nammanee et al., and many others.

KEY WORDS. asymptotically nonexpansive mapping, uniformly convex Banach space, Mann-type iteration, Ishikawa-type iteration, Noor-type iteration

1 Introduction

Let X be a real Banach space and C be a nonempty subset of X. A mapping $T : C \to C$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ of real numbers with $k_n \geq 1$ and $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|,$$

for all $x, y \in C$ and all $n \geq 1$. The mapping T is called *uniformly L-Lipschitzian* if there exists a positive constant L such that

$$\|T^n x - T^n y\| \leq L\|x - y\|,$$

for all $x, y \in C$ and all $n \geq 1$. It is easy to see that if T is asymptotically nonexpansive, then it is uniformly L-Lipschitzian with the uniform Lipschitz constant $L = \sup\{k_n : n \geq 1\}$.

In 2002, Xu and Noor [10] introduced and studied a three-step scheme to approximate fixed points of asymptotically nonexpansive mappings in a Banach space. Glowinski and Le Tallec [2] used three-step iterative schemes to find the approximate solutions of the elastoviscoplasticity problem, liquid crystal theory, and eigenvalue computation. It has been shown in [2] that the three-step iterative scheme gives better numerical results than the two-step and one-step approximate iterations. Haubruge, Nguyen and Strodiot [3] studied the convergence analysis of three-step schemes of Glowinski and Le Tallec [2] and applied these schemes to obtain new splitting-type algorithms for solving variation inequalities, separable convex programming and minimization of a sum of convex functions. They also proved that three-step iterations lead to highly parallelized algorithms under certain conditions. Thus we conclude that three-step scheme plays an important and significant part in solving various problems, which arise in pure and applied sciences. In 2004, Cho, Zhou, and Guo [1], and Liu and Kang [4] extended the preceding scheme to the three-step iterative scheme with errors and gave weak and strong convergence theorems for asymptotically nonexpansive mappings in a Banach space. Recently, Nammanee, Noor and Suantai [5] defined a three-step iterative scheme with errors which is an extension of schemes in [1] and [4] iterations and gave some weak and strong convergence theorems for asymptotically nonexpansive mappings in a uniformly convex Banach space. The authors of the present paper [6] defined a new three-step iterative schemes and gave some strong convergence theorems for asymptotically nonexpansive mappings. Inspired by the preceding iteration scheme, we define a new iteration scheme with errors as follows.

Let C be a nonempty convex subset of a real Banach space X and $T : C \to C$ be a mapping.
Algorithm 1 For a given $x_1 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative schemes, for all $n \geq 1$,

$$\begin{aligned} z_n &= a'_n x_n + b'_n T^n x_n + r_n u_n, \\ y_n &= a_n x_n + b_n T^n x_n + c_n T^n z_n + s_n v_n, \\ x_{n+1} &= \alpha_n x_n + \beta_n T^n x_n + \gamma_n T^n y_n + \delta_n T^n z_n + t_n w_n, \end{aligned} \tag{1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{r_n\}$, $\{s_n\}$ and $\{t_n\}$ are appropriate sequences in $[0, 1]$ with $\alpha_n + \beta_n + \gamma_n + \delta_n + t_n = a_n + b_n + c_n + s_n = a'_n + b'_n + r_n = 1$, and $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in C. The iterative schemes (1) is called the *three-step mean value iterative scheme with errors.*

If $\beta_n \equiv 0$, then Algorithm 1 reduces to
Algorithm 2 [5] For a given $x_1 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative schemes, for all $n \geq 1$,

$$\begin{aligned} z_n &= a'_n x_n + b'_n T^n x_n + r_n u_n, \\ y_n &= a_n x_n + b_n T^n x_n + c_n T^n z_n + s_n v_n, \\ x_{n+1} &= \alpha_n x_n + \gamma_n T^n y_n + \delta_n T^n z_n + t_n w_n, \end{aligned} \tag{2}$$

where $\{\alpha_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{r_n\}$, $\{s_n\}$ and $\{t_n\}$ are appropriate sequences in $[0, 1]$ with $\alpha_n + \gamma_n + \delta_n + t_n = a_n + b_n + c_n + s_n = a'_n + b'_n + r_n = 1$, and $\{u_n\}$,

$\{v_n\}$ and $\{w_n\}$ are bounded sequences in C. The iterative schemes (2) is called the *modified Noor iterative scheme with errors.*

If $\beta_n = \delta_n = b_n \equiv 0$, then Algorithm 1 reduces to

Algorithm 3 [1, 4] For a given $x_1 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative schemes, for all $n \geq 1$,

$$\begin{aligned} z_n &= a'_n x_n + b'_n T^n x_n + r_n u_n, \\ y_n &= a_n x_n + c_n T^n z_n + s_n v_n, \\ x_{n+1} &= \alpha_n x_n + \gamma_n T^n y_n + t_n w_n, \end{aligned} \tag{3}$$

where $\{\alpha_n\}$, $\{\gamma_n\}$, $\{a_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{r_n\}$, $\{s_n\}$ and $\{t_n\}$ are appropriate sequences in $[0,1]$ with $\alpha_n + \gamma_n + t_n = a_n + c_n + s_n = a'_n + b'_n + r_n = 1$, and $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in C. The iterative schemes (3) is called the *Noor iterative scheme with errors.*

2 Auxiliary Lemmas

For convenience, we use the notations $\lim_n \equiv \lim_{n\to\infty}$, $\liminf_n \equiv \liminf_{n\to\infty}$, and $\limsup_n \equiv \limsup_{n\to\infty}$. In the sequel, we shall need the following lemmas.

Lemma 1 ([7], Lemma 1) *Let $\{a_n\}, \{b_n\}$ and $\{\lambda_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+\lambda_n)a_n + b_n, \quad n \geq 1.$$

If $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_n a_n$ exists.

Lemma 2 *Let X be a real Banach space and C be a nonempty closed convex subset of X. Let $T : C \to C$ be an asymptotically nonexpansive mapping with the nonempty fixed-point set $F(T)$ (i.e., $F(T) := \{x \in C : x = Tx\} \neq \varnothing$) and a sequence $\{k_n\}$ of real numbers such that $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be a sequence defined by Algorithm 1 with the restrictions that $\sum_{n=1}^{\infty} t_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n s_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n c_n r_n < \infty$ and $\sum_{n=1}^{\infty} \delta_n r_n < \infty$. Then we have the following conclusions.*

(i) $\lim_n \|x_n - p\|$ *exists for any $p \in F(T)$.*

(ii) $\lim_n d(x_n, F(T))$ *exists, where $d(x, F(T))$ denotes the distance from x to the fixed-point set $F(T)$.*

Proof: Let $p \in F(T)$. We note that $\{u_n - p\}$, $\{v_n - p\}$, and $\{w_n - p\}$ are bounded sequences in C. Let

$$L = \sup\{k_n : n \geq 1\} \text{ and } M = \sup\{\|u_n - p\|, \|v_n - p\|, \|w_n - p\| : n \geq 1\}.$$

By using (1), we have

$$\begin{aligned} \|z_n - p\| &\leq a'_n \|x_n - p\| + b'_n \|T^n x_n - p\| + r_n \|u_n - p\| \\ &\leq (1 - b'_n)\|x_n - p\| + b'_n k_n \|x_n - p\| + M r_n \\ &\leq (1 + b'_n(k_n - 1))\|x_n - p\| + M r_n \\ &\leq k_n \|x_n - p\| + M r_n, \end{aligned} \tag{4}$$

$$\begin{aligned} \|y_n - p\| &\leq a_n \|x_n - p\| + b_n \|T^n x_n - p\| + c_n \|T^n z_n - p\| + s_n \|v_n - p\| \\ &\leq (1 - b_n - c_n)\|x_n - p\| + b_n k_n \|x_n - p\| + c_n k_n \|z_n - p\| + M s_n \\ &\leq (1 + (b_n + c_n + c_n k_n)(k_n - 1))\|x_n - p\| + M(s_n + c_n r_n k_n) \\ &\leq (1 + (L + 2)(k_n - 1))\|x_n - p\| + M(s_n + L c_n r_n), \end{aligned} \tag{5}$$

and so

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n \|x_n - p\| + \beta_n \|T^n x_n - p\| + \gamma_n \|T^n y_n - p\| \\ &\quad + \delta_n \|T^n z_n - p\| + t_n \|w_n - p\| \\ &\leq (1 - \beta_n - \gamma_n - \delta_n)\|x_n - p\| + \beta_n k_n \|x_n - p\| \\ &\quad + \gamma_n k_n \|y_n - p\| + \delta_n k_n \|z_n - p\| + M t_n \\ &\leq (1 + (\beta_n + \gamma_n + \gamma_n k_n (L + 2) + \delta_n (k_n + 1))(k_n - 1))\|x_n - p\| \\ &\quad + M(t_n + \gamma_n k_n s_n + L \gamma_n k_n c_n r_n + \delta_n k_n r_n) \\ &\leq (1 + (L^2 + 3L + 3)(k_n - 1))\|x_n - p\| \\ &\quad + M(t_n + L \gamma_n s_n + L^2 \gamma_n c_n r_n + L \delta_n r_n). \end{aligned}$$

By assumption, the conclusions of the lemma follow from Lemma 1. This completes the proof. □

We also need the following lemma proved by Schu [8].

Lemma 3 *Let X be a uniformly convex Banach space, let $\{\lambda_n\}$ be a sequence of real numbers such that $0 < b \leq \lambda_n \leq c < 1$ for all $n \geq 1$, and let $\{x_n\}$ and $\{y_n\}$ be sequences of X such that $\limsup_n \|x_n\| \leq a$, $\limsup_n \|y_n\| \leq a$ and $\lim_n \|\lambda_n x_n + (1 - \lambda_n) y_n\| = a$ for some $a \geq 0$. Then $\lim_n \|x_n - y_n\| = 0$.*

By Schu's Lemma, we have the following lemma.

Lemma 4 *Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences in a uniformly convex Banach space X. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ with $\alpha_n+\beta_n+\gamma_n=1$, $\limsup_n \|x_n\|\le a$, $\limsup_n \|y_n\|\le a$, $\limsup_n \|z_n\|\le a$, and $\lim_n \|\alpha_n x_n+\beta_n y_n+\gamma_n z_n\|=a$, where $a\ge 0$. If $\liminf_n \alpha_n>0$ and $\liminf_n \beta_n>0$, then $\lim_n \|x_n-y_n\|=0$.*

Proof: We may assume without loss of generality that $\alpha_n>0$ and $\beta_n>0$ for all $n\in\mathbb{N}$. Let $\{n_k\}$ be a subsequence of $\{n\}$ such that

$$\lim_k \left\|\frac{\alpha_{n_k}}{\alpha_{n_k}+\beta_{n_k}}x_{n_k}+\frac{\beta_{n_k}}{\alpha_{n_k}+\beta_{n_k}}y_{n_k}\right\| = \liminf_n \left\|\frac{\alpha_n}{\alpha_n+\beta_n}x_n+\frac{\beta_n}{\alpha_n+\beta_n}y_n\right\|.$$

Then

$$\begin{aligned} a &= \liminf_k \|\alpha_{n_k}x_{n_k}+\beta_{n_k}y_{n_k}+\gamma_{n_k}z_{n_k}\| \\ &\le \liminf_k \Big((\alpha_{n_k}+\beta_{n_k})\left\|\frac{\alpha_{n_k}}{\alpha_{n_k}+\beta_{n_k}}x_{n_k}+\frac{\beta_{n_k}}{\alpha_{n_k}+\beta_{n_k}}y_{n_k}\right\|+\gamma_{n_k}\|z_{n_k}\|\Big) \\ &\le \liminf_k (\alpha_{n_k}+\beta_{n_k})\left\|\frac{\alpha_{n_k}}{\alpha_{n_k}+\beta_{n_k}}x_{n_k}+\frac{\beta_{n_k}}{\alpha_{n_k}+\beta_{n_k}}y_{n_k}\right\|+\limsup_k \gamma_{n_k}\|z_{n_k}\| \\ &\le \liminf_k (\alpha_{n_k}+\beta_{n_k})\liminf_n \left\|\frac{\alpha_n}{\alpha_n+\beta_n}x_n+\frac{\beta_n}{\alpha_n+\beta_n}y_n\right\|+a\limsup_k \gamma_{n_k}. \end{aligned}$$

This implies that

$$\begin{aligned} &\liminf_k (\alpha_{n_k}+\beta_{n_k})a \\ &= (1-\limsup_k \gamma_{n_k})a \\ &\le \liminf_k (\alpha_{n_k}+\beta_{n_k})\liminf_n \left\|\frac{\alpha_n}{\alpha_n+\beta_n}x_n+\frac{\beta_n}{\alpha_n+\beta_n}y_n\right\|. \end{aligned}$$

Since $\liminf_n(\alpha_n+\beta_n)\ge \liminf_n \alpha_n+\liminf_n \beta_n>0$, it follows that

$$a\le \liminf_n \left\|\frac{\alpha_n}{\alpha_n+\beta_n}x_n+\frac{\beta_n}{\alpha_n+\beta_n}y_n\right\| \le \limsup_n \left\|\frac{\alpha_n}{\alpha_n+\beta_n}x_n+\frac{\beta_n}{\alpha_n+\beta_n}y_n\right\|\le a.$$

We now observe that

$$\liminf_n \frac{\alpha_n}{\alpha_n+\beta_n}\ge \liminf_n \alpha_n>0 \quad \text{and} \quad \liminf_n \frac{\beta_n}{\alpha_n+\beta_n}\ge \liminf_n \beta_n>0.$$

By Lemma 3, we have $\lim_n \|x_n-y_n\|=0$. This completes the proof. □

The following lemmas are the important ingredients for proving our main results in the next section.

Lemma 5 *Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $T: C\to C$ be an asymptotically nonexpansive mapping with the nonempty* *fixed-point set $F(T)$ and a sequence $\{k_n\}$ of real numbers such that $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be a sequence defined by Algorithm 1 with the restrictions that $\sum_{n=1}^{\infty} t_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n s_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n c_n r_n < \infty$ and $\sum_{n=1}^{\infty} \delta_n r_n < \infty$. Then we have the following assertions.*

(i) *If $0 < \liminf_n \gamma_n \leq \limsup_n(\beta_n + \gamma_n + \delta_n) < 1$ and $\limsup_n(b_n + c_n) < 1$, then $\lim_n \|T^n x_n - x_n\| = 0$.*

(ii) *If $0 < \liminf_n \delta_n \leq \limsup_n(\beta_n + \gamma_n + \delta_n) < 1$ and $\limsup_n b'_n < 1$, then $\lim_n \|T^n x_n - x_n\| = 0$.*

(iii) *If $0 < \liminf_n \beta_n \leq \limsup_n(\beta_n + \gamma_n + \delta_n) < 1$, then $\lim_n \|T^n x_n - x_n\| = 0$.*

Proof: Let $p \in F(T)$. By Lemma 2, we have $\lim_n \|x_n - p\| = a$ for some $a \geq 0$. Since $\lim_n t_n = 0$,

$$\begin{aligned} a &= \lim_n \|x_{n+1} - p\| \\ &= \lim_n \|(1 - \beta_n - \gamma_n - \delta_n)(x_n - p) + \beta_n(T^n x_n - p) + \gamma_n(T^n y_n - p) \\ &\quad + \delta_n(T^n z_n - p) + t_n(w_n - x_n)\| \\ &= \lim_n \|(1 - \beta_n - \gamma_n - \delta_n)(x_n - p) + \beta_n(T^n x_n - p) \\ &\quad + \gamma_n(T^n y_n - p) + \delta_n(T^n z_n - p)\|. \end{aligned} \tag{6}$$

We first observe that

$$\limsup_n \|T^n x_n - p\| \leq \limsup_n k_n \|x_n - p\| = a. \tag{7}$$

To prove (i), let $\{m_j\}$ be a subsequence of $\{n\}$. We show that there is a subsequence $\{n_k\}$ of $\{m_j\}$ such that $\lim_k \|T^{n_k} y_{n_k} - x_{n_k}\| = 0$.
As $\liminf_n \gamma_n > 0$, $\sum_{n=1}^{\infty} \gamma_n s_n < \infty$, and $\sum_{n=1}^{\infty} \gamma_n c_n r_n < \infty$, $\lim_n s_n = c_n r_n = 0$. By using (5), we have

$$\limsup_j \|T^{m_j} y_{m_j} - p\| \leq \limsup_j k_{m_j} \|y_{m_j} - p\| \leq a. \tag{8}$$

If $\liminf_j \delta_{m_j} > 0$, then $\lim_j r_{m_j} = 0$. By (4), we gives

$$\limsup_j \|T^{m_j} z_{m_j} - p\| \leq \limsup_j k_{m_j} \|z_{m_j} - p\| \leq a. \tag{9}$$

It follows from (6)-(9) and Lemma 4 that

$$\lim_j \|T^{m_j} y_{m_j} - x_{m_j}\| = 0.$$

On the other hand, if $\liminf_j \delta_{m_j} = 0$, then we may extract a subsequence $\{\delta_{n_k}\}$ of $\{\delta_{m_j}\}$ so that $\lim_k \delta_{n_k} = 0$, it follows that

$$\lim_k \delta_{n_k} \|x_{n_k} - p\| = 0 = \lim_k \delta_{n_k} \|T^{n_k} z_{n_k} - p\|.$$

This together with (6) gives

$$\begin{aligned} a = \lim_k \|&(1 - \beta_{n_k} - \gamma_{n_k})(x_{n_k} - p) \\ &+ \beta_{n_k}(T^{n_k} x_{n_k} - p) + \gamma_{n_k}(T^{n_k} y_{n_k} - p)\|. \end{aligned} \tag{10}$$

It follows from (7), (8), (10), and Lemma 4 that

$$\lim_k \|T^{n_k} y_{n_k} - x_{n_k}\| = 0.$$

By double extract subsequence principle,

$$\lim_n \|(x_n - p) - (T^n y_n - p)\| = \lim_n \|T^n y_n - x_n\| = 0. \tag{11}$$

It follows that $\lim_n \|T^n y_n - p\| = a$. Also

$$a = \liminf_n \|T^n y_n - p\| \leq \liminf_n k_n \|y_n - p\| = \liminf_n \|y_n - p\|.$$

From (5), we gives $\limsup_n \|y_n - p\| \leq a$, so that $\lim_n \|y_n - p\| = a$.
Next we prove that

$$\lim_n \|T^n x_n - x_n\| = 0, \tag{12}$$

let $\{\ell_j\}$ be a subsequence of $\{n\}$. It suffices to show that there is a subsequence $\{n_k\}$ of $\{\ell_j\}$ such that $\lim_k \|T^{n_k} x_{n_k} - x_{n_k}\| = 0$. Since $\lim_n s_n = 0$,

$$\begin{aligned} a &= \lim_j \|y_{\ell_j} - p\| \\ &= \lim_j \|(1 - b_{\ell_j} - c_{\ell_j})(x_{\ell_j} - p) + b_{\ell_j}(T^{\ell_j} x_{\ell_j} - p) \\ &\quad + c_{\ell_j}(T^{\ell_j} z_{\ell_j} - p) + s_{\ell_j}(v_{\ell_j} - x_{\ell_j})\| \\ &= \lim_j \|(1 - b_{\ell_j} - c_{\ell_j})(x_{\ell_j} - p) + b_{\ell_j}(T^{\ell_j} x_{\ell_j} - p) + c_{\ell_j}(T^{\ell_j} z_{\ell_j} - p)\|. \end{aligned}$$

If $\liminf_j c_{\ell_j} > 0$, by Lemma 4 and $\limsup_n (b_n + c_n) < 1$, then

$$\lim_j \|T^{\ell_j} z_{\ell_j} - x_{\ell_j}\| = 0. \tag{13}$$

On the other hand, if $\liminf_j c_{\ell_j} = 0$, then we may extract a subsequence $\{c_{n_k}\}$ of $\{c_{\ell_j}\}$ so that $\lim_k c_{n_k} = 0$, it follows that

$$\lim_k c_{n_k} \|T^{n_k} z_{n_k} - x_{n_k}\| = 0. \tag{14}$$

By using (1), we have

$$\begin{aligned}\|T^{n_k}x_{n_k} - x_{n_k}\| &\le \|T^{n_k}x_{n_k} - T^{n_k}y_{n_k}\| + \|T^{n_k}y_{n_k} - x_{n_k}\| \\ &\le k_{n_k}\|x_{n_k} - y_{n_k}\| + \|T^{n_k}y_{n_k} - x_{n_k}\| \\ &\le k_{n_k}b_{n_k}\|T^{n_k}x_{n_k} - x_{n_k}\| + k_{n_k}c_{n_k}\|T^{n_k}z_{n_k} - x_{n_k}\| \\ &\quad + k_{n_k}s_{n_k}\|v_{n_k} - x_{n_k}\| + \|T^{n_k}y_{n_k} - x_{n_k}\|.\end{aligned}$$

This together with (11), (13), and (14) gives

$$\lim_k (1 - k_{n_k}b_{n_k})\|T^{n_k}x_{n_k} - x_{n_k}\| = 0.$$

As $\liminf_n(1 - k_n b_n) = 1 - \limsup_n b_n \ge 1 - \limsup_n(b_n + c_n) > 0$, we have

$$\lim_k \|T^{n_k}x_{n_k} - x_{n_k}\| = 0.$$

By double extract subsequence principle, we obtain (12) and the proof of (i) is finished.

By using a similar method, it can be shown that (ii) is satisfied.

(iii) To show that

$$\lim_n \|T^n x_n - x_n\| = 0, \tag{15}$$

let $\{m_j\}$ be a subsequence of $\{n\}$. It suffices to show that there is a subsequence $\{n_k\}$ of $\{m_j\}$ such that $\lim_k \|T^{n_k}x_{n_k} - x_{n_k}\| = 0$. We consider the following cases.

Case 1: $\liminf_j \gamma_{m_j} > 0$.

Subcase 1.1: $\liminf_j \delta_{m_j} > 0$. Then we obtain (6)-(9). It follows from Lemma 4 that $\lim_j \|T^{m_j}x_{m_j} - x_{m_j}\| = 0$.

Subcase 1.2: $\liminf_j \delta_{m_j} = 0 = \lim_k \delta_{n_k}$, where $\{\delta_{n_k}\} \subset \{\delta_{m_j}\}$. Then we obtain (10), and so

$$\lim_k \|T^{n_k}x_{n_k} - x_{n_k}\| = 0.$$

Case 2: $\liminf_j \gamma_{m_j} = 0$. Choose $\{\gamma_{\ell_k}\} \subset \{\gamma_{m_j}\}$ such that $\lim_k \gamma_{\ell_k} = 0$, it follows that

$$\lim_k \gamma_{\ell_k}\|x_{\ell_k} - p\| = 0 = \lim_k \gamma_{\ell_k}\|T^{\ell_k}y_{\ell_k} - p\|.$$

This together with (6) gives

$$a = \lim_k \|(1 - \beta_{\ell_k} - \delta_{\ell_k})(x_{\ell_k} - p) + \beta_{\ell_k}(T^{\ell_k}x_{\ell_k} - p) + \delta_{\ell_k}(T^{\ell_k}z_{\ell_k} - p)\|. \tag{16}$$

Subcase 2.1: $\liminf_k \delta_{\ell_k} > 0$. By (4), we have $\limsup_k \|T^{\ell_k}z_{\ell_k} - p\| \le a$. It follows from (7), (16) and Lemma 4,

$$\lim_k \|T^{\ell_k}x_{\ell_k} - x_{\ell_k}\| = 0.$$

Subcase 2.2: $\liminf_k \delta_{\ell_k} = 0 = \lim_i \delta_{n_i}$, where $\{\delta_{n_i}\} \subset \{\delta_{\ell_k}\}$. It follows that

$$\lim_i \delta_{n_i} \|T^{n_i} z_{n_i} - p\| = 0.$$

This together with (16) gives

$$a = \lim_i \|(1-\beta_{n_i})(x_{n_i} - p) + \beta_{n_i}(T^{n_i} x_{n_i} - p)\|.$$

It follows from Lemma 3, $\lim_i \|T^{n_i} x_{n_i} - x_{n_i}\| = 0$. By double extract subsequence principle, we obtain (15). This completes the proof. □

Lemma 6 *Let X be a real Banach space and C be a nonempty closed convex subset of X. Let $T : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\}$ of real numbers such that $k_n \geq 1$ and $\lim_n k_n = 1$ and, $\{x_n\}$ be a sequence defined in C by Algorithm 1 with the restrictions that $\lim_n t_n = \lim_n \gamma_n s_n = \lim_n \gamma_n c_n r_n = \lim_n \delta_n r_n = 0$. If $\lim_n \|T^n x_n - x_n\| = 0$, then $\lim_n \|T x_n - x_n\| = 0$.*

Proof: Using (1), we have

$$\begin{aligned}
\|T^n z_n - x_n\| &\leq \|T^n z_n - T^n x_n\| + \|T^n x_n - x_n\| \\
&\leq k_n \|z_n - x_n\| + \|T^n x_n - x_n\|, \\
&\leq (b_n' k_n + 1)\|T^n x_n - x_n\| + r_n k_n \|u_n - x_n\|,
\end{aligned}$$

$$\begin{aligned}
\|T^n y_n - x_n\| &\leq \|T^n y_n - T^n x_n\| + \|T^n x_n - x_n\| \\
&\leq k_n \|y_n - x_n\| + \|T^n x_n - x_n\|, \\
&\leq b_n k_n \|T^n x_n - x_n\| + c_n k_n \|T^n z_n - x_n\| \\
&\quad + s_n k_n \|v_n - x_n\| + \|T^n x_n - x_n\| \\
&\leq (b_n k_n + c_n b_n' k_n^2 + c_n k_n + 1)\|T^n x_n - x_n\| \\
&\quad + s_n k_n \|v_n - x_n\| + c_n r_n k_n^2 \|u_n - x_n\|,
\end{aligned}$$

and so

$$\begin{aligned}
&\|x_{n+1} - T^n x_{n+1}\| \\
&\le \|x_{n+1} - x_n\| + \|T^n x_{n+1} - T^n x_n\| + \|T^n x_n - x_n\| \\
&\le (1 + k_n)\|x_{n+1} - x_n\| + \|T^n x_n - x_n\| \\
&\le \beta_n(1 + k_n)\|T^n x_n - x_n\| + \gamma_n(1 + k_n)\|T^n y_n - x_n\| \\
&\quad + \delta_n(1 + k_n)\|T^n z_n - x_n\| + t_n(1 + k_n)\|w_n - x_n\| + \|T^n x_n - x_n\| \\
&\le \beta_n(1 + k_n)\|T^n x_n - x_n\| \\
&\quad + \gamma_n(1 + k_n)(b_n k_n + c_n b'_n k_n^2 + c_n k_n + 1)\|T^n x_n - x_n\| \\
&\quad + \gamma_n s_n k_n(1 + k_n)\|v_n - x_n\| + \gamma_n c_n r_n(1 + k_n)k_n^2\|u_n - x_n\| \\
&\quad + \delta_n(1 + k_n)(b'_n k_n + 1)\|T^n x_n - x_n\| + \delta_n r_n(1 + k_n)k_n\|u_n - x_n\| \\
&\quad + t_n(1 + k_n)\|w_n - x_n\| + \|T^n x_n - x_n\| \to 0.
\end{aligned}$$

Thus

$$\begin{aligned}
\|x_{n+1} - Tx_{n+1}\| &\le \|x_{n+1} - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - Tx_{n+1}\| \\
&\le \|x_{n+1} - T^{n+1}x_{n+1}\| + k_1\|T^n x_{n+1} - x_{n+1}\| \to 0,
\end{aligned}$$

which implies $\lim_n \|Tx_n - x_n\| = 0$. This completes the proof. □

3 Main results

In this section, we establish several strong convergence theorems of the three-step mean value iterative scheme with errors for asymptotically nonexpansive mappings.

Theorem 7 *Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $T : C \to C$ be an asymptotically nonexpansive mapping with the nonempty fixed-point set $F(T)$ and a sequence $\{k_n\}$ of real numbers such that $k_n \ge 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be a sequence in C defined by Algorithm 1 with the following restrictions:*

(i) $0 < \liminf_n \gamma_n \le \limsup_n(\beta_n + \gamma_n + \delta_n) < 1$,

(ii) $\limsup_n(b_n + c_n) < 1$, *and*

(iii) $\sum_{n=1}^{\infty} t_n < \infty$, $\sum_{n=1}^{\infty} s_n < \infty$, $\sum_{n=1}^{\infty} c_n r_n < \infty$, $\sum_{n=1}^{\infty} \delta_n r_n < \infty$.

If T satisfies Condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.

Let $\{x_n\}$ be a given sequence in C. Recall that a mapping $T : C \to C$ with the nonempty fixed-point set $F(T)$ in C satisfies *Condition (A) with respect to the sequence* $\{x_n\}$ ([9]) if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$f(d(x_n, F(T))) \leq \|x_n - Tx_n\|, \text{ for all } n \geq 1.$$

Proof. By Lemma 5(i) and Lemma 6, we have

$$\lim_n \|Tx_n - x_n\| = 0.$$

Let f be a nondecreasing function corresponding to Condition (A) with respect to $\{x_n\}$. Then

$$f(d(x_n, F(T))) \leq \|Tx_n - x_n\| \to 0,$$

and so

$$d(x_n, F(T)) \to 0.$$

Therefore, the conclusion of the theorem follows exactly from [6]. This completes the proof. □

Remark 8 Suppose we rewrite our scheme by treating the additional terms as error terms in the sense of Xu [11] in this way: $x_1 \in C$,

$$\begin{aligned} z_n &= a'_n x_n + b'_n T^n x_n + r_n u_n, \\ y_n &= a_n x_n + c_n T^n z_n + (b_n + s_n)\left(\frac{b_n}{b_n + s_n} T^n x_n + \frac{s_n}{b_n + s_n} v_n\right), \\ x_{n+1} &= \alpha_n x_n + \gamma_n T^n y_n + (\beta_n + \delta_n + t_n) \\ &\quad \times \left(\frac{\beta_n}{\beta_n + \delta_n + t_n} T^n x_n + \frac{\delta_n}{\beta_n + \delta_n + t_n} T^n z_n + \frac{t_n}{\beta_n + \delta_n + t_n} w_n\right), \end{aligned}$$

for all $n \geq 1$. To obtain a strong convergence theorem by Theorem 2.4 of [1], we are restricted to the following

$$\sum_{n=1}^{\infty} (\beta_n + \delta_n + t_n) < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} (b_n + s_n) < \infty,$$

from which $\lim_n \beta_n = \lim_n \delta_n = \lim_n b_n = 0$, $\sum_{n=1}^{\infty} s_n < \infty$, and $\sum_{n=1}^{\infty} t_n < \infty$. But our Theorem 7 still gives the result for more general restriction. For example, our result is applicable to the case of $\beta_n = \delta_n = b_n = 1/4$ and $s_n = t_n = 1/2^n$.

Consequently, we obtain the following corollaries. When $\beta_n \equiv 0$, we have

Corollary 9 *Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $T : C \to C$ be an asymptotically nonexpansive mapping with* *the nonempty fixed-point set and a sequence* $\{k_n\}$ *of real numbers such that* $k_n \geq 1$ *and* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. *Let* $\{x_n\}$ *be a sequence in* C *defined by Algorithm 2 with the following restrictions:*

(i) $0 < \liminf_n \gamma_n \leq \limsup_n(\gamma_n + \delta_n) < 1$,

(ii) $\limsup_n(b_n + c_n) < 1$, *and*

(iii) $\sum_{n=1}^{\infty} t_n < \infty$, $\sum_{n=1}^{\infty} s_n < \infty$, $\sum_{n=1}^{\infty} c_n r_n < \infty$, $\sum_{n=1}^{\infty} \delta_n r_n < \infty$.

If T *satisfies Condition (A) with respect to the sequence* $\{x_n\}$, *then* $\{x_n\}$ *converges strongly to a fixed point of* T.

When $\beta_n = \delta_n = b_n \equiv 0$ in Theorem 7, we also have

Corollary 10 *Let* X *be a uniformly convex Banach space and* C *be a nonempty closed convex subset of* X. *Let* $T : C \to C$ *be an asymptotically nonexpansive mapping with the nonempty fixed-point set and a sequence* $\{k_n\}$ *of real numbers such that* $k_n \geq 1$ *and* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. *Let* $\{x_n\}$ *be a sequence in* C *defined by Algorithm 3 with the following restrictions:*

(i) $0 < \liminf_n \gamma_n \leq \limsup_n \gamma_n < 1$,

(ii) $\limsup_n c_n < 1$, *and*

(iii) $\sum_{n=1}^{\infty} t_n < \infty$, $\sum_{n=1}^{\infty} s_n < \infty$, $\sum_{n=1}^{\infty} c_n r_n < \infty$.

If T *satisfies Condition (A) with respect to the sequence* $\{x_n\}$, *then* $\{x_n\}$ *converges strongly to a fixed point of* T.

Remark 11

1. Corollary 9 extends and improves Theorem 2.3 of [5] in the following ways:
   (i) The condition $\liminf_n c_n > 0$ is removed.
   (ii) The restriction $\sum_{n=1}^{\infty} r_n < \infty$ is weakened and replaced by $\sum_{n=1}^{\infty} c_n r_n < \infty$ and $\sum_{n=1}^{\infty} \delta_n r_n < \infty$.
   (iii) The complete continuity imposed on T is replaced by the more general Condition (A) with respect to $\{x_n\}$ (see also [1, Corollary 2.5]).
2. Corollary 10 extends and improves Theorem 2.4 of [1]. The restriction $\sum_{n=1}^{\infty} r_n < \infty$ is weakened and replaced by $\sum_{n=1}^{\infty} c_n r_n < \infty$.

3. Corollary 10 also extends and improves Theorem 3.2 of [4] in the following ways:

   (i) The semi-compactness imposed on T is weakened by assuming that T satisfies Condition (A) with respect to $\{x_n\}$ [1, Corollary 2.5].

   (ii) The condition $\lim_n c_n = 0$ is weakened and replaced by $\limsup_n c_n < 1$.

Next, as consequences of Lemma 5(ii), (iii) and Lemma 6, we have the following theorems.

Theorem 12 *Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $T : C \to C$ be an asymptotically nonexpansive mapping with the nonempty fixed-point set and a sequence $\{k_n\}$ of real numbers such that $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be a sequence in C defined by Algorithm 1 with the following restrictions:*

(i) $0 < \liminf_n \delta_n \leq \limsup_n(\beta_n + \gamma_n + \delta_n) < 1$,

(ii) $\limsup_n b'_n < 1$, *and*

(iii) $\sum_{n=1}^{\infty} t_n < \infty$, $\sum_{n=1}^{\infty} \alpha_n s_n < \infty$, $\sum_{n=1}^{\infty} r_n < \infty$.

If T satisfies Condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.

Theorem 13 *Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $T : C \to C$ be an asymptotically nonexpansive mapping with the nonempty fixed-point set and a sequence $\{k_n\}$ of real numbers such that $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be a sequence in C defined by Algorithm 1 with the following restrictions:*

(i) $0 < \liminf_n \beta_n \leq \limsup_n(\beta_n + \gamma_n + \delta_n) < 1$ *and*

(ii) $\sum_{n=1}^{\infty} t_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n s_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n c_n r_n < \infty$, $\sum_{n=1}^{\infty} \delta_n r_n < \infty$.

If T satisfies Condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.

Remark 14 By using the same ideas and techniques, we can also discuss the weak convergence for asymptotically nonexpansive mappings with errors and thereby improve the corresponding results obtained by Cho, Zhou and Guo [1], Liu and Kang [4], and Nammanee, Noor and Suantai [5].

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