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Integral Bifurcation Method and Its Application for Solving the Modified Equal Width Wave Equation and Its Variants

ABSTRACT. In this paper, a improved method named the integral bifurcation is introduced. In order to demonstrate its effectiveness for obtaining travelling waves of the nonlinear wave equations, we studied the modified equal width wave equation and its variants by this new method. Under the different parameter conditions, many integral bifurcations are obtained. According to these integral bifurcations, different kinds of travelling wave solutions are figured out. Compared with [1], many new travelling wave solutions are obtained.

KEY WORDS. integral bifurcation method, the modified equal width equation, integral bifurcations, travelling wave solutions

1 Introduction

In recent years, the sine-cosine method (see Refs. [1–4] and cited therein), the tanh-function method (see Refs. [5–8] and cited therein) and the bifurcation theory of the planar dynamical system (see Refs. [11–21] and cited therein) have been often used to study the problem of all kinds of travelling wave solutions in the nonlinear wave equation domain. These mathematical methods have been, and continue to be, popular tools for nonlinear analysis. However, by using the sine-cosine and tanh-function methods to solve nonlinear wave equations, we cannot obtain the solutions of the type of elliptic function. Among these three mathematical methods, the bifurcation theory of the planar dynamical system is acceptable on discussion of the existence of travelling wave solutions, using this method, we can obtain all kinds of travelling wave solutions, but its analysis of phase portraits and discussion of bifurcation are more complicated. Therefore, in this paper, we shall introduce a improved method (are also called simplified method) named the integral bifurcation based on the bifurcation theory of the planar dynamical system. This improved method needn't make complicated analysis of phase portraits like the bifurcation theory and is easy enough in practice. In order to demonstrate its effectiveness for obtaining travelling waves of the nonlinear wave equations, we shall consider the following nonlinear modified equal width (MEW) wave equation

$$u_t + a(u^3)_x + bu_{xxt} = 0\,, \tag{1.1}$$

and its variants

$$u_t + a(u^n)_x - b(u^n)_{xxt} = 0\,, \tag{1.2}$$

and

$$u_t + a(u^{-n})_x - b(u^{-n})_{xxt} = 0\,, \tag{1.3}$$

where a, b nonzero real parameters, n is positive integer and $n > 1$.

The modified equal width (MEW) wave equation has been discussed in Refs. [1, 9, 10]. Just as A. M. Wazwaz said, the MEW equation, which is related to the regularized long wave (RLW) equation [22], has solitary waves with both positive and negative amplitudes, all of which have the same width. The MEW equation is a nonlinear wave equation with cubic nonlinearity with a pulse-like solitary wave solution. The MEW equation's variants, (1.2) and (1.3) can be reduced to $K(m, n)$ type equations which is well known, so these two equations are also very good application.

By using tanh and sine-cosine methods, A. M. Wazwaz studied the equations (1.1), (1.2) and (1.3), many solutions including compactons and periodic solutions are given (see [1]). In fact, by using the integral bifurcation method, we shall obtain more exact travelling wave solutions.

Making the transformation $u(x, t) = \phi(x - ct) = \phi(\xi)$, then substituting $\phi(x - ct)$ into (1.1), (1.2) and (1.3) respectively, we obtain the following three nonlinear ODE equations

$$-c\phi' + a(\phi^3)' - bc\phi''' = 0\,, \tag{1.4}$$

and

$$-c\phi' + a(\phi^n)' + bc(\phi^n)''' = 0\,, \tag{1.5}$$

and

$$-c\phi' + a(\phi^{-n})' + bc(\phi^{-n})''' = 0\,, \tag{1.6}$$

where "′" is the derivative with respect to ξ (i.e. $\phi' = \phi_\xi$) and c is wave speed.

Integrating (1.4), (1.5) and (1.6) once and setting the integral constant as zero, we obtain the following three wave equations, respectively

$$-c\phi + a\phi^3 - bc\phi'' = 0\,, \tag{1.7}$$

and

$$-c\phi + a\phi^n + bcn(n-1)\phi^{n-2}(\phi')^2 + bcn\phi^{n-1}\phi'' = 0\,, \tag{1.8}$$

and

$$-c\phi + a\phi^{-n} + bcn(n+1)\phi^{-(n+2)}(\phi')^2 - bcn\phi^{-(n+1)}\phi'' = 0\,. \tag{1.9}$$

Letting $\phi' = y$, the equations (1.7), (1.8) and (1.9) become the following three two-dimensional systems, respectively

$$\frac{d\phi}{d\xi} = y, \qquad \frac{dy}{d\xi} = -\frac{1}{b}\phi + \frac{a}{bc}\phi^3\,, \tag{1.10}$$

and

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{c\phi - a\phi^n - bcn(n-1)\phi^{n-2}y^2}{bcn\phi^{n-1}}\,, \tag{1.11}$$

and

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{-c\phi^{n+3} + a\phi^2 + bcn(n+1)y^2}{bcn\phi}\,. \tag{1.12}$$

Systems (1.10), (1.11) and (1.12) are all integral systems. Clearly, system (1.10) has the following first integral

$$y^2 = -\frac{1}{b}\phi^2 + \frac{a}{2bc}\phi^4 + C\,, \tag{1.13}$$

where C is integral constant. We define

$$F_1(\phi, y^2) = y^2 + \frac{1}{b}\phi^2 - \frac{a}{2bc}\phi^4\,. \tag{1.14}$$

System (1.11) has the following first integral

$$y^2 = \frac{2}{bn(n+1)}\phi^{3-n} - \frac{a}{bcn^2}\phi^2 + C\phi^{2-2n}\,. \tag{1.15}$$

Similarly we define

$$F_2(\phi, y^2) = \phi^{2n-2}y^2 + \frac{a}{bcn^2}\phi^{2n} - \frac{2}{bn(n+1)}\phi^{n+1}\,. \tag{1.16}$$

System (1.12) has the following first integral

$$y^2 = \frac{2}{bn(n-1)}\phi^{n+3} - \frac{a}{bcn^2}\phi^2 + C\phi^{2n+2}\,. \tag{1.17}$$

We also define

$$F_3(\phi, y^2) = \frac{y^2}{\phi^{2n+2}} + \frac{a}{bcn^2}\phi^{-2n} - \frac{2}{bn(n-1)}\phi^{-(n-1)}\,. \tag{1.18}$$

In the next,we shall introduce integral bifurcation method. By using this method, under the different parameter conditions and choosing the proper integral constant C and using (1.13), (1.15) and (1.17), we shall derive all kinds of integral bifurcations. Utilizing these integral bifurcations, we can obtain all kinds of travelling wave solutions of (1.1), (1.2) and (1.3).

The rest of this paper is organized as follows: In Section 2, we shall introduce the integral bifurcation method. In section 3, by using the integral bifurcation method, we shall derive the travelling wave solutions of equation (1.1). In section 4, by using the integral bifurcation method, we will derive the travelling wave solutions of equation (1.2). In section 5, by using the integral bifurcation method, we shall derive the travelling wave solutions of equation (1.3).

2 Integral bifurcation method

For a given $(n+1)-$dimensional nonlinear partial differential equation

$$E[t, x_i, u_{x_i}, u_{x_ix_i}, u_{x_ix_j}, u_{tt}, \cdots] = 0, \quad (i, j = 1, 2, \cdots, n)\,. \tag{2.1}$$

The integral bifurcation method simply proceeds as follows:
Step1. Making a transformation $u(t, x_1, x_2, \cdots, x_n) = \phi(\xi)$, $\xi = \sum_{i=1}^{n} \mu_i x_i - ct$, (2.1) can be reduced to a nonlinear ODE

$$P(\xi, \phi, \phi_\xi, \phi_{\xi\xi}, \phi_{\xi\xi\xi} \cdots) = 0\,, \tag{2.2}$$

where μ_i, $(i = 1, 2, \cdots, n)$ are arbitrary nonzero constants. After integrating Eq. (2.2) several times, if it can be reduced to the following second-order nonlinear ODE

$$G(\phi, \phi_\xi, \phi_{\xi\xi}) = 0\,, \tag{2.3}$$

then we go on the next process.
Step2. Let $\phi_\xi = \frac{d\phi}{d\xi} = y$. Eq. (2.3) can be reduced to a two-dimensional planar systems

$$\frac{d\phi}{d\xi} = y, \qquad \frac{dy}{d\xi} = f(\phi, y)\,, \tag{2.4}$$

where $f(\phi, y)$ is an integral expression or a fraction. If $f(\phi, y)$ is a fraction such as $f(\phi, y) = \frac{f^*(\phi,y)}{g(\phi)}$ and $g(\phi_s) = 0$, then $\phi_{\xi\xi}$ (i.e $\frac{dy}{d\xi}$) does not exist when $\phi = \phi_s$. In this case, we make a transformation $d\xi = g(\phi)d\tau$, Eq. (2.4) can be rewritten as

$$\frac{d\phi}{d\tau} = g(\phi)y, \qquad \frac{dy}{d\tau} = f^*(\phi, y)\,, \tag{2.5}$$

where τ is a parameter. If the system (2.4) is an integral system, then Eqs. (2.4) and (2.5) have the same first integral as the follows

$$H(\phi, y) = h\,, \tag{2.6}$$

where h is integral constant. Commonly, the function y of (2.6) is satisfied the following relationship:

$$y = y(\phi, h)\,. \tag{2.7}$$

Substituting (2.7) into the first equation of (2.4) and integrating it, we obtain

$$\int_{\phi(0)}^{\phi} \frac{d\varphi}{y(\varphi, h)} = \int_0^{\xi} d\nu\,, \tag{2.8}$$

where $\phi(0)$ and 0 are initial constants. Taking proper initial constants and integrating equation (2.8), we can obtain exact travelling wave solution of Eq. (2.1). In fact, the initial constants can be taken by some extreme points or inflection points of the travelling waves. In other words, $\phi(0)$ is root of the equation (2.7) when $y = 0$ or the equation $\frac{dy}{d\xi} = 0$. Particularly, the initial constants can be also taken by $(\phi_s, 0)$ and a beforehand given $(\phi(0), \xi_0)$.

As the value of parameters of Eq. (2.1) and constant h of Eqs. (2.6), (2.7) are varied, so are the integral expression (2.8). Therefore, we call these integral expressions integral bifurcations. The different integral bifurcations correspond to different travelling wave solutions. This is the whole process of the integral bifurcation method. Using this method, we shall investigate travelling wave solutions of the equations (1.1), (1.2) and (1.3). See the below computations.

3 Travelling wave solutions of the equation (1.1)

It is easy to see that the system (1.10) has three equilibrium points $(0, 0)$ and $(\pm\sqrt{\frac{c}{a}}, 0)$ as $ac > 0$. From (1.14), we have

$$F_1(0, 0) = 0, \qquad F_1\left(\sqrt{\frac{c}{a}}, 0\right) = F_1\left(-\sqrt{\frac{c}{a}}, 0\right) = \frac{c}{2ab}\,. \tag{3.1}$$

According to the analysis of the section 2, we shall calculate the explicit expressions of all kinds of travelling wave solutions of (1.1).

3.1 Under the conditions of $ac > 0,\ b > 0$,

(1) taking $C = F_1(\pm\sqrt{\frac{c}{a}}, 0) = \frac{c}{2ab}$ and substituting it into (1.13), it yields

$$y = \pm\sqrt{\frac{a}{2bc}}\left(\phi^2 - \frac{c}{a}\right)\,. \tag{3.2}$$

Letting $\frac{dy}{d\xi} = 0$ in (1.10), we obtain $\phi(0) = 0$. Under the initial condition $\phi(0) = 0$, substituting (3.2) into (2.8), we obtain the following integral bifurcations

$$\int_0^{\phi} \frac{d\phi}{\frac{c}{a} - \phi^2} = \sqrt{\frac{a}{2bc}} \int_0^{\xi} d\xi, \quad \text{for} \quad \xi \geq 0\,, \tag{3.3}$$

$$\int_{\phi}^{0} \frac{d\phi}{\frac{c}{a} - \phi^2} = -\sqrt{\frac{a}{2bc}} \int_{\xi}^{0} d\xi, \quad \text{for} \quad \xi < 0\,. \tag{3.4}$$

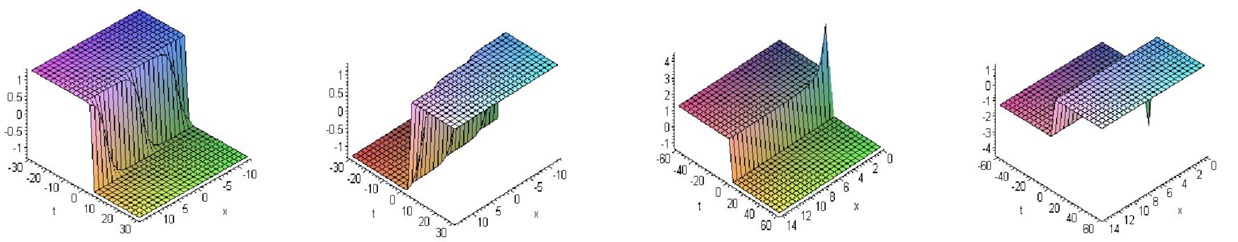
Integrating (3.3) and (3.4), we obtain two couple of kink and anti-kink wave solutions,

$$u_1(x,t) = \phi(x - ct) = \pm\sqrt{\frac{c}{a}}\ \tanh \frac{1}{\sqrt{2b}}(x - ct)\,, \tag{3.5}$$

and

$$u_2(x,t) = \phi(x - ct) = \pm\sqrt{\frac{c}{a}}\ \coth \frac{1}{\sqrt{2b}}(x - ct)\,, \tag{3.6}$$

the 3D graphs of kink and anti-kink wave solutions are shown in Fig. 1. In the graphs, the abscissa axis is t, the ordinate axis is x and the vertical axis is u.



(a) *kink wave 1* (b) *anti-kink wave 1* (c) *kink wave 2* (d) *anti-kink wave 2*

Fig. 1: The 3D graphs of (3.5) and (3.6) as $a = 2.5,\ b = 2,\ c = 4,\ C = 0$.

(2) When $0 < C < \frac{c}{2ab}$, from (1.13), it yields

$$y = \pm\sqrt{\frac{a}{2bc}}\sqrt{\frac{2bcC}{a} - \frac{2c}{a}\phi^2 + \phi^4} = \pm\sqrt{\frac{a}{2bc}}\sqrt{(\alpha^2 - \phi^2)(\beta^2 - \phi^2)}\,, \tag{3.7}$$

where $\alpha^2 = \frac{c}{a}(1+\sqrt{1 - \frac{2abC}{c}})$, $\beta^2 = \frac{c}{a}(1-\sqrt{1 - \frac{2abC}{c}})$ and $\alpha > \beta > \phi > 0$. Taking the initial conditions $\phi(0) = 0$ and substituting (3.7) into the (2.8), we obtain the following integral bifurcations,

$$\int_0^{\phi} \frac{d\phi}{\sqrt{(\alpha^2 - \phi^2)(\beta^2 - \phi^2)}} = \sqrt{\frac{a}{2bc}} \int_0^{\xi} d\xi \quad \text{for } \xi \geq 0\,. \tag{3.8}$$

$$\int_{\phi}^{0} \frac{d\phi}{\sqrt{(\alpha^2 - \phi^2)(\beta^2 - \phi^2)}} = -\sqrt{\frac{a}{2bc}} \int_{\xi}^{0} d\xi, \quad \text{for } \xi < 0\,. \tag{3.9}$$

By using the elliptic integral formulas, we obtain the following a family of periodic wave solutions,

$$u(x,t) = \phi(x - ct) = \beta\ sn(\omega_1(x - ct), k_1)\,, \tag{3.10}$$

where $\omega_1 = \pm\alpha\sqrt{\frac{a}{2bc}}$, $k_1 = \frac{\beta}{\alpha}$.

(3) Taking $C = 0$, from (1.13) we have

$$y = \pm\sqrt{\frac{a}{2bc}}\phi\sqrt{\phi^2 - \frac{2c}{a}}\,, \tag{3.11}$$

and $\phi_{1,2}(0) = \pm\sqrt{\frac{2c}{a}}$. Similarly, Under the initial condition $(\phi(0), \xi_0) = (\pm\sqrt{\frac{2c}{a}}, \xi_0)$, substituting (3.11) into (2.8) and integrating it, we obtain a periodic wave solution:

$$u(x,t) = \phi(\xi) = \sqrt{\frac{2c}{a}}\sec\frac{1}{\sqrt{b}}(x - ct - \xi_0)\,, \tag{3.12}$$

where ξ_0 is an arbitrary constant. Especially, when $\xi_0 = 0$ or $\xi_0 = \frac{\pi}{2}$, we obtain the following two results which are the same as in Ref. [1],

$$u(x,t) = \phi(\xi) = \sqrt{\frac{2c}{a}}\sec\frac{1}{\sqrt{b}}(x - ct)\,, \tag{3.13}$$

or

$$u(x,t) = \phi(\xi) = \sqrt{\frac{2c}{a}}\csc\frac{1}{\sqrt{b}}(x - ct)\,. \tag{3.14}$$

3.2 Under the conditions of $ac > 0,\ b < 0$,

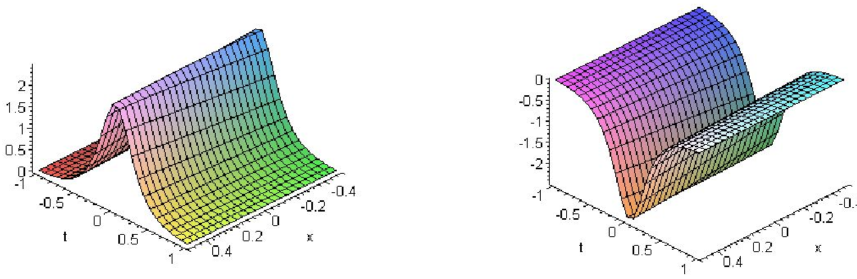
(1) taking $C = 0$, from (1.13), it yields

$$y = \pm\sqrt{-\frac{a}{2bc}}\phi\sqrt{\frac{2c}{a} - \phi^2}\,, \tag{3.15}$$

and $\phi(0) = \pm\sqrt{\frac{2c}{a}}$. Under the initial condition $\phi(0) = \pm\sqrt{\frac{2c}{a}}$, substituting (3.15) into (2.8) and integrating it, we obtain two smooth solitary wave solutions,

$$u(x,t) = \phi(\xi) = \pm\sqrt{\frac{2c}{a}}\operatorname{sech}\frac{1}{\sqrt{-b}}(x - ct)\,, \tag{3.16}$$

the 3D graphs of solitary wave solutions of (3.16) are shown in Fig. 2. In the graphs, the abscissa axis is t, the ordinate axis is x and the vertical axis is u.



(a) *solitary wave of peak form* (b) *solitary wave of valley form*

Fig. 2: The 3D graphs of (3.16) as $a = 5,\ b = -8,\ c = 15,\ C = 0,\ x \in (-0.5, 0.5)$.

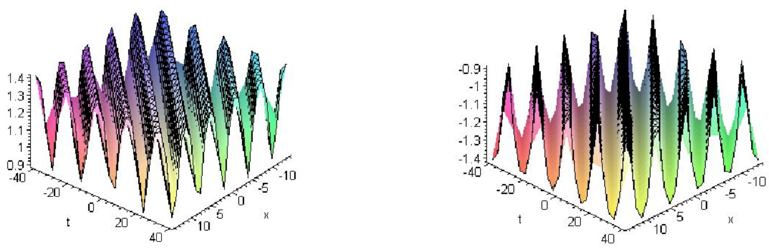
(2) When $\frac{c}{2ab} < C < 0$, from (3.13), it yields

$$y = \pm\sqrt{-\frac{a}{2bc}}\sqrt{(\alpha^2 - \phi^2)(\phi^2 - \beta^2)}\,, \tag{3.17}$$

and $\phi(0) = \alpha$, where $\alpha^2,\ \beta^2$ are given above. Corresponding to the Eq. (3.17), we obtain two families of periodic wave solutions of the type of elliptic function,

$$u(x,t) = \phi(x - ct) = \pm\sqrt{\alpha^2 - (\alpha^2 - \beta^2)sn^2[\omega_2(x - ct), k_2]}\,, \tag{3.18}$$

where $\omega_2 = \alpha\sqrt{-\frac{a}{2bc}},\ k_2 = \sqrt{\frac{\alpha^2-\beta^2}{\alpha^2}}$ and the 3D graphs of (3.18) are shown in Fig. 3. In the graphs, the abscissa axis is t, the ordinate axis is x and the vertical axis is u.



(a) + (b) −

Fig. 3: The 3D graphs of (3.18) as $a = 2.5,\ b = -2,\ c = 4,\ C = -0.3,\ x \in (-14, 14)$.

(3) When $C > 0$, from (1.13), we obtain

$$y = \pm\sqrt{-\frac{a}{2bc}}\sqrt{(\beta^2 + \phi^2)(\alpha^2 - \phi^2)}\,, \tag{3.19}$$

and $\phi(0)=\alpha$, where $\alpha^2>0,\ \beta^2<0$ are given above. Corresponding to the Eq. (3.19), we obtain a family of periodic wave solutions,

$$u(x,t)=\phi(x-ct)=\alpha\ cn[\omega_3(x-ct),k_3]\,, \tag{3.20}$$

where $\omega_3=\sqrt{-\frac{a(\alpha^2+\beta^2)}{2bc}},\ k_3=\sqrt{\frac{\alpha^2}{\alpha^2+\beta^2}}$.

Under the other parameter conditions, according to the above results, we can obtain the other travelling wave solutions without difficulty:

(i) When $ac>0,\ b<0,\ C=\frac{c}{2ab}$, from (3.5) and (3.6), we obtain

$$u(x,t)=\phi(x-ct)=\pm i\sqrt{\frac{c}{a}}\ \tan\frac{1}{\sqrt{-2b}}(x-ct)\,, \tag{3.21}$$

or

$$u(x,t)=\phi(x-ct)=\mp i\sqrt{\frac{c}{a}}\ \cot\frac{1}{\sqrt{-2b}}(x-ct)\,, \tag{3.22}$$

where $i=\sqrt{-1}$.

(ii) When $ac<0,\ b>0,\ C=0$, from (3.13), we obtain

$$u(x,t)=\phi(\xi)=\pm i\sqrt{-\frac{2c}{a}}\ \mathrm{sech}\,\frac{1}{\sqrt{b}}(x-ct)\,. \tag{3.23}$$

(iii) When $ac<0,\ b>0,\ C=\frac{a}{2bc}$, from (3.5) and (3.6), we obtain

$$u(x,t)=\phi(x-ct)=\pm i\sqrt{-\frac{c}{a}}\ \tanh\frac{1}{\sqrt{2b}}(x-ct)\,, \tag{3.24}$$

or

$$u(x,t)=\phi(x-ct)=\mp i\sqrt{-\frac{c}{a}}\ \coth\frac{1}{\sqrt{2b}}(x-ct)\,. \tag{3.25}$$

(iv) When $ac<0,\ b<0,\ C=0$, from (3.16), we obtain

$$u(x,t)=\phi(\xi)=\pm i\sqrt{-\frac{2c}{a}}\ \mathrm{sech}\,\frac{1}{\sqrt{-b}}(x-ct)\,. \tag{3.26}$$

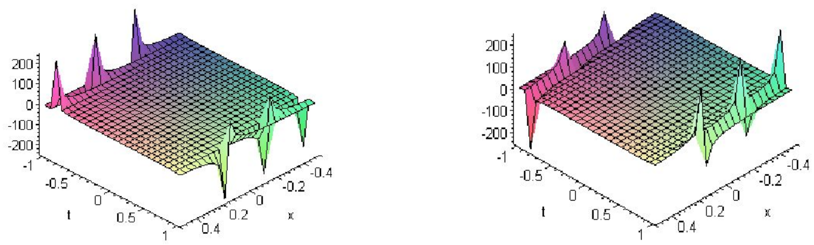
(v) When $ac<0,\ b<0,\ C=\frac{a}{2bc}$, from (3.5) and (3.6), we obtain

$$u(x,t)=\phi(x-ct)=\pm\sqrt{-\frac{c}{a}}\ \tan\frac{1}{\sqrt{-2b}}(x-ct)\,, \tag{3.27}$$

or

$$u(x,t) = \phi(x-ct) = \mp\sqrt{-\frac{c}{a}}\ \cot\frac{1}{\sqrt{-2b}}(x-ct)\,. \tag{3.28}$$

The 3D graphs of travelling wave solutions of the type of tangent function are shown in Fig. 4. In the graphs, the abscissa axis is t, the ordinate axis is x and the vertical axis is u.



(a) $+$ (b) $-$

Fig. 4: The 3D graphs of (3.27) as $a=-2,\ b=-3,\ c=4,\ C=0,\ x\in(-0.5,0.5)$.

(vi) When $ac>0,\ b<0,\ C=0$, from (3.14) yields

$$u(x,t) = \phi(x-ct) = i\sqrt{\frac{2c}{a}}\ \operatorname{csch}\frac{1}{\sqrt{-b}}(x-ct)\,. \tag{3.29}$$

4 Travelling wave solutions of the equation (1.2)

Because the equation (1.15) has a term of $C\phi^{2-2n}$, we only consider those travelling wave solutions in which the integral constant C is zero.

4.1 Under the conditions of $ac>0,\ b>0,\ C=0$, from (1.15), we obtain

$$y^2 = \frac{\frac{2}{bn(n+1)}\phi^{n-1} - \frac{a}{bcn^2}\phi^{2n-2}}{\phi^{2n-4}}\,, \tag{4.1}$$

i.e.

$$y = \pm\frac{\frac{1}{n}\sqrt{\frac{a}{bc}}\sqrt{\frac{2cn}{a(n+1)}\phi^{n-1} - (\phi^{n-1})^2}}{\phi^{n-2}}\,, \tag{4.2}$$

and $\phi_1(0)=0,\ \phi_2(0)=[\frac{2cn}{a(n+1)}]^{\frac{1}{n-1}}$. Where $\phi_1(0)=0\neq\phi_s$ as $n=2$, and $\phi_1(0)=0=\phi_s$ as

$n > 2$. Substituting (4.2) into (2.8), we obtain the following integral bifurcations

$$\int_0^{\phi} \frac{d\phi^{n-1}}{\sqrt{\frac{2cn}{a(n+1)}\phi^{n-1} - (\phi^{n-1})^2}} = \frac{n-1}{n}\sqrt{\frac{a}{bc}}\int_0^{\xi} d\xi \quad \text{for } \xi \geq 0\,, \tag{4.3}$$

$$-\int_{\phi}^{0} \frac{d\phi^{n-1}}{\sqrt{\frac{2cn}{a(n+1)}\phi^{n-1} - (\phi^{n-1})^2}} = \frac{n-1}{n}\sqrt{\frac{a}{bc}}\int_{\xi}^{0} d\xi \quad \text{for } \xi < 0\,, \tag{4.4}$$

and

$$\int_{[\frac{2cn}{a(n+1)}]^{\frac{1}{n-1}}}^{\phi} \frac{d\phi^{n-1}}{\sqrt{\frac{2cn}{a(n+1)}\phi^{n-1} - (\phi^{n-1})^2}} = \frac{n-1}{n}\sqrt{\frac{a}{bc}}\int_0^{\xi} d\xi \quad \text{for } \xi \geq 0\,, \tag{4.5}$$

$$-\int_{\phi}^{[\frac{2cn}{a(n+1)}]^{\frac{1}{n-1}}} \frac{d\phi^{n-1}}{\sqrt{\frac{2cn}{a(n+1)}\phi^{n-1} - (\phi^{n-1})^2}} = \frac{n-1}{n}\sqrt{\frac{a}{bc}}\int_{\xi}^{0} d\xi \quad \text{for } \xi < 0\,. \tag{4.6}$$

Integrating (4.3) and (4.4), we obtain

$$\phi^{n-1} = \frac{cn}{a(n+1)}\left[1 - \cos\frac{n-1}{n}\sqrt{\frac{a}{bc}}(x-ct)\right]\,, \tag{4.7}$$

or

$$\phi^{n-1} = \frac{2cn}{a(n+1)}\sin^2\frac{n-1}{2n}\sqrt{\frac{a}{bc}}(x-ct)\,. \tag{4.8}$$

Integrating (4.5) and (4.6), we obtain

$$\phi^{n-1} = \frac{cn}{a(n+1)}\left[1 + \cos\frac{n-1}{n}\sqrt{\frac{a}{bc}}(x-ct)\right]\,, \tag{4.9}$$

or

$$\phi^{n-1} = \frac{2cn}{a(n+1)}\cos^2\frac{n-1}{2n}\sqrt{\frac{a}{bc}}(x-ct)\,. \tag{4.10}$$

Thus, when $n = 2$, we obtain two smooth periodic wave solutions

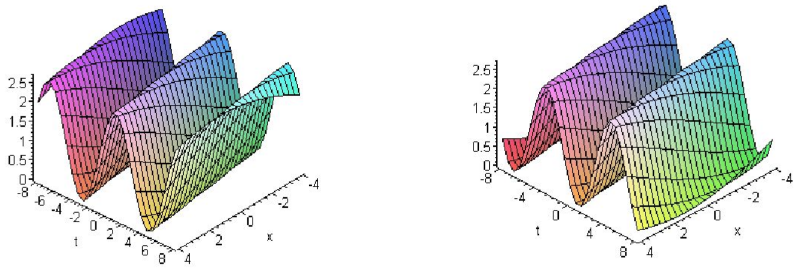
$$u(x,t) = \phi(x-ct) = \frac{cn}{a(n+1)}\left[1 \pm \cos\frac{n-1}{n}\sqrt{\frac{a}{bc}}(x-ct)\right]\,, \tag{4.11}$$

or

$$u(x,t) = \phi(x-ct) = \frac{2cn}{a(n+1)}\sin^2\frac{n-1}{2n}\sqrt{\frac{a}{bc}}(x-ct)\,, \tag{4.12}$$

$$u(x,t) = \phi(x-ct) = \frac{2cn}{a(n+1)}\cos^2\frac{n-1}{2n}\sqrt{\frac{a}{bc}}(x-ct)\,, \tag{4.13}$$

for $x - ct \in (-\infty, +\infty)$. The 3D graphs of periodic wave solutions of (4.11) are shown in Fig. 5. In the graphs, the abscissa axis is t, the ordinate axis is x and the vertical axis is u.



(a) + (b) −

Fig. 5: The 3D graphs of (4.11) as $n = 2,\ a = -2,\ b = -3,\ c = 4,\ C = 0,\ x \in (-0.5, 0.5)$.

When n is an even number and $n > 2$, we obtain two periodic cusp wave solutions

$$u(x,t) = \phi(x - ct) = \left\{ \frac{cn}{a(n+1)} \left[1 \pm \cos \frac{n-1}{n} \sqrt{\frac{a}{bc}} (x - ct) \right] \right\}^{\frac{1}{n-1}}, \tag{4.14}$$

or

$$u(x,t) = \phi(x - ct) = \left[\frac{2cn}{a(n+1)} \sin^2 \frac{n-1}{2n} \sqrt{\frac{a}{bc}} (x - ct) \right]^{\frac{1}{n-1}}, \tag{4.15}$$

$$u(x,t) = \phi(x - ct) = \left[\frac{2cn}{a(n+1)} \cos^2 \frac{n-1}{2n} \sqrt{\frac{a}{bc}} (x - ct) \right]^{\frac{1}{n-1}}. \tag{4.16}$$

When n is an odd number and $n > 1$, we obtain four periodic cusp wave solutions

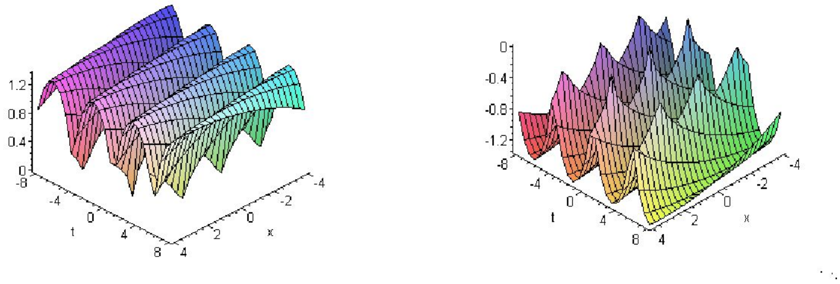
$$u(x,t) = \phi(x - ct) = \pm \left\{ \frac{cn}{a(n+1)} \left[1 \pm \cos \frac{n-1}{n} \sqrt{\frac{a}{bc}} (x - ct) \right] \right\}^{\frac{1}{n-1}}, \tag{4.17}$$

or

$$u(x,t) = \phi(x - ct) = \pm \left[\frac{2cn}{a(n+1)} \sin^2 \frac{n-1}{2n} \sqrt{\frac{a}{bc}} (x - ct) \right]^{\frac{1}{n-1}}, \tag{4.18}$$

$$u(x,t) = \phi(x - ct) = \pm \left[\frac{2cn}{a(n+1)} \cos^2 \frac{n-1}{2n} \sqrt{\frac{a}{bc}} (x - ct) \right]^{\frac{1}{n-1}}. \tag{4.19}$$

The 3D graphs of periodic cusp wave solutions of (4.18) are shown in Fig. 6. In the graphs, the abscissa axis is t, the ordinate axis is x and the vertical axis is u.



(a) + (b) −

Fig. 6: The 3D graphs of (4.18) as $n = 5,\ a = 2,\ b = 3,\ c = 4,\ C = 0,\ x \in (-4, 4)$.

When n is an arbitrary positive integer, from (4.15), (4.16), (4.18) and (4.19) we can obtain two compacton solutions of peak type which have been given in reference [1]:

$$\begin{cases} u(x,t) = \left[\frac{2cn}{a(n+1)} \sin^2 \frac{n-1}{2n} \sqrt{\frac{a}{bc}}(x-ct)\right]^{\frac{1}{n-1}}, & 0 \le (x-ct) \le \frac{2n\pi}{n-1}\sqrt{\frac{bc}{a}}, \\ 0, & otherwise. \end{cases} \tag{4.20}$$

and

$$\begin{cases} u(x,t) = \left[\frac{2cn}{a(n+1)} \cos^2 \frac{n-1}{2n} \sqrt{\frac{a}{bc}}(x-ct)\right]^{\frac{1}{n-1}}, & |(x-ct)| \le \frac{n\pi}{n-1}\sqrt{\frac{bc}{a}}, \\ 0, & otherwise. \end{cases} \tag{4.21}$$

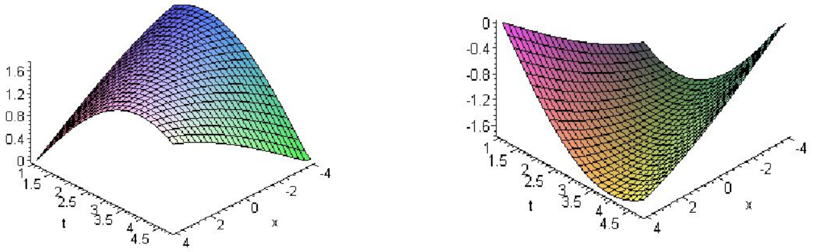
In fact, when n is an odd number, we also obtain two compacton solutions of valley type

$$\begin{cases} u(x,t) = -\left[\frac{2cn}{a(n+1)} \sin^2 \frac{n-1}{2n} \sqrt{\frac{a}{bc}}(x-ct)\right]^{\frac{1}{n-1}}, & 0 \le (x-ct) \le \frac{2n\pi}{n-1}\sqrt{\frac{bc}{a}}, \\ 0, & otherwise. \end{cases} \tag{4.22}$$

and

$$\begin{cases} u(x,t) = -\left[\frac{2cn}{a(n+1)} \cos^2 \frac{n-1}{2n} \sqrt{\frac{a}{bc}}(x-ct)\right]^{\frac{1}{n-1}}, & |(x-ct)| \le \frac{n\pi}{n-1}\sqrt{\frac{bc}{a}}, \\ 0, & otherwise. \end{cases} \tag{4.23}$$

The 3D graphs of compacton solutions of (4.20) and (4.22) are shown in Fig. 7. In the graphs, the abscissa axis is t, the ordinate axis is x and the vertical axis is u.



(a) compacton of (4.20) (b) compacton of (4.22)

Fig. 7: The 3D graphs of (4.18) and (4.22) as $n=3,\ a=2,\ b=3,\ c=4, C=0, x\in(-4,4)$.

4.2 Suppose that $ac<0,\ b>0,\ C=0$, from (1.15), it yields

$$y=\pm\frac{\frac{1}{n}\sqrt{-\frac{a}{bc}}\sqrt{-\frac{2cn}{a(n+1)}\phi^{n-1}+(\phi^{n-1})^2}}{\phi^{n-2}}, \tag{4.24}$$

$$\tag{4.25}$$

and $\phi(0)=0$. Thus, corresponding the Eq. (4.24), we obtain

$$\phi^{n-1}=-\frac{cn}{a(n+1)}\left[\cosh\frac{n-1}{n}\sqrt{-\frac{a}{bc}}(x-ct)-1\right]. \tag{4.26}$$

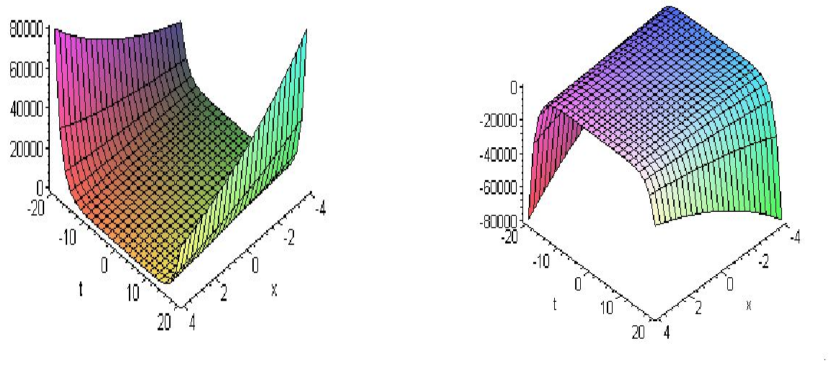
When n is an even number, we obtain a unbounded travelling wave solution of hyperbolic cosine type

$$u(x,t)=\phi(x-ct)=\left\{-\frac{cn}{a(n+1)}\left[\cosh\frac{n-1}{n}\sqrt{-\frac{a}{bc}}(x-ct)-1\right]\right\}^{\frac{1}{n-1}}. \tag{4.27}$$

When n is an odd number, we obtain two unbounded travelling wave solutions of hyperbolic cosine type

$$u(x,t)=\phi(x-ct)=\pm\left\{-\frac{cn}{a(n+1)}\left[\cosh\frac{n-1}{n}\sqrt{-\frac{a}{bc}}(x-ct)-1\right]\right\}^{\frac{1}{n-1}}. \tag{4.28}$$

The 3D graphs of unbounded travelling wave solutions of (4.28) are shown in Fig. 8. In the graphs, the abscissa axis is t, the ordinate axis is x and the vertical axis is u.



(a) $+$ (b) $-$

Fig. 8: The 3D graphs of (4.28) as $n = 5,\ a = -2,\ b = 3,\ c = 4,\ C = 0,\ x \in (-4, 4)$.

Similarly, under the other parameter conditions, according to the above results, we can obtain the other travelling wave solutions without difficulty:

(1) When n is an even number and $ac > 0,\ b < 0,\ C = 0$, from (4.14) and (4.15), we obtain three unbounded travelling wave solutions:

$$u(x,t) = \phi(x-ct) = \left\{ \frac{cn}{a(n+1)} \left[1 \pm \cosh \frac{n-1}{n} \sqrt{-\frac{a}{bc}}(x-ct) \right] \right\}^{\frac{1}{n-1}}, \tag{4.29}$$

and

$$u(x,t) = \phi(x-ct) = \left[-\frac{2cn}{a(n+1)} \sinh^2 \frac{n-1}{2n} \sqrt{-\frac{a}{bc}}(x-ct) \right]^{\frac{1}{n-1}}. \tag{4.30}$$

(2) When n is an odd number and $ac > 0,\ b < 0,\ C = 0$, from (4.17), (4.18) and (4.19), we obtain six unbounded travelling wave solutions:

$$u(x,t) = \phi(x-ct) = \pm \left\{ \frac{cn}{a(n+1)} \left[1 + \cosh \frac{n-1}{n} \sqrt{-\frac{a}{bc}}(x-ct) \right] \right\}^{\frac{1}{n-1}}, \tag{4.31}$$

and

$$u(x,t) = \phi(x-ct) = \pm \left[-\frac{2cn}{a(n+1)} \sinh^2 \frac{n-1}{2n} \sqrt{-\frac{a}{bc}}(x-ct) \right]^{\frac{1}{n-1}}, \tag{4.32}$$

$$u(x,t) = \phi(x-ct) = \pm \left[\frac{2cn}{a(n+1)} \cosh^2 \frac{n-1}{2n} \sqrt{-\frac{a}{bc}}(x-ct) \right]^{\frac{1}{n-1}}. \tag{4.33}$$

5 Travelling wave solutions of the equation (1.3)

Since (1.16) is a high order equation, we only consider the case of integral constant $C=0$ in this section.

5.1 Suppose that $ac>0,\ b>0,\ C=0$ or $ac<0,\ b<0,\ C=0$, from (1.17), we obtain

$$y=\pm\sqrt{\frac{2}{bn(n-1)}}\phi\sqrt{\phi^{n+1}-A^2}\,, \tag{5.1}$$

and $\phi_1(0)=A^{\frac{2}{n+1}}$ or $\phi_{1,2}(0)=\pm A^{\frac{2}{n+1}}$, where $A=\sqrt{\frac{a(n-1)}{2cn}}$. Substituting (5.1) into (2.8), we obtain the following integral bifurcations

$$\int_{\pm[A^{\frac{2}{n+1}}]}^{\phi}\frac{d\phi}{\phi\sqrt{\phi^{n+1}-A^2}}=\frac{n-1}{n}\sqrt{\frac{a}{bc}}\int_{\xi_k}^{\xi}d\xi,\quad \text{for } \xi\geq 0\,, \tag{5.2}$$

$$-\int_{\phi}^{\pm[A^{\frac{2}{n+1}}]}\frac{d\phi}{\phi\sqrt{\phi^{n+1}-A^2}}=\frac{n-1}{n}\sqrt{\frac{a}{bc}}\int_{\xi}^{\xi_k}d\xi,\quad \text{for } \xi<0\,, \tag{5.3}$$

where ξ_k is an arbitrary constant. Integrating (5.2) and (5.3), we obtain

$$\phi^{n+1}=\frac{a(n-1)}{2cn}\sec^2\frac{n+1}{2n}\sqrt{\frac{a}{bc}}(\xi-\xi_k)\,. \tag{5.4}$$

When n is an even number, we obtain a family of periodic wave solutions

$$u(x,t)=\phi(x-ct)=\left[\frac{a(n-1)}{2cn}\sec^2\frac{n+1}{2n}\sqrt{\frac{a}{bc}}(\xi-\xi_k)\right]^{\frac{1}{n+1}}\,. \tag{5.5}$$

Taking $\xi_k=0$ and $\xi_k=\frac{\pi}{2}$ respectively, we obtain the following two periodic wave solutions which are the same as Ref. [1],

$$u(x,t)=\phi(x-ct)=\left[\frac{a(n-1)}{2cn}\sec^2\frac{n+1}{2n}\sqrt{\frac{a}{bc}}(x-ct)\right]^{\frac{1}{n+1}}\,, \tag{5.6}$$

and

$$u(x,t)=\phi(x-ct)=\left[\frac{a(n-1)}{2cn}\csc^2\frac{n+1}{2n}\sqrt{\frac{a}{bc}}(x-ct)\right]^{\frac{1}{n+1}}\,. \tag{5.7}$$

When n is an odd number, we obtain two families of periodic wave solutions:

$$u(x,t)=\phi(x-ct)=\pm\left[\frac{a(n-1)}{2cn}\sec^2\frac{n+1}{2n}\sqrt{\frac{a}{bc}}(\xi-\xi_k)\right]^{\frac{1}{n+1}}\,. \tag{5.8}$$

Taking $\xi_k = 0$ and $\xi_k = \frac{\pi}{2}$ respectively, we obtain the following four periodic wave solutions:

$$u(x,t) = \phi(x-ct) = \pm\left[\frac{a(n-1)}{2cn}\sec^2\frac{n+1}{2n}\sqrt{\frac{a}{bc}}(x-ct)\right]^{\frac{1}{n+1}}. \tag{5.9}$$

and

$$u(x,t) = \phi(x-ct) = \pm\left[\frac{a(n-1)}{2cn}\csc^2\frac{n+1}{2n}\sqrt{\frac{a}{bc}}(x-ct)\right]^{\frac{1}{n+1}}. \tag{5.10}$$

5.2 Suppose that $ac > 0,\ b < 0,\ C = 0$ or $ac < 0,\ b > 0,\ C = 0$. According to the results (5.7), (5.8), (5.9), (5.10), we can obtain the other traveling wave solutions without difficulty:

(i) When n is an even number, we obtain two solitary wave solutions:

$$u(x,t) = \phi(x-ct) = \left[\frac{a(n-1)}{2cn}\ \operatorname{sech}^2\frac{n+1}{2n}\sqrt{-\frac{a}{bc}}(x-ct)\right]^{\frac{1}{n+1}}, \tag{5.11}$$

and

$$u(x,t) = \phi(x-ct) = \left[-\frac{a(n-1)}{2cn}\ \operatorname{csch}^2\frac{n+1}{2n}\sqrt{-\frac{a}{bc}}(x-ct)\right]^{\frac{1}{n+1}}. \tag{5.12}$$

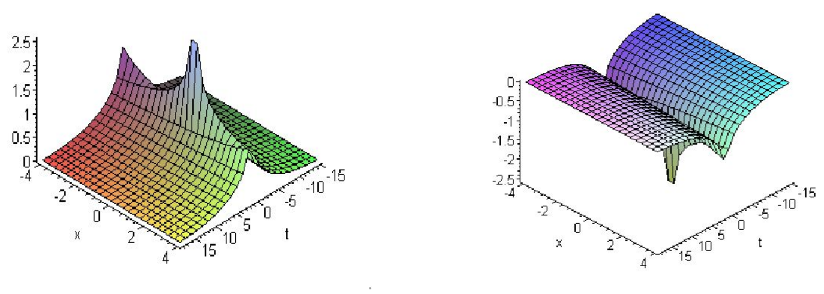
(ii) When n is an odd number, we obtain four solitary wave solutions:

$$u(x,t) = \phi(x-ct) = \pm\left[\frac{a(n-1)}{2cn}\ \operatorname{sech}^2\frac{n+1}{2n}\sqrt{-\frac{a}{bc}}(x-ct)\right]^{\frac{1}{n+1}}, \quad \text{for } ac > 0, \tag{5.13}$$

and

$$u(x,t) = \phi(x-ct) = \pm\left[-\frac{a(n-1)}{2cn}\ \operatorname{csch}^2\frac{n+1}{2n}\sqrt{-\frac{a}{bc}}(x-ct)\right]^{\frac{1}{n+1}}, \quad \text{for } ac < 0. \tag{5.14}$$

The 3D graphs of solitary wave solutions of (5.14) are shown in Fig. 9. In the graphs, the abscissa axis is t, the ordinate axis is x and the vertical axis is u.



(a) solitary of peak form (b) solitary of valley form

Fig. 9: The 3D graphs of (5.14) as $n = 5,\ a = -2,\ b = 3,\ c = 4,\ C = 0,\ x \in (-4, 4)$.

From the above process of deriving, it is easy to see that this method is also available to many nonlinear integral systems.

6 Conclusion

In this paper, we introduced a new method named integral bifurcation. By using this method, we studied the modified equal width wave equation and its variants and obtained many new traveling wave solutions in addition to the results in reference [1]. Clearly, this method is available to many nonlinear partial equations. However, when we solve the universal nonlinear partial equations by this method, are all the effects good? We will continue to thoroughly pay attention to this question.

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