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Implicit fixed point iterations

ABSTRACT. Let K be a compact convex subset of a real Hilbert space H; $T : K \to K$ a continuous hemicontractive map. Let $\{\alpha_n\}$ be a real sequence in $[0, 1]$ satisfying appropriate conditions, then for arbitrary $x_0 \in K$ and $\{v_n\}$ in K, the sequence $\{x_n\}$ defined iteratively by $x_n = \alpha_n x_{n-1} + (1 - \alpha_n)Tv_n$, $n \geq 1$ converges strongly to a fixed point of T.

We also establish a strong convergence of an implicit iteration process to a common fixed point for a finite family of $\psi-$uniformly pseudocontractive and $\psi-$uniformly accretive mappings in real Banach spaces.

The results presented in this paper extend and improve the corresponding results of Refs. [4, 9, 19, 20, 22, 25, 44].

KEY WORDS. Implicit iteration process, Mann iteration, $\psi-$uniformly pseudocontractive and $\psi-$uniformly accretive mappings, Common fixed point, Banach space, Hilbert Space

1 Fundamentals

We assume that E is a real Banach space and K be a nonempty convex subset of E. Let J denote the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = ||x||^2 \ \text{ and } \ ||f^*|| = ||x||\},$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. We shall denote the single-valued duality map by j.

Let $\Psi := \{\psi \mid \psi : [0, \infty) \to [0, \infty)$ is a strictly increasing mapping such that $\psi(0) = 0\}$.

Definition 1 *A mapping $T : K \to K$ is called $\psi-$uniformly pseudocontractive if there exist mapping $\psi \in \Psi$ and $j(x - y) \in J(x - y)$ such that*

$$\langle Tx - Ty, j(x - y) \rangle \leq ||x - y||^2 - \psi(||x - y||), \ \forall x, \ y \in K. \tag{1.1}$$

Definition 2 *A mapping $S : D(S) \subset E \to E$ is called $\psi-$uniformly accretive if there exist mapping $\psi \in \Psi$ and $j(x-y) \in J(x-y)$ such that*

$$\langle Sx - Sy, j(x-y)\rangle \geq \psi(||x-y||),\ \forall x,\ y \in E. \tag{1.2}$$

Remark 1 a) Taking $\psi(a) := \psi(a)a$, $\forall a \in [0,\infty)$, $(\psi \in \Psi)$, we get the usual definitions of $\psi-$ pseudocontractive and $\psi-$ accretive mappings.

b) Taking $\psi(a) := \gamma a^2; \gamma \in (0,1)$, $\forall a \in [0,\infty)$, $(\psi \in \Psi)$, we get the usual definitions of strongly pseudocontractive and strongly accretive mappings.

c) T is $\psi-$uniformly pseudocontractive iff $S = I - T$ is $\psi-$uniformly accretive.

d) It is known that T is strongly pseudocontractive if and only if $(I-T)$ is strongly accretive.

Let H be a Hilbert space.

Definition 3 *A mapping $T : H \to H$ is said to be* pseudocontractive *(see e.g.,* [1, 2]*) if*

$$||Tx - Ty||^2 \leq ||x-y||^2 + ||(I-T)x - (I-T)y||^2,\ \forall x, y \in H \tag{1.3}$$

and is said to be strongly pseudocontractive *if there exists $k \in (0,1)$ such that*

$$||Tx - Ty||^2 \leq ||x-y||^2 + k||(I-T)x - (I-T)y||^2,\ \forall x, y \in H. \tag{1.4}$$

Definition 4 *Let $F(T) := \{x \in H : Tx = x\}$ and let K be a nonempty subset of H. A map $T : K \to K$ is called* hemicontractive *if $F(T) \neq \emptyset$ and*

$$||Tx - x^*||^2 \leq ||x - x^*||^2 + ||x - Tx||^2 \ \ \forall\ x \in H,\ x^* \in F(T). \tag{1.5}$$

Remark 2 It is easy to see that the class of pseudocontractive maps with fixed points is a subclass of the class of hemicontractions. The following example, due to Rhoades [35], shows that the inclusion is proper. For $x \in [0,1]$, define $T : [0,1] \to [0,1]$ by $Tx = (1 - x^{\frac{2}{3}})^{\frac{3}{2}}$. It is shown in [35] that T is not Lipschitz and so cannot be nonexpansive. A straightforward computation (see e.g., [38]) shows that T is pseudocontractive. For the importance of fixed points of pseudocontractions the reader may consult [2].

We shall make use of the following results.

Lemma 1 [40] *Suppose that $\{\rho_n\}, \{\sigma_n\}$ are two sequences of nonnegative numbers such that for some real number $N_0 \geq 1$,*

$$\rho_{n+1} \leq \rho_n + \sigma_n\ \forall n \geq N_0.$$

(a) *If* $\sum \sigma_n < \infty$, *then,* $\lim \rho_n$ *exists.*

(b) *If* $\sum \sigma_n < \infty$ *and* $\{\rho_n\}$ *has a subsequence converging to zero, then* $\lim \rho_n = 0$.

Lemma 2 [20] *For all* $x, y \in H$ *and* $\lambda \in [0, 1]$, *the following well-known identity holds:*

$$||(1-\lambda)x + \lambda y||^2 = (1-\lambda)||x||^2 + \lambda||y||^2 - \lambda(1-\lambda)||x-y||^2.$$

Lemma 3 [42] *Let* $J : E \to 2^E$ *be the normalized duality mapping. Then for any* $x, y \in E$, *we have*

$$||x+y||^2 \le ||x||^2 + 2\langle y, j(x+y)\rangle, \quad \forall j(x+y) \in J(x+y).$$

Lemma 4 [23] *Let* $\{\theta_n\}$ *be a sequence of nonnegative real numbers,* $\{\lambda_n\}$ *be a real sequence satisfying*

$$0 \le \lambda_n \le 1, \quad \sum_{n=0}^{\infty} \lambda_n = \infty$$

and let $\psi \in \Psi$. *If there exists a positive integer* n_0 *such that*

$$\theta_{n+1}^2 \le \theta_n^2 - \lambda_n \psi(\theta_{n+1}) + \sigma_n,$$

for all $n \ge n_0$, *with* $\sigma_n \ge 0$, $\forall n \in \mathbb{N}$, *and* $\sigma_n = 0(\lambda_n)$, *then* $\lim_{n\to\infty} \theta_n = 0$.

2 Implicit Mann iteration process in Hilbert spaces

In the last ten years or so, numerous papers have been published on the iterative approximation of fixed points of Lipschitz *strongly* pseudocontractive (and correspondingly Lipschitz *strongly* accretive) maps using the *Mann iteration process* (see e.g., [22]). Results which had been known only in *Hilbert spaces* and only for *Lipschitz maps* have been extended to more general Banach spaces (see e.g., [5–16, 21, 30–38, 40, 41, 43, 45] and the references cited therein) and to more general classes of maps (see e.g., [6–16, 19, 21, 27–34, 36–38, 40, 41, 43, 45] and the references cited therein). This success, however, has not carried over to arbitrary *Lipschitz pseudocontraction* T even when the domain of the operator T is a *a compact convex subset of a Hilbert space.* In fact, it is still an open question whether or not the Mann iteration process converges under this setting. In 1974, Ishikawa introduced an iteration process which, in some sense, is more general than that of Mann and which converges, under this setting, to a fixed point of T. He proved the following theorem.

Theorem 1 *If* K *is a compact convex subset of a Hilbert space* H, $T : K \mapsto K$ *is a Lipschitzian pseudocontractive map and* x_0 *is any point in* K, *then the sequence* $\{x_n\}$

converges strongly to a fixed point of T*, where* x_n *is defined iteratively for each positive integer* $n \geq 0$ *by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \end{aligned} \tag{2.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *are sequences of positive numbers satisfying the conditions*

$$(i)\ 0 \leq \alpha_n \leq \beta_n < 1; \ \ (ii)\ \lim_{n\to\infty} \beta_n = 0; \ \ (iii)\ \sum_{n\geq 0} \alpha_n \beta_n = \infty.$$

Since its publication in 1974, Theorem 1, as far as we know, has never been extended to more general Banach spaces. In [27], Qihou extended the theorem to the slightly more general class of Lipschitz *hemicontractions* and in [28] he proved, under the setting of Theorem 1, that the convergence of the recursion formula (2.1) to a fixed point of T when T is a continuous hemicontractive map, *under the additional hypothesis that the number of fixed points of* T *is finite*. The iteration process (2.1) is generally referred to as the *Ishikawa iteration process* in light of [20]. Another iteration process which has been studied extensively in connection with fixed points of pseudocontractive maps is the following:

For K a convex subset of a real normed space H, and $T : K \to K$, the sequence $\{x_n\}$ is defined iteratively by $x_1 \in K$,

$$x_{n+1} = (1-c_n)x_n + c_n T x_n, \ n \geq 1, \tag{2.2}$$

where $\{c_n\}$ is a real sequence satisfying the following conditions:

$$(iv)\ 0 \leq c_n < 1; (v)\ \lim_{n\to\infty} c_n = 0; (vi)\ \sum_{n=1}^{\infty} c_n = \infty.$$

The iteration process (2.2) is generally referred to as the *Mann iteration process* in light of [22].

In 1995, Liu [21] introduced what he called *Ishikawa and Mann iteration processes with errors* as follows:

(1-a) For K a nonempty subset of H and $T : K \to E$, the sequence $\{x_n\}$ defined by

$$\begin{aligned} x_1 &\in K, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n + u_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n + v_n, \ n \geq 1, \end{aligned} \tag{2.3}$$

where, $\{\alpha_n\}, \{\beta_n\}$ are sequences in [0,1] satisfying appropriate conditions and $\sum ||u_n|| < \infty$, $\sum ||v_n|| < \infty$ is called the *Ishikawa Iteration process with errors.*

(1-b) With K, H and T as in part (1-a), the sequence $\{x_n\}$ defined by

$$\begin{aligned} x_1 &\in K, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T x_n + u_n, \ n \geq 1, \end{aligned} \tag{2.4}$$

where $\{\alpha_n\}$ is a sequence in [0,1] satisfying appropriate conditions and $\sum ||u_n|| < \infty$, is called the *Mann iteration process with errors.*

While it is known that consideration of error terms in iterative processes is an important part of the theory, it is also clear that the iteration processes with errors introduced by Liu in (1-a) and (1-b) are unsatisfactory. The occurrence of errors is random so that the conditions imposed on the error terms in (1-a) and (1-b) which imply, in particular, that they tend to zero as n tends to infinity are, therefore, unreasonable. In 1997, Y. Xu [43] introduced the following more satisfactory definitions.

(1-c) Let K be a nonempty convex subset of H and $T : K \to K$ a mapping. For any given $x_1 \in K$, the sequence $\{x_n\}$ defined iteratively by

$$\begin{aligned} x_{n+1} &= a_n x_n + b_n T y_n + c_n u_n, \\ y_n &= a_n^{'} x_n + b_n^{'} T x_n + c_n^{'} v_n, \ n \geq 1, \end{aligned} \tag{2.5}$$

where $\{u_n\}, \{v_n\}$ are bounded sequences in K and $\{a_n\}, \{b_n\}, \{c_n\}, \{a_n^{'}\}, \{b_n^{'}\}$ and $\{c_n^{'}\}$ are sequences in $[0,1]$ such that $a_n + b_n + c_n = a_n^{'} + b_n^{'} + c_n^{'} = 1 \ \forall \ n \geq 1$ is called the *Ishikawa iteration sequence with errors in the sense of Xu.*

(1-d) If, with the same notations and definitions as in (1-c), $b_n^{'} = c_n^{'} = 0$, for all integers $n \geq 1$, then the sequence $\{x_n\}$ now defined by

$$\begin{aligned} x_1 &\in K \\ x_{n+1} &= a_n x_n + b_n T x_n + c_n u_n, \ n \geq 1, \end{aligned} \tag{2.6}$$

is called the *Mann iteration sequence with errors in the sense of Xu.* We remark that if K is bounded (as is generally the case), the error terms u_n, v_n are *arbitrary* in K.

In [9], Chidume and Chika Moore proved the following theorem.

Theorem 2 *Let K be a compact convex subset of a real Hilbert space H; $T : K \to K$ a continuous hemicontractive map. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{a_n^{'}\}, \{b_n^{'}\}$ and $\{c_n^{'}\}$ be real sequences in $[0,1]$ satisfying the following conditions:*

(vii) $a_n + b_n + c_n = 1 = a_n^{'} + b_n^{'} + c_n^{'} \ \forall \ n \geq 1$;

(viii) $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} b_n^{'} = 0$;

(ix) $\sum c_n < \infty; \sum c_n^{'} < \infty$;

(x) $\sum \alpha_n \beta_n = \infty; \sum \alpha_n \beta_n \delta_n < \infty, where\ \delta_n := ||Tx_n - Ty_n||^2$;

(xi) $0 \leq \alpha_n \leq \beta_n < 1\ \forall\ n \geq 1, where\ \alpha_n := b_n + c_n; \beta_n := b_n^{'} + c_n^{'}$.

For arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ iteratively by

$$\begin{aligned} x_{n+1} &= a_n x_n + b_n T y_n + c_n u_n, \\ y_n &= a_n^{'} x_n + b_n^{'} T x_n + c_n^{'} v_n, n \geq 1, \end{aligned}$$

where $\{u_n\}, \{v_n\}$ are arbitrary sequences in K. Then, $\{x_n\}$ converges strongly to a fixed point of T.

They also gave the following remark in [9].

Remark 3 d) In connection with the iterative approximation of fixed points of pseudocontractions, the following question is still open. Does the Mann iteration process always converge for continuous pseudocontractions, or for even Lipschitz pseudocontractions?

e) Let H be a Banach space and K be a nonempty compact convex subset of H. Let $T : K \to K$ be a Lipschitz pseudocontractive map. Under this setting, even for H, as a Hilbert space, the answer to the above question is not known. There is, however, an example [19] of a *discontinuous* pseudocontractive map T with a unique fixed point for which the Mann iteration process does not always converge to the fixed point of T. Let H be the complex plane and $K := \{z \in H : |z| \leq 1\}$. Define $T : K \to K$ by

$$T(re^{i\theta}) = \begin{cases} 2re^{i(\theta + \frac{\pi}{3})}, & \text{for } 0 \leq r \leq \frac{1}{2}, \\ e^{i(\theta + \frac{2\pi}{3})}, & \text{for } \frac{1}{2} < r \leq 1. \end{cases}$$

Then, zero is the only fixed point of T. It is shown in [15] that T is pseudocontractive and that with $c_n = \frac{1}{n+1}$, the sequence $\{z_n\}$ defined by $z_{n+1} = (1 - c_n) z_n + c_n T z_n$, $z_0 \in K$, $n \geq 1$, does not converge to zero. Since the T in this example is not continuous, the above question remains open.

In [10], Chidume and Mutangadura, provide an example of a Lipschitz pseudocontractive map with a unique fixed point for which the Mann iteration sequence failed to converge and they stated there ”This resolves a long standing open problem”.

We introduce the following Mann type implicit iteration process associated with continuous hemicontractive mappings to have a strong convergence in the setting of Hilbert spaces.

Let K be a closed convex subset of a real normed space H and $T : K \to K$ be a mapping. For a sequence $\{v_n\}$ in K, define $\{x_n\}$ in the following way:

$$\begin{aligned} x_0 &\in K, \\ x_n &= \alpha_n x_{n-1} + (1-\alpha_n) T v_n, \end{aligned} \tag{2.7}$$

where $\{\alpha_n\}$ be a real sequence in $[0,1]$ satisfying some appropriate conditions.

Now we prove our main results.

Theorem 3 *Let K be a compact convex subset of a real Hilbert space H; $T : K \to K$ a continuous hemicontractive map. Let $\{\alpha_n\}$ be a real sequence in $[0,1]$ satisfying $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$. For arbitrary $x_0 \in K$ and $\{v_n\}$ in K, define the sequence $\{x_n\}$ by (2.7) satisfying $\sum\limits_{n\geq 1} \|v_n - x_n\| < \infty$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof: The existence of a fixed point of T follows from Schauders fixed point theorem. Let $x^* \in K$ be a fixed point of T and $M = dim(K)$. Using the fact that T is hemicontractive we obtain

$$\|Tv_n - x^*\|^2 \leq \|v_n - x^*\|^2 + \|v_n - Tv_n\|^2. \tag{2.8}$$

With the help of (2.7), Lemma 2 and (2.8), we obtain the following estimates:

$$\begin{aligned} \|x_n - x^*\|^2 &= \|\alpha_n x_{n-1} + (1-\alpha_n)Tv_n - x^*\|^2 \\ &= \|\alpha_n(x_{n-1} - x^*) + (1-\alpha_n)(Tv_n - x^*)\|^2 \\ &= \alpha_n \|x_{n-1} - x^*\|^2 + (1-\alpha_n)\|Tv_n - x^*\|^2 \\ &\quad -\alpha_n(1-\alpha_n)\|x_{n-1} - Tv_n\|^2 . \end{aligned} \tag{2.9}$$

Substituting (2.8) in (2.9), we get

$$\begin{aligned} \|x_n - x^*\|^2 \leq\; & \alpha_n \|x_{n-1} - x^*\|^2 + (1-\alpha_n)\|v_n - x^*\|^2 \\ & +(1-\alpha_n)\|v_n - Tv_n\|^2 - \alpha_n(1-\alpha_n)\|x_{n-1} - Tv_n\|^2 . \end{aligned} \tag{2.10}$$

Also

$$\begin{aligned} \|v_n - x^*\|^2 &\leq \|v_n - x_n\|^2 + \|x_n - x^*\|^2 \\ &\quad +2M\|x_n - x^*\|\|v_n - x_n\| \\ &\leq \|v_n - x_n\|^2 + \|x_n - x^*\|^2 \\ &\quad +2M\|v_n - x_n\| , \end{aligned} \tag{2.11}$$

$$\begin{aligned} \|v_n - Tv_n\|^2 &\leq \|v_n - x_n\|^2 + \|x_n - Tv_n\|^2 \\ &\quad +2M\|x_n - Tv_n\|\|v_n - x_n\| \\ &\leq \|v_n - x_n\|^2 + \|x_n - Tv_n\|^2 \\ &\quad +2M\|v_n - x_n\| , \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} \|x_n - Tv_n\|^2 &= \|\alpha_n x_{n-1} + (1-\alpha_n)Tv_n - Tv_n\|^2 \\ &= \alpha_n^2 \|x_{n-1} - Tv_n\|^2 . \end{aligned} \tag{2.13}$$

Substituting (2.11-2.13) in (2.10), we get

$$\begin{aligned} \|x_n - x^*\|^2 &\leq \alpha_n \|x_{n-1} - x^*\|^2 + (1-\alpha_n)(\|v_n - x_n\|^2 \\ &\quad + \|x_n - x^*\|^2 + 2M\|v_n - x_n\|) \\ &\quad +(1-\alpha_n)(\|v_n - x_n\|^2 + \alpha_n^2\|x_{n-1} - Tv_n\|^2 + 2M\|v_n - x_n\|) \\ &\quad -\alpha_n(1-\alpha_n)\|x_{n-1} - Tv_n\|^2 , \end{aligned}$$

implies

$$\begin{aligned} \|x_n - x^*\|^2 &\leq \|x_{n-1} - x^*\|^2 + 2\frac{1-\alpha_n}{\alpha_n}\|v_n - x_n\|^2 + 4M\frac{1-\alpha_n}{\alpha_n}\|v_n - x_n\| \\ &\quad -(1-\alpha_n)^2\|x_{n-1} - Tv_n\|^2 , \end{aligned}$$

and from the condition $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$, we conclude that the inequality

$$\|x_n - x^*\|^2 \leq \|x_{n-1} - x^*\|^2 - \delta^2\|x_{n-1} - Tv_n\|^2 + \delta_n, \tag{2.14}$$

holds for all fixed points x^* of T provided

$$\delta_n = 2\frac{1-\delta}{\delta}\|v_n - x_n\|^2 + 4M\frac{1-\delta}{\delta}\|v_n - x_n\| .$$

Moreover

$$\delta^2\|x_{n-1} - Tv_n\|^2 \leq \|x_{n-1} - x^*\|^2 - \|x_n - x^*\|^2 + \delta_n,$$

and thus

$$\begin{aligned} \delta^2\sum_{j=1}^{\infty}\|x_{j-1} - Tv_j\|^2 &\leq \sum_{j=1}^{\infty}(\|x_{j-1} - x^*\|^2 - \|x_j - x^*\|^2) + \sum_{j=1}^{\infty}\delta_j \\ &= \|x_0 - x^*\|^2 + \sum_{j=1}^{\infty}\delta_j. \end{aligned}$$

Hence due to the condition $\sum_{n\geq 1}\|v_n - x_n\| < \infty$, we obtain

$$\sum_{j=1}^{\infty}\|x_{j-1} - Tv_j\|^2 < \infty. \tag{2.15}$$

It implies that

$$\lim_{n\to\infty}\|x_{n-1} - Tv_n\| = 0.$$

From (2.13) it further implies that

$$\lim_{n\to\infty} \|x_n - Tv_n\| = 0.$$

Also the condition $\sum_{n\geq 1} \|v_n - x_n\| < \infty$ implies $\lim_{n\to\infty} \|v_n - x_n\| = 0$ and the continuity of T further implies that $\lim_{n\to\infty} \|Tv_n - Tx_n\| = 0$. Now from

$$\|x_n - Tx_n\| \leq \|x_n - Tv_n\| + \|Tv_n - Tx_n\|,$$

implies that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

By compactness of K this immediately implies that there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to a fixed point of T, say y^*. Since (2.14) holds for all fixed points of T we have

$$\|x_n - y^*\|^2 \leq \|x_{n-1} - y^*\|^2 - \delta^2 \|x_{n-1} - Tv_n\|^2 + \delta_n,$$

and in view of (2.15) and Lemma 1 we conclude that $\|x_n - y^*\| \to 0$ as $n \to \infty$, i.e., $x_n \to y^*$ as $n \to \infty$. The proof is complete. □

Corollary 1 *Let H, K, T, be as in Theorem 3 and $\{\alpha_n\}$ be a real sequence in $[0,1]$ satisfying $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$. Let $P_K : H \to K$ be the projection operator of H onto K. For arbitrary $x_0 \in K$ and $\{v_n\}$ in K, define the sequence $\{x_n\}$ by*

$$x_n = P_K(\alpha_n x_{n-1} + (1-\alpha_n)Tv_n), \; n \geq 1,$$

satisfying $\sum_{n\geq 1} \|v_n - x_n\| < \infty$. Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof: The operator P_K is nonexpansive (see e.g., [1]). K is a Chebyshev subset of H so that, P_K is a single-valued map. Hence, we have the following estimate:

$$\begin{aligned}
\|x_n - x^*\|^2 &= \|P_K(\alpha_n x_{n-1} + (1-\alpha_n)Tv_n) - P_K x^*\|^2 \\
&\leq \|\alpha_n x_{n-1} + (1-\alpha_n)Tv_n - x^*\|^2 \\
&= \|\alpha_n(x_{n-1} - x^*) + (1-\alpha_n)(Tv_n - x^*)\|^2 \\
&\leq \alpha_n \|x_{n-1} - x^*\|^2 + (1-\alpha_n)(\|v_n - x_n\|^2 \\
&\quad + \|x_n - x^*\|^2 + 2M\|v_n - x_n\|) \\
&\quad +(1-\alpha_n)(\|v_n - x_n\|^2 + \alpha_n^2 \|x_{n-1} - Tv_n\|^2 + 2M\|v_n - x_n\|) \\
&\quad -\alpha_n(1-\alpha_n)\|x_{n-1} - Tv_n\|^2,
\end{aligned}$$

implies

$$\begin{aligned}
\|x_n - x^*\|^2 &\leq \|x_{n-1} - x^*\|^2 + 2\frac{1-\alpha_n}{\alpha_n}\|v_n - x_n\|^2 + 4M\frac{1-\alpha_n}{\alpha_n}\|v_n - x_n\| \\
&\quad -(1-\alpha_n)^2 \|x_{n-1} - Tv_n\|^2.
\end{aligned}$$

The set $K \cup T(K)$ is compact and so the sequence $\{\|x_n - Tx_n\|\}$ is bounded. The rest of the argument follows exactly as in the proof of Theorem 3 and the proof is complete. □

3 Multi-step iterations in Hilbert spaces

Let K be a nonempty closed convex subset of a real normed space H and T_1, T_2, ..., T_p : $K \to K$ $(p \geq 2)$ be a family of selfmappings.

Algorithm 1 *For a given $x_0 \in K$, compute the sequence $\{x_n\}$ by the implicit iteration process of arbitrary fixed order $p \geq 2$,*

$$\begin{aligned} x_n &= \alpha_n x_{n-1} + (1-\alpha_n) T_1 y_n^1, \\ y_n^i &= \beta_n^i x_{n-1} + (1-\beta_n^i) T_{i+1} y_n^{i+1};\ i = 1,\ 2,\ ...,\ p-2, \\ y_n^{p-1} &= \beta_n^{p-1} x_{n-1} + (1-\beta_n^{p-1}) T_p x_n,\ n \geq 1, \end{aligned} \tag{3.1}$$

which is called the multi-step implicit iteration process, where $\{\alpha_n\}$, $\{\beta_n^i\} \subset [0,1]$, $i = 1$, 2, ..., $p-1$.

For $p = 3$, we obtain the following three-step implicit iteration process:

Algorithm 2 *For a given $x_0 \in K$, compute the sequence $\{x_n\}$ by the iteration process*

$$\begin{aligned} x_n &= \alpha_n x_{n-1} + (1-\alpha_n) T_1 y_n^1,\ , \\ y_n^1 &= \beta_n^1 x_{n-1} + (1-\beta_n^1) T_2 y_n^2, \\ y_n^2 &= \beta_n^2 x_{n-1} + (1-\beta_n^2) T_3 x_n,\ n \geq 1, \end{aligned} \tag{3.2}$$

where $\{\alpha_n\}$, $\{\beta_n^1\}$ and $\{\beta_n^2\}$ are three real sequences in $[0,1]$ satisfying some certain conditions.

For $p = 2$, we obtain the following two-step implicit iteration process:

Algorithm 3 *For a given $x_0 \in K$, compute the sequence $\{x_n\}$ by the iteration process*

$$\begin{aligned} x_n &= \alpha_n x_{n-1} + (1-\alpha_n) T_1 y_n^1, \\ y_n^1 &= \beta_n^1 x_{n-1} + (1-\beta_n^1) T_2 x_n, \quad n \geq 1, \end{aligned} \tag{3.3}$$

where $\{\alpha_n\}$ and $\{\beta_n^1\}$ are two real sequences in $[0,1]$ satisfying some certain conditions.

If $T_1 = T$, $T_2 = I$, $\beta_n^1 = 0$ in (3.3), we obtain the implicit Mann iteration process:

Algorithm 4 *For any given $x_0 \in K$, compute the sequence $\{x_n\}$ by the iteration process*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T x_n,\ n \geq 1, \tag{3.4}$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$ satisfying some certain conditions.

Theorem 4 *Let K be a compact convex subset of a real Hilbert space H and $T_1, T_2, \ldots, T_p$ $(p \geq 2)$ be selfmappings of K. Let T_1 be a continuous hemicontractive map. Let $\{\alpha_n\}$, $\{\beta_n^i\} \subset [0,1]$, $i = 1, 2, \ldots, p-1$ be real sequences in $[0,1]$ satisfying $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$ and $\sum_{n\geq 1}(1-\beta_n^1) < \infty$. For arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by (3.1). Then $\{x_n\}$ converges strongly to the common fixed point of $\bigcap_{i=1}^{p} F(T_i) \neq \phi$.*

Proof: By applying Theorem 3 under assumption that T_1 is continuous hemicontractive, we obtain Theorem 4 which proves strong convergence of the iteration process defined by (3.1). Consider by taking $T_1 = T$ and $v_n = y_n^1$,

$$\begin{aligned} \|v_n - x_n\| &= \left\|y_n^1 - x_n\right\| \\ &= \left\|\beta_n^1 x_{n-1} + (1-\beta_n^1)T_2 y_n^2 - x_n\right\| \\ &= \left\|\beta_n^1 (x_{n-1} - x_n) + (1-\beta_n^1)\left(T_2 y_n^2 - x_n\right)\right\| \\ &\leq \beta_n^1 \|x_{n-1} - x_n\| + (1-\beta_n^1)\left\|T_2 y_n^2 - x_n\right\| \\ &\leq \beta_n^1 \|x_{n-1} - x_n\| + M(1-\beta_n^1), \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \|x_{n-1} - x_n\| &= \|x_{n-1} - \alpha_n x_{n-1} - (1-\alpha_n)Tv_n\| \\ &= (1-\alpha_n)\|x_{n-1} - Tv_n\|. \end{aligned} \tag{3.6}$$

From (3.5) and (3.6), we have

$$\begin{aligned} \|v_n - x_n\| &\leq \beta_n^1(1-\alpha_n)\|x_{n-1} - Tv_n\| + M(1-\beta_n^1) \\ &\leq \beta_n^1(1-\delta)\|x_{n-1} - Tv_n\| + M(1-\beta_n^1). \end{aligned}$$

Now from (2.15) and the condition $\sum_{n\geq 1}(1-\beta_n^1) < \infty$, it can be easily seen that $\sum_{n\geq 1}\|v_n - x_n\| < \infty$. □

Corollary 2 *Let K be a compact convex subset of a real Hilbert space H; $T : K \to K$ a hemicontractive map. Let $\{\alpha_n\}$ be a real sequence in $[0,1]$ satisfying $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$. For arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by (3.4). Then $\{x_n\}$ converges strongly to a fixed point of T.*

4 Implicit iteration process for a finite family of ψ-uniformly pseudocontractive mappings

Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $\{T_i : i \in I\}$ be N self-mappings of K.

In 2001, Xu and Ori [44] introduced the following implicit iteration process for a finite family of nonexpansive mappings $\{T_i : i \in I\}$ (here $I = \{1, 2, ..., N\}$), with $\{\alpha_n\}$ a real sequence in $(0, 1)$, and an initial point $x_0 \in K$:

$$\begin{aligned} x_1 &= \alpha_1 x_0 + (1-\alpha_1)T_1 x_1, \\ x_2 &= \alpha_2 x_1 + (1-\alpha_2)T_2 x_2, \\ &\vdots \\ x_N &= \alpha_N x_{N-1} + (1-\alpha_N)T_N x_N, \\ x_{N+1} &= \alpha_{N+1} x_N + (1-\alpha_{N+1})T_1 x_{N+1}, \\ &\vdots \end{aligned}$$

which can written in the following compact form:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n, \ \forall n \geq 1, \tag{4.1}$$

where $T_n = T_{n \pmod N}$ (here the $\pmod N$ function takes values in I). Xu and Ori proved the weak convergence of this process to a common fixed point of the finite family defined in a Hilbert space. They further remarked that it is yet unclear what assumptions on the mappings and/or the parameters $\{\alpha_n\}$ are sufficient to guarantee the strong convergence of the sequence $\{x_n\}$.

In [24], Oslilike proved the following theorem.

Theorem 5 *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $\{T_i : i \in I\}$ be N strictly pseudocontractive self-mappings of K with $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence satisfying the conditions:*

(i) $0 < \alpha_n < 1$,

(ii) $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$,

(iii) $\sum_{n=1}^{\infty}(1-\alpha_n)^2 < \infty$.

From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by the implicit iteration process (4.1). Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in I\}$ if and only if $\lim_{n\to\infty} d(x_n, F) = 0$.

Definition 5 [24] *A normed space E is said to satisfy Opial's condition if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$.*

In [4], Chen et al proved the following theorem.

Theorem 6 *Let K be a nonempty closed convex subset of a q-uniformly smooth and p-uniformly convex Banach space E that has the Opial property. Let s be any element in $(0,1)$ and let $\{T_i\}_{i=1}^N$be a finite family of strictly pseudocontractive self-maps of K such that $\{T_i\}_{i=1}^N$, have at least one common fixed point. For any point x_0 in K and any sequence $\{\alpha_n\}_{n=1}^\infty$ in $(0,s)$, define the sequence $\{x_n\}$ by the implicit iteration process (4.1). Then $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^N$.*

The purpose of this section is to study the strong convergence of the implicit iteration process (4.1) to a common fixed point for a finite family of $\psi-$uniformly pseudocontractive and $\psi-$uniformly accretive mappings in real Banach spaces.

Theorem 7 *Let $\{T_1, T_2, \ldots, T_N\} : K \to K$ be N, $\psi-$uniformly pseudocontractive mappings with $F = \bigcap_{i=1}^N F(T_i) \neq \phi$. From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by the implicit iteration process (4.1) satisfying $\sum_{n=1}^\infty (1-\alpha_n) = \infty$ and $\lim_{n\to\infty}(1-\alpha_n) = 0$. If the sequence $\{T_n x_n\}$ is bounded, then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, ..., T_N\}$.*

Proof: Since each T_i is $\psi-$uniformly pseudocontractive, we have from (1.1)

$$\langle T_i x - T_i y, j(x-y)\rangle \leq ||x-y||^2 - \psi(||x-y||), \quad i = 1, 2, \ldots, N\,. \tag{4.2}$$

We know that the mappings $\{T_1, T_2, \ldots, T_N\}$ have a common fixed point in K, say w, then the fixed point set $F = \bigcap_{i=1}^N F(T_i) \neq \phi$ is nonempty. We will show that w is the unique fixed point of F. Suppose there exists $q \in F(T_1)$ such that $w \neq q$ i.e., $\|w-q\| > 0$. Then

$$\psi(\|w-q\|) > 0. \tag{AR}$$

Since ψ is strictly increasing with $\psi(0) = 0$. Then, from the definition of $\psi-$uniformly pseudocontractive mapping,

$$\begin{aligned} ||w-q||^2 &= \langle w-q, j(w-q)\rangle = \langle T_1 w - T_1 q, j(w-q)\rangle \\ &\leq ||w-q||^2 - \psi(||w-q||), \end{aligned}$$

implies

$$\psi(||w-q||) \leq 0,$$

contracditing (AR), which implies the uniqueness. Hence $F(T_1) = \{w\}$. Similarly we can prove that $F(T_i) = \{w\}$; $i = 2, 3, \ldots, N$. Thus $F = \{w\}$.

Since the sequence $\{T_n x_n\}$ is bounded, we set

$$M_1 = ||x_0 - w|| + \sup_{n\geq 1} ||T_n x_n - w||.$$

Obviously $M_1 < \infty$.

It is clear that $||x_0 - w|| \leq M_1$. Let $||x_{n-1} - w|| \leq M_1$. Next we will prove that $||x_n - w|| \leq M_1$.

Consider

$$\begin{aligned} ||x_n - w|| &= ||\alpha_n x_{n-1} + (1-\alpha_n) T_n x_n - w|| \\ &= ||\alpha_n (x_{n-1} - w) + (1-\alpha_n)(T_n x_n - w)|| \\ &\leq \alpha_n ||x_{n-1} - w|| + (1-\alpha_n)||T_n x_n - w|| \\ &\leq \alpha_n M_1 + (1-\alpha_n) M_1 \\ &= M_1. \end{aligned}$$

So, from the above discussion, we can conclude that the sequence $\{x_n - w\}$ is bounded. Let $M_2 = \sup_{n\geq 1} ||x_n - w||$.

Denote $M = M_1 + M_2$. Obviously $M < \infty$.

The real function $f : [0, \infty) \to [0, \infty)$, defined by $f(t) = t^2$ is increasing and convex. For all $\lambda \in [0, 1]$ and t_1, $t_2 > 0$ we have

$$((1-\lambda)t_1 + \lambda t_2)^2 \leq (1-\lambda)t_1^2 + \lambda t_2^2. \tag{4.3}$$

Consider

$$\begin{aligned} ||x_n - w||^2 &= ||\alpha_n x_{n-1} + (1-\alpha_n) T_n x_n - w||^2 \\ &= ||\alpha_n (x_{n-1} - w) + (1-\alpha_n)(T_n x_n - w)||^2 \\ &\leq [\alpha_n \|x_{n-1} - w\| + (1-\alpha_n) \|T_n x_n - w\|]^2 \\ &\leq \alpha_n \|x_{n-1} - w\|^2 + (1-\alpha_n) \|T_n x_n - w\|^2 \\ &\leq \|x_{n-1} - w\|^2 + M^2(1-\alpha_n). \end{aligned} \tag{4.4}$$

From Lemma 3 and (4.1), we have

$$\begin{aligned} \|x_n - w\|^2 &= \|\alpha_n x_{n-1} + (1-\alpha_n) T_n x_n - w\|^2 \\ &= \|\alpha_n (x_{n-1} - w) + (1-\alpha_n)(T_n x_n - w)\|^2 \\ &\leq {\alpha_n}^2 ||x_{n-1} - w||^2 + 2(1-\alpha_n) \langle T_n x_n - w, j(x_n - w)\rangle \\ &\leq {\alpha_n}^2 ||x_{n-1} - w||^2 + 2(1-\alpha_n) \|x_n - w\|^2 \\ &\quad -2(1-\alpha_n)\psi(\|x_n - w\|), \end{aligned} \tag{4.5}$$

Substituting (4.4) in (4.5), we get

$$\begin{aligned}
\|x_n - w\|^2 &\leq [\alpha_n^2 + 2(1-\alpha_n)]||x_{n-1} - w||^2 - 2(1-\alpha_n)\psi(\|x_n - w\|) \\
&\quad +2M^2(1-\alpha_n)^2 \\
&= \left[1 + (1-\alpha_n)^2\right] ||x_{n-1} - w||^2 - 2(1-\alpha_n)\psi(\|x_n - w\|) \\
&\quad +2M^2(1-\alpha_n)^2 \\
&\leq ||x_{n-1} - w||^2 - 2(1-\alpha_n)\psi(\|x_n - w\|) + 3M^2(1-\alpha_n)^2. \qquad (4.6)
\end{aligned}$$

Denote

$$\begin{aligned}
\theta_n &= ||x_{n-1} - w||, \\
\lambda_n &= 2(1-\alpha_n), \\
\sigma_n &= 3M^2(1-\alpha_n)^2.
\end{aligned}$$

Condition $\lim_{n\to\infty}(1-\alpha_n) = 0$ assures the existence of $n_0 \in \mathbb{N}$ such that $\lambda_n = 2(1-\alpha_n) \leq 1$, for all $n \geq n_0$. Now with the help of $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$ and Lemma 4, we obtain from (4.6) that

$$\lim_{n\to\infty} ||x_n - w|| = 0,$$

completing the proof. □

Remark 4 Theorem 7 extend and improve the Theorems 5-6 in the following directions:

- The strictly pseudocontractive mappings are replaced by the more general $\psi-$uniformly pseudocontractive and $\psi-$uniformly accretive mappings;
- Theorem 7 holds in real Banach space whereas the results of Theorem 6 are in q-uniformly smooth and p-uniformly convex Banach space;
- We do not need the assumption $\lim_{n\to\infty} d(x_n, F)$ as in Theorem 5.
- One can easily see that if we take $\alpha_n = 1 - \frac{1}{n^\sigma}$; $0 < \sigma < 1$, then $\sum(1-\alpha_n) = \infty$, but $\sum(1-\alpha_n)^2 = \infty$. Hence the conclusion of Theorem 5 is false.

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