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A New Criterion of Stability for Stochastic Networks With Two Stations and Two Heterogeneous Servers

ABSTRACT. We introduce and study a new notion of stability of a stochastic fluid model in terms of random stopping times (partially building on ideas used by Stolyar [10] in his deterministic setting). It may be viewed as an analog of the original criterion for random T's (which may differ for different φ's). In particular, it is shown that our notion of stability is equivalent to L_p-stability for some $p > 1$. We consider an example of a polling system with tow stations and two servers in which the corresponding fluid model may be unstable in the sense as it was written in ([10]) but stable from the generalised viewpoint that we adopt.

KEY WORDS. stability, queueing networks, polling systems.

1 Introduction

In a number of papers, the fluid approximation approach was used for the instability analysis of queueing models. Dai [2] and Meyn [9] proved that if all fluid limits are unstable, then the underlying Markov process is transient. Bramson [1] showed that a Markov process may be transient even if some of its fluid limits are stable. One should note that in order to establish the positive recurrence of a Markov process, it is sufficient (and in certain sense necessary) to show some weak stability of all corresponding fluid limits.

Kumar and Meyn [7] considered stochastic fluid limits and proposed the following notion of stability : a fluid model is L_p-stable, $p > 0$ if

$$sup_{\varphi}\mathbb{E}|\varphi(t)|^p \to 0 \quad \text{as} \quad t \to +\infty\,.$$

They showed the equivalence of the L_2-stability of the fluid model and several notions of stability for the underlying Markov process.

This paper is organised as follows. We introduce and study the notion of stability of a stochastic fluid model in terms of random stopping times (partially building on ideas used by Stolyar in his deterministic setting). It may be viewed as an analog of the original criterion for random T's (which may differ for different φ's). In particular, it is shown that our notion of stability is equivalent to L_p-stability for some $p > 1$.

We consider an example of a polling system with tow stations and two servers in which the corresponding fluid model may be unstable in the sense as it was written in ([10]), i.e.

$$\exists T' > 0,\ \exists \varepsilon \in (0,1]\ :\ |\varphi(T')| \leq 1 - \varepsilon \text{ a.s.}, \tag{1}$$

for any fluid limit φ, but stable from the generalised viewpoint that we adopt.

It shows that even simple queueing systems may exhibit a kind of fluid behavior (basically random bifurcations) that cannot be captured by deterministic fluid models, but nevertheless, is essential for stability analysis. This kind of behavior has already been described by Malyshev et al. in a number of papers about "random walks" in $\mathbb{Z}_+^N$ (see [8] and the list of references therein).

1.1 Positive recurrence of a Markov process via stability of its fluid limits

Let χ be a complete metric space with a metric ρ, and $\mathcal{B}$ the $\sigma - algebra$ generated by open sets. Let $\mathbf{0} \in \chi$ be a fixed element. For $x \in \chi$, put $|x| = \rho(x, \mathbf{0})$. In what follows, we make the following assumptions.

Assumption 1.1 (i) *for any constant $K \geq 0$, the set*

$$A(K) = \{x \in \chi : |x| \leq K\}$$

is compact;

(ii) *for any constant $c \geq 0$, a mapping $x \to c * x$ is defined such that*

(1) *$c * \mathbf{0} = \mathbf{0}$ for any $c \geq 0$;*

(2) *$\rho(c * x_1, c * x_2) = c\rho(x_1, x_2)$ for any $c \geq 0$ and $x_1, x_2 \in \chi$;*

(3) *if $c_n \to c$, then $c_n * x \to c * x$ for any x (and, therefore, the convergence is uniform on any set $A(K)$.)*

In fact, in (i) it is sufficient to assume that the set $A(1)$ is compact. For simplicity, we will write $\dfrac{x}{c}$ instead $\dfrac{1}{c} * x$.

Let $\mathcal{Z} = \{z_1, ..., z_d\}$ be a finite set with natural discrete topology. For each $M > 0$, denote by $\mathcal{D}[0, M]$ the space of $\chi \times \mathcal{Z}$-valued cadlag (right-continuous with LHS limits) functions

$$f(t) = (f^1(t), f^2(t)), \quad t \in [0, M]$$

endowed with the skorohod J_1-metric :

$$d_M(f_1, f_2) = \inf_{g\in\Delta}\{ \sup_{t\in[0,M]} [|g(t) - t| + \rho(f_1^1(g(t)), f_2^1(t)) + \mathbb{1}_{(f_1^2(g(t))\neq f_2^2(t))}]\},$$

where

$$\Delta = \{g : [0, M] \to [0, M],\ g \text{ monotone continious}, g(0) = 0,\ g(M) = M\}.$$

Let $\mathcal{D}[0, \infty)$ denote a space of $\chi \times \mathcal{Z}$-valued cadlag functions on $[0, \infty)$ with the metric

$$d(f_1, f_2) = \sum_1^\infty 2^{-M} \frac{d_M(f_{1,M}, f_{2,M})}{1 + d_M(f_{1,M}, f_{2,M})},$$

where $f_{i,M}$ is a restriction of f_i in $[0, M], i = 1, 2$.

Let $\chi^{'}$ be a closed subset of χ, and $P(t, x, z, B)$ a probabilistic transition kernel. Here $t \geq 0, x \in \chi^{'}, z \in \mathcal{Z}, B \in \mathcal{Z}'$, where $\mathcal{Z}'$ is a σ-algebra in $\chi^{'} \times \mathcal{Z}$ generated by open sets.

For $(x, z) \in \chi^{'} \times \mathcal{Z}$, let

$$(X, Z)^{(x,z)} = \{(X, Z)^{(x,z)}(t), t \geq 0\}$$

be a $\chi^{'} \times \mathcal{Z}$-valued time-homogeneous Markov process with transition kernel P, a.s. cadlag paths, and the initial state $(X, Z)^{(x,z)}(0) = (x, z)$. We assume further that the process satisfies the strong Markov property.

Remark 1.1 One can introduce a more general description of a Markov process with infinite (either countable or not) "index set" $\mathcal{Z}$. In this case, a lot of additional technicalities arise. Within this paper, we decided to confine ourselves only to finite set $\mathcal{Z}$.

Definition 1.1 *A Markov process $(X, Z) = \{(X, Z)^{(x,z)}\}$ is positive recurrent (with respect to the semi norm $|.|$) if there exists a finite K such that the set*

$$B = B(K) = \{(x, z) : |x| \ \leq K\} \subset \chi^{'}$$

is positive recurrent, i.e for some $\delta > 0$,

1. *for all $(x, z) \in \chi^{'} \times \mathcal{Z}$,*

$$\eta^{(x,z)}(B) = inf\{t \geq \delta : (X, Z)^{(x,z)}(t) \in B\} < \infty \quad a.s.;$$

2. $\sup_{(x,z)\in B} \mathbb{E}\eta^{(x,z)}(B) < \infty.$

For $x \in \chi^{'}, z, r \in \mathcal{Z}$, let

$$Y_r^{(x,z)}(t) = \int_0^t 1\!\!1_{(Z^{(x,z)}(u)=r)} du, \quad t \geq 0$$

be the process couting the sejourn time of the 2nd coordinate at r.

For each $(x,z) \in \chi^{'} \times \mathcal{Z}, |x| > 0$, introduce a family of scaled processes

$$\tilde{X}^{(x,z)} = \{\tilde{X}^{(x,z)}(t) = \frac{X^{(x,z)}(|x|t)}{|x|}, \quad t \geq 0\}$$

and for each $r \in \mathcal{Z}$,

$$\tilde{Y}_r^{(x,z)} = \{\tilde{Y}_r^{(x,z)}(t) = \frac{Y_r^{(x,z)}(|x|t)}{|x|}, \quad t \geq 0\}$$

Definition 1.2 *We call the family*

$$(\tilde{X}, \tilde{Y}) = \{\tilde{X}^{(x,z)}, \tilde{Y}_r^{(x,z)}, \quad r \in \mathcal{Z}\}_{x \in \chi^{'}, |x| \geq 1, z \in \mathcal{Z}}$$

relatively compact (at infinity) if, for each sequence

$$(\tilde{X}^{(x_n,z_n)}, \tilde{Y}_r^{(x_n,z_n)}, \quad r \in \mathcal{Z}), \quad |x_n| \to \infty, \quad z_n \in \mathcal{Z}$$

there exists a subsequence $(\tilde{X}^{(x_{n_k},z_{n_k})}, \tilde{Y}_r^{(x_{n_k},z_{n_k})}, \quad r \in \mathcal{Z})$ *that converges weakly (in Skorohod topology) to some limit process*

$$\varphi = \{\varphi(t), t \geq 0\},$$

which is called a fluid limit.

For any $t \geq 0$ and fluid limit φ, the values of $\varphi(t)$ lie in $\chi \times \mathbb{R}_+^d$.

Put $\varphi(t) = (x(t), y(t))$ and $y(t) = \{y_z(t)\}_{z \in \mathcal{Z}}$, where $x(t) \in \chi$ and $y(t) \in \mathbb{R}_+^d$.

Note that $\sum_z y_z(t) = t$ for any fluid limit and for any t.

Denote by $\Phi = \varphi$ the family of all fluid limits φ (or, equivalently, the family of their distributions).

Lemma 1.1 *If the family* $(\tilde{X}, \tilde{Y})$ *is relatively compact, then the family* Φ *is compact (i.e. any sequence of fluid limits contains a convergent subsequence).*

The following assumption applies for the rest of this section.

Assumption 1.2 *The family of processes* $\{X^{(x,z)}, \ (x,z) \in \chi^{'} \times \mathcal{Z}\}$ *is such that*

1. *for all* $t > 0$ *and* $(x, z) \in \chi' \times \mathcal{Z}$,

$$\mathbb{E}|X^{(x,z)}(t)| < \infty$$

and moreover, for any K *and any* z,

$$\sup_{|x| \leq K} \mathbb{E}|X^{(x,z)}(t)| < \infty;$$

2. *for all* $0 \leq u < t$, *the family of random variables*

$$\{\rho(\tilde{X}^{(x,z)}(u), \tilde{X}^{(x,z)}(t));\ |x| \geq 1;\ z \in \mathcal{Z}\}$$

is uniformly integrable (U.I), and for any z,

$$\limsup_{|x| \to \infty} \mathbb{P}\{\sup_{u',t' \in [u,t]} \rho(\tilde{X}^{(x,z)}(u'), \tilde{X}^{(x,z)}(t')) > C(t-u)\} = 0, \tag{2}$$

where C *is a finite constant that does not depend on* u, t.

Theorem 1.1 [4] *Assume that for some* $\varepsilon > 0$, *there exists a finite constant* T *such that*

$$\sup_{\varphi \in \Phi} \mathbb{E}|x(T)| < 1 - \varepsilon.$$

Then, the underlying Markov process (X, Z) *is positive recurrent.*

Remark 1.2 The L_p-stability implies conditions of theorem 1.1. Indeed, take any $T > 0$ such that $\sup_\varphi \mathbb{E}|x(T)|^p < 1$ and apply the Hölder inequality.

Fix $t \geq 0$, and on the event $\{|x(t)| > 0\}$, introduce the shift transformation $\varphi \to \varphi^t = (x^t, y^t)$ as follows :

$$x^t(u) = \frac{x(t + u|x(t)|)}{|x(t)|};$$

for $r \in \mathcal{Z}$,

$$y_r^t(u) = \frac{y_r(t + u|x(t)|) - y_r(t)}{|x(t)|}.$$

Along with Markov processes with fixed initial values, we consider processes with random initial values (x, z). For such processes, one can define fluid limits as follows. Consider a sequence

$$(\tilde{X}^{(x_n,z_n)}, \tilde{Y}_r^{(x_n,z_n)}, \quad r \in \mathcal{Z}), \text{ where } |x_n| \to \infty,$$

in probability, $z_n \in \mathcal{Z}$. By assumption 1.2, it contains a subsequence

$$(\tilde{X}^{(x_{n_k},z_{n_k})}, \tilde{Y}_r^{(x_{n_k},z_{n_k}}, \ r \in \mathcal{Z})$$

that converges weakly (in the Skorohod topology) to some limit process, which is also called a fluid limit. The family of all such fluid limits (or, equivalently, of their distributions) is denoted by $\tilde{\Phi}$. Introduce the following :

Assumption 1.3 *For any* $\varphi \in \tilde{\Phi}, t \geq 0, z \in \mathcal{Z}$, *the right derivative* $v_z(t) = y_z'(t+0)$ *exists a.s. on the event* $\{|x(t) > 0|\}$.

Put $v(t) = \{v_z(t)\}_{z \in \mathcal{Z}}$ and define the set

$$\mathbf{V} = \{\{v_z\}_{z \in \mathcal{Z}} : v_z \geq 0, \forall z \text{ and } \sum_{z \in \mathcal{Z}} v_z = 1\}.$$

For any stopping time τ, put

$$v^{\tau}(t) = v(\tau + t), t \geq 0, \text{ if } |x(\tau + t)| > 0.$$

For a set U and a fluid limit $\varphi \in \tilde{\Phi}$, put

$$\beta = \beta_{\varphi} = inf\{t \geq 0 : |x(t)| = 0 \vee (x^t(0), v^t(0)) \in U\}. \tag{3}$$

Denote $\chi_1 = \{x \in \chi : |x| = 1\}$.

We are ready now to formulate and prove the main result which we will make use in **section 1.2**.

Theorem 1.2 *Let Assumption 1.3 hold. Assume that there exist* $\varepsilon > 0$ *and a measurable set* $U \subseteq \chi_1 \times \boldsymbol{V}$ *such that for each* $\varphi \in \Phi$,

1. *the stopping time* β_{φ} *is admissible ;*
2. *if* $(x(0), v(0)) \in U$ *a.s., then* $\mathbb{E}|x(\beta)| \leq 1 - \varepsilon$;
3. *the family of random variables* $\{\beta_{\varphi},\ \varphi \in \Phi\}$ *is uniformly integrable.*

Then, for some $\varepsilon > 0$ *and for any* $\varphi \in \Phi$, *there exists a stopping time* τ_{φ} *such that*

$$\mathbb{E}|x(\tau_{\varphi})| \leq 1 - \varepsilon \tag{4}$$

and

$$\lim_{K \to \infty} \sup_{\varphi \in \Phi} K\mathbb{P}\{\tau_{\varphi} > K\} = 0.$$

In particular, the conditions of theorem 1.1 are satisfied and therefore the underlying Markov process (X, Z) *is positive recurrent.*

Proof: By the total the total probability law, conditions of the theorem imply that

- a stopping time β_{φ} is admissible for any $\varphi \in \tilde{\Phi}$
- the family $\{\beta_{\varphi}, \varphi \in \tilde{\Phi}\}$ is uniformly integrable.

Let $\Phi_0 = \{\varphi \in \tilde{\Phi} : (x(0), v(0)) \in U\}$ and $\Phi_1 = \tilde{\Phi} \setminus \Phi_0$.

If $\varphi \in \Phi_0$, put $\tau_\varphi = \beta_\varphi$.

Otherwise, put $c = \sup_{\varphi \in \Phi_1} \mathbb{E}|x(\beta_\varphi)|$ and

$$l = min\{n \geq 0 : \ (1-\varepsilon)^n c \leq 1\} + 2.$$

For $\varphi \in \Phi_1$, on define τ_φ via the following recursive procedure.

Put $T_0 = 0$, $\varphi^{(0)} = \varphi$, $T_1 = \beta_\varphi$, $\varphi^{(1)} = (x^{(1)}, y^{(1)}) = \varphi^{T_1}$

and for $i \in \{1, \ldots, l-1\}$, if $|x^{(i)}(0)| > 0$ then put

$$T_{i+1} = T_i + |x^{(i)}(0)|\beta_{\varphi^{(i)}}, \ \varphi^{(i+1)} = \varphi^{T_{i+1}}.$$

Denote $\Gamma = min(l, min\{i : |x^{(i)}(0)| = 0\})$ and $\tau_\varphi = T_\Gamma$.

Then (4) follows from inequalities

$$\begin{aligned}
\mathbb{E}|x(T_\Gamma)| &\equiv \mathbb{E}\{|x(T_\Gamma)|\mathbb{1}_{(|x(T_\Gamma)|>0)}\} \\
&= \mathbb{E}\{|x(T_l)|\mathbb{1}_{(|x(T_l)|>0)}\} \\
&\leq \mathbb{E}\{|x(T_l)|\mathbb{1}_{(|x(T_{l-1})|>0)}\} \\
&\leq (1-\varepsilon)\mathbb{E}\{|x(T_{l-1})|\mathbb{1}_{(|x(T_{l-1})|>0)}\} \\
&\vdots \\
&\leq (1-\varepsilon)^{l-1}\mathbb{E}|x(T_1)| \\
&\leq (1-\varepsilon)^{l-1}c \\
&\leq 1-\varepsilon.
\end{aligned}$$

Let us show uniform integrability of $\{\tau_\varphi, \varphi \in \Phi\}$. For any $u \geq 1$,

$$\mathbb{E}\{T_\Gamma \mathbb{1}_{(T_\Gamma > u)}\} \leq \sum_{i=1}^{l} \mathbb{E}\{T_i \mathbb{1}_{(\Gamma \geq i)} \mathbb{1}_{(T_i \geq u)}\}.$$

Set $g(u) = \sup_\varphi \mathbb{E}\{\beta_\varphi \mathbb{1}_{(\beta_\varphi \geq u)}\}$. Then, $g(u) \to 0$ as $u \to \infty$.

One can see that for $i = 1$,

$$\mathbb{E}\{T_1 \mathbb{1}_{(\Gamma \geq 1)} \mathbb{1}_{(T_1 \geq u)}\} \leq g(u).$$

Further, for any $v \geq 1$ and $u \geq v(2+C)$,

$$\begin{aligned}
&\mathbb{E}\{T_2 1\!\!1_{(\Gamma\geq 2)} 1\!\!1_{(T_2\geq u)}\} \leq \mathbb{E}\{[T_1 + (1+C.T_1)\beta_{\varphi T_1}] 1\!\!1_{(\Gamma\geq 2)} 1\!\!1_{(T_1+(1+CT_1)\beta_{\varphi T_1}\geq u)}\} \\
&\quad \leq \mathbb{E}\{v[1+(1+C).T_1] 1\!\!1_{(T_1\geq \frac{u-v}{v(1+C)})}\} + \\
&\quad + \mathbb{E}\{\mathbb{E}\{\beta_{\varphi T_1} 1\!\!1_{(\beta_{\varphi T_1}\geq v)}/T_1, x(T_1)\}[1+(1+C)T_1] 1\!\!1_{(\Gamma\geq 2)}\} \\
&\quad \leq v\mathbb{P}\{T_1 \geq \frac{u-v}{v(1+C)}\} + (1+C)v\mathbb{E}\{T_1 1\!\!1_{(T_1\geq \frac{u-v}{v(1+C)})}\} + \\
&\quad + g(v)\mathbb{E}\{1+(1+C)T_1\} \\
&\quad \leq (2+C)v.g(\frac{u-v}{v(1+C)}) + g(v)[1+(1+C)g(0)]
\end{aligned}$$

uniformly in $\tilde{\Phi}$. Take $v = v(u)$ such that

$$v \to \infty \text{ and } v.g(\frac{u-v}{v(1+C)}) \to 0 \text{ as } u \to \infty.$$

The proof is completed by induction. □

1.2 Application to the study of the stability of a polling system

Consider an open polling system with two stations and two "heterogenous" servers. With each station $i = 1, 2$ an input stream of customers is associated, that has i.i.d interarrival times with common distribution function $F_i^{(0)}(t)$ and finite positive mean λ_i^{-1}. The inputs to different stations $i = 1, 2$ are mutually independent. For $i, m \in \{1, 2\}$, server m has a station i i.i.d service times with common distribution function $F_i^{(m)}(t)$ and finite positive mean $(\mu_i^{(m)})^{-1}$. Both servers follow the so-called exhaustive service policy : after completing a service, a server either starts the service of a new customer (if there is any), or leaves the station ; after a finite "walking" ("switch-over") period, the server arrives to the other station. For server m, walking times from station i_1 to station i_2 form an i.i.d sequence of non-negative random variables with finite mean $W^{(m)}(i_1, i_2)$ (either $i_1 = 1$; $i_2 = 2$, or $i_1 = 2$; $i_2 = 1$). If a server arrives to a station with empty queue, it becomes "passive" and waits there for the first customer. If during this period the other server arrives to this station, it becomes passive, too, and waits for the second customer to arrive to this station.

This system can be analysed via the fluid approximation approach. In order to avoid the surplus of technical details, we make the following

Assumption 1.4 *The distribution functions $F_i^{(m)}, i = 1, 2; m = 0, 1, 2$ are exponential; $\lambda_1 = \lambda_2 = 1$; all the walking times are equal to zero a.s. ($W^{(m)}(i_1, i_2) = 0, m = 1, 2$).*

Consider a right-continuous time-homogeneous Markov process

$$\{X(t); Z(t)\} = \{(Q_1(t), Q_2(t)), (Z^{(1)}, Z^{(2)}(t))\},\ t \geq 0,$$

where

- $Q_i(t)$ is the queue length at station $i = 1, 2$ (including the customers being served),
- for $m = 1, 2, Z^{(m)}(t) \in \{-2, -1, 1, 2\}$ is the position of server m at time instant t; $Z^{(m)}(t) = i$ means that server m is serving ("active") at station i; $Z^{(m)}(t) = -i$ means that server m is waiting ("passive") at station i.

With necessity, we have to assume that $Q_i(t) \geq 1$ if at least one of $Z^{(m)}(t)$ equals i, and $Q_i(t) \geq 2$ if $Z^{(1)}(t) = Z^{(2)}(t) = i$.

Under Assumption 1.4, a server cannot become passive if there at least two customers in the whole system.

Here $X(t)$ take its values in

$$\chi^{'} = \{0, 1, \cdots\} \times \{0, 1, \cdots\} \subset \chi \equiv \mathbb{R}_+^2$$

and $Z(t)$ in

$$\mathcal{Z} = \{-2, -1, 1, 2\} \times \{-2, -1, 1, 2\}.$$

Put $\mathbf{0} = ((0, 0))$. For $x^{(m)} = (x_1^{(m)}, x_2^{(m)}) \in \chi$, $m = 1, 2$, introduce the metric

$$\rho(x^{(1)}, x^{(2)}) = \sum_{i=1}^{2} |x_i^{(1)} - x_i^{(2)}|.$$

Then, $|x| = x_1 + x_2$ for $x = (x_1, x_2) \in \chi$.

The process (X, Z) is piecewise deterministic (in fact, piecewise constant) and, therefore, possesses the strong Markov property.

Put

$$C = 3 + \mu_1^{(1)} + \mu_1^{(2)} + \mu_2^{(1)} + \mu_2^{(2)}.$$

Assumption 1.2 holds for the process (X, Z). Indeed, for any $t \geq 0$, $K > 0$, and $z \in \mathcal{Z}$,

$$\sup_{|x| \leq K} \mathbb{E}|X^{(x,z)}(t)| \leq K + 2t < \infty;$$

For any $0 \leq u < t$, $z \in \mathcal{Z}$, $|x| \geq 1$,

$$\rho(\tilde{X}^{(x,z)}(u), \tilde{X}^{(x,z)}(t)) \leq_{st} \frac{\sum_{i=1}^{|x|} \pi_i}{|x|},$$

where r.v.'s $\pi_1, \pi_2, \ldots$ are i.i.d and have Poisson distribution with parameter $(C-1)(t-u)$. Therefore, the family

$$\{\rho(\tilde{X}^{(x,z)}(u), \tilde{X}^{(x,z)}(t)), z \in \mathcal{Z}, |x| \geq 1\}$$

is uniformly integrable.

Since

$$\frac{\sum_{i=1}^{|x|}\pi_i}{|x|} \longrightarrow (C-1)(t-u) \text{ a.s.} \quad \text{as } |x| \to \infty,$$

then, (2) holds.

Note that if $\mu_i^{(1)} + \mu_i^{(2)} \leq 1$ for some $i = 1, 2$, then the polling system cannot be stable. Indeed, for $i = 1, 2$ and for any $\Delta > 0$ and $K > 0$, we denote by τ the first moment after Δ when the queue length at station i becomes smaller than K. If both servers start at station i with the queue length $Q \geq K$, then τ is either infinite with positive probability (if $\mu_i^{(1)} + \mu_i^{(2)} < 1$) or finite, but with infinite mean (if $\mu_i^{(1)} + \mu_i^{(2)} = 1$).

Similarly, the polling system cannot be stable if either

$$max(\mu_1^{(1)}, \mu_2^{(2)}) \leq 1 \quad \text{or} \quad max(\mu_1^{(2)}, \mu_2^{(1)}) \leq 1.$$

Let us number the stations and the servers so that

$$\mu_1^{(1)} = min\{\mu_i^{(m)};\ i, m = 1, 2\}.$$

Then, the polling model may be stable in one of the following cases :

(A1) $\mu_1^{(1)} > 1$,

(A2.1) $\mu_1^{(1)} \leq 1,\ \mu_1^{(2)} > 1,\ \mu_2^{(1)} > 1,\ \mu_2^{(2)} > 1$,

(A3.1) $\mu_1^{(1)} \leq 1,\ \mu_1^{(2)} > 1,\ \mu_2^{(1)} \leq 1,\ \mu_2^{(2)} > 1$,

(A4.1) $\mu_1^{(1)} \leq 1,\ \mu_1^{(2)} \leq 1,\ \mu_1^{(1)} + \mu_1^{(2)} > 1,\ \mu_2^{(1)} > 1,\ \mu_2^{(2)} > 1$.

We need some additional notations. First, for $m, i = 1, 2$, let

$$p_i^{(m)} = (1 - \frac{\mu_i^{(m)}}{\mu_i^{(1)} + \mu_i^{(2)}}).max(0, 1 - \mu_i^{(m)}).$$

Then put

$$c_{11} = \frac{1}{\mu_1^{(1)} + \mu_1^{(2)} - 1}, \quad c_{22} = \frac{1}{\mu_2^{(1)} + \mu_2^{(2)} - 1}, \quad c_{12} = \frac{1 - \mu_1^{(1)}}{\mu_2^{(2)} - 1},$$

$$c_{21} = \frac{1 - \mu_2^{(1)}}{\mu_1^{2} - 1}\mathbb{1}_{(\mu_1^{(2)} > 1 \geq \mu_2^{1})} + \frac{1 - \mu_1^{(2)}}{\mu_2^{1} - 1}\mathbb{1}_{(\mu_2^{(1)} > 1 \geq \mu_1^{2})}.$$

Introduce the families of conditions as follows :

Condition (A2). Inequalities **(A2.1)** and

$$c_{11}(c_{22}(1-p_1^{(1)})+c_{12}p_1^{(1)})<1.$$

Condition (A3). Inequalities **(A3.1)** and

$$(1-c_{11}c_{12}p_1^{(1)})^+.(1-c_{22}c_{21}p_2^{(1)})^+ > c_{11}c_{22}(1-p_1^{(1)})(1-p_2^{(1)}),$$

where $a^+ = max(a,o)$.

Condition (A4). Inequalities **(A4.1)** and

$$c_{11}(c_{22}(1-p_1^{(1)}-p_1^{(2)})+c_{12}p_1^{(1)}+c_{21}p_1^{(2)})<1. \tag{5}$$

Theorem 1.3 [4] *Under Assumption 1.4, if one of conditions* **A1-A4** *is satisfied, then the process* (X,Z) *is positive recurrent and ergodic.*

Remark 1.3 If the service times are "server-independent" (i.e. $\mu_i^{(1)} = \mu_i^{(2)} = \mu_i,\ i=1,2$), then, the stability condition

$$\frac{1}{\mu_1}+\frac{1}{\mu_2}<2 \tag{6}$$

is well known and may be easily obtained via the criterion (1). In this case, one can check that (6) holds if and only if either **(A1)**, or **(A4)** is satisfied. Similarly, if the service times are "station-independent" (i.e. $\mu_1^{(m)} = \mu_2^{(m)} = \mu^m,\ m=1,2$), then the stability condition $\mu^{(1)}+\mu^{(2)}>2$ holds if and only if **(A1)** or **(A3)** is validated.

We complete this paper with the following theorem, we assume that the set U consists only of one point $(1,0,1,1)$ and the r.v.'s β and β_0 are defined by (3).

Theorem 1.4 *Assume that any of conditions* **(A2.1)-(A4.1)** *holds. If*

$$1 \le \mathbb{E}|x_0(\beta_0)| < \infty \ \text{and}\ \mathbb{E}\log|x_0(\beta_0)| < 0, \tag{7}$$

then

1. *for any fluid limit* φ

$$\gamma = \gamma_\varphi = inf\{t>0:\ |x(t)|=0\} < \infty\ a.s.; \tag{8}$$

and for any non-flashing fluid limit,

$$\mathbb{E}\gamma = \infty; \tag{9}$$

2. *with any fluid limit* $\varphi = (\boldsymbol{x}, \boldsymbol{y})$ *one can associate an infinite sequence* $\{{\gamma_\varphi}^{(l)}\}$ *of r.v.'s such that* $\gamma_\varphi^{(l)} \to \infty$ *a.s. as* $l \to \infty$*, and*

$$|\boldsymbol{x}(\gamma_\varphi^{(l)})| = 0 \quad a.s. \quad for\ any\ n; \tag{10}$$

3. *the underlying Markov process is recurrent.*

Proof: 1. Put $|\mathbf{x}_0(\beta_0)| = q_1(\beta_0) = u_0 > 0$ a.s. for any $n = 0, 1, ...,$ set

$$\beta_{n+1} = inf\{t > \beta_n : \ q_2(t) = 0, \ \mathbf{z}^{(1)}(t) = \mathbf{z}^{(2)}(t) = 1\}$$

and put $\gamma_0 = \lim_{n\to\infty} \beta_n \leq \infty$. It follows from Theorem 1.3 that $|\mathbf{x}_0(t)| > 0$ for any $t < \gamma_0$ and $q_1(\beta_n)$ may be represented in the form

$$q_1(\beta_n) = \prod_{j=0}^{n} u_j,$$

where $\{u_j\}$ are i.i.d.

Put $\delta_0 = \beta_0$ and, for $n \geq 1$,

$$\delta_n = \frac{\beta_n - \beta_{n-1}}{q_1(\beta_{n-1})}.$$

Then, $\{\delta_n\}$ form an i.i.d sequence, δ_n does not depend on $u_0, \cdots, u_{n-1}$ for any n, and

$$\gamma_0 = \delta_0 + \sum_{i=1}^{\infty} \delta_i \prod_{j=0}^{i-1} u_j.$$

Since all r.v's are a.s. strictly positive,

$$\mathbb{E}\gamma_0 = \mathbb{E}\delta_0 (1 + \sum_{j=1}^{\infty} (\mathbb{E}u_0)^j)$$

is infinite if $\mathbb{E}u_0 \geq 1$. On the other hand, if $\mathbb{E}\log u_0 = -c < 0$, then

$$\prod_{j=0}^{i} u_j = exp\{\sum_{j=0}^{i} \log u_j\} \to 0 \quad \text{a.s.} \quad \text{as} \quad i \to \infty.$$

Set

$$\nu = min\{i : \sum_{j=0}^{k-1} \log u_j \leq -ck/2, \ \forall k \geq i\} < \infty \quad \text{a.s.}$$

Then

$$\gamma_0 \leq \delta_0 + \sum_{i=1}^{\nu-1} \delta_i \prod_{j=0}^{i-1} u_j + \sum_{i=1}^{\infty} \delta_i exp\{-c(i-1)/2\} < \infty \quad \text{a.s.}$$

and

$$|\mathbf{x}_0(\gamma_0)| = \lim_{n\to\infty} |\mathbf{x}_0(\beta_n)| = \lim_{n\to\infty} \prod_{j=0}^{n} u_j = 0 \quad \text{a.s.}$$

Take now any other non-flashing fluid limit φ. Note that either $\gamma_\varphi = \beta$ or φ^β has the same distribution as φ_0. Therefore, one can represent $\gamma = \gamma_\varphi$ in the form

$$\gamma = \beta + |\mathbf{x}(\beta)|.\tilde{\gamma}_0,$$

where $\tilde{\gamma}_0$ is distributed like γ_0 and does not depend on $\mathbf{x}(\beta)$.

Note that

$$\mathbb{P}(\gamma > \beta) = \mathbb{P}(|\mathbf{x}(\beta)| > 0) > 0.$$

Therefore, under conditions (7), γ is finite a.s. and $\mathbb{E}\gamma = \infty$.

2. Consider any fluid limit $\varphi = (\mathbf{x}, \mathbf{y})$. For any $t > 0$, if $|\mathbf{x}(t)| = q > 0$, then $|\mathbf{x}(t + \gamma_{\varphi^t}.q)| = 0$ a.s., that proves (10).

3. Consider the ordinary birth-death process $U(t)$ with constant birth and death intensities λ and μ respectively, and initial size $U(0) = 1$.

Set

$$\eta = inf\{t > 0 : \ U(t) = 0\} \text{ and } H(\lambda, \mu) = \mathbb{E}(\eta 1\!\!1_{(\delta<\infty)}).$$

The following is well-known :

1. If $\lambda \neq \mu$, then $H(\lambda, \mu) < \infty$.
2. If $\lambda < \mu$, then

$$\sup_{t\geq 0} \mathbb{E}\{U(t) 1\!\!1_{(\delta>t)}\} \leq H(\lambda, \mu) < \infty.$$

The state space of the Markov process (X, Z) is countable, and any two states communicate. It is sufficient to show that there exists a constant K such that the random variable

$$\beta^{(1)}(x, z, K) = inf\{t > 0 : \ Q_1^{(x,z)}(t) + Q_2^{(x,z)}(t) \leq K\}$$

is finite a.s. for any couple $x = (n_1, n_2)$ of non-negative integers and for any $z \in \mathcal{Z}$.

It is easy to see that $\beta^{(2)}(x, z) = inf\{t > 0 : \ Q_2^{(x_n,z)}(t) = 0,$

$$Z^{(1)}(t) = Z^{(2)}(t) = 1, \ Z^{(1)}(t-0) = 2 \ \vee \ Z^{(2)}(t-0) = 2\}$$

is finite a.s. (here the symbol $\vee$ stands for the union of two events).

Set

$$\beta(x, z) = min\{\beta^{(1)}(x, z, 1), \beta^{(2)}(x, z)\} \text{ and } \beta_n = \beta((n, 0), (1, 1)).$$

Set $z = (1,1),\ x_n = (n,0)$, tend n to infinity, and denote by φ the weak limit of the process $(\tilde{X}^{(x_n,z)}, \tilde{Y}^{(x_n,z)})$.

For any constant $c > 0$,

$$min(\frac{\beta_n}{n}, c) \to min(\beta, c) \text{ weakly.}$$

Here, $\beta \equiv \beta_0$ is defined in the statement of the theorem).

Therefore, $\beta_n/n \to \beta$ and

$$\frac{Q_1^{(x_n,z)}(\beta_n)}{n} \longrightarrow q_1(\beta) \quad \text{weakly.}$$

Assume we have proved uniform integrability of $(\log(Q_1^{(x_n,z)}(\beta_n)/n))^+$.

Then,

$$\lim_{n\to\infty} sup \mathbb{E} \log(\frac{max(1, Q_1^{(x_n,z)}(\beta_n) + Q_2^{(x_n,z)}(\beta_n))}{n}) \le \mathbb{E} \log q_1(\beta)$$

(since $Q_2^{(x_n,z)}(\beta_n) \le 1$ a.s.) and there exists K such that

$$\sup_{n\ge K} \mathbb{E} \log(\frac{max(1, Q_1^{(x_n,z)}(\beta_n) + Q_2^{(x_n,z)}(\beta_n))}{n}) \le -\varepsilon$$

for some $\varepsilon > 0$.

Start with any initial value (x,z), put $\kappa_1 = \beta^{(1)}(x,z,K)$ and for $k = 1, 2, \ldots,$

$\kappa_{k+1} = inf\{t > \kappa_k :\ (Q_2^{(x_n,z)}(t) = 0,\ Z^{(1)}(t) = Z^{(2)}(t) = 1,$

$(Z^{(1)}(t-0) = 2 \vee Z^{(2)}(t-0) = 2)) \vee (Q_1^{(x_n,z)}(t) + Q_2^{(x_n,z)}(t) \le 1)\}.$

Denote $Q(k) = Q_1^{(x,z)}(\kappa_k) + Q_2^{(x,z)}(\kappa_k)$. Then

$$\mathbb{E}(\log \frac{Q(k+1)}{Q(k)} / Q(k) > K) \le -\varepsilon$$

and, therefore,

$\gamma(x,z) = min\{k :\ Q(k) \le K\}$ and $\beta^{(1)}(x,z,K) \le \kappa_{\gamma(x,z)}$ are finite a.s. Thus, the Markov process is recurrent.

We give now the proof of uniform integrability for a sequence of random variables

$\{(\log(Q_1^{(x_n,z)}(\beta_n)/n))^+,\ n \ge 1\}$.

The proof is based on similar arguments in cases **A2.1-A4.1**.

Consider case (**A3.1**), as the most complicated one.

Let $\alpha_n = inf\{t > 0 :\ Z^{(1)}(t) \ne 1 \vee Z^{(2)}(t) \ne 1\}$. Then $\alpha_n \le \beta_n$ a.s. and

$$\mathbb{E}Q_2^{(x_n,z)}(\alpha_n) = (n-1)c_{11}.$$

Put $D = \{Q_1^{(x,z)}(\beta_n) + Q_2^{(x,z)}(\beta_n) \le 1\}$. Denote by $\overline{D}$ the complement of D, that is,

$$\overline{D} = \{\text{both servers stop not on the time interval } [0, \beta_n]\}.$$

From the total probability law,

$$\begin{aligned}\mathbb{E}Q_1^{(x_n,z)}(\beta_n) = \mathbb{E}\{Q_1^{(x_n,z)}(\beta_n)\mathbb{1}_D\} + \mathbb{E}\{Q_1^{(x_n,z)}(\beta_n)\mathbb{1}_{B_{-1}}\}+ \\ + \mathbb{E}\{Q_1^{(x_n,z)}(\beta_n)\mathbb{1}_{\tilde{B}_{-1}}\} + \sum_{i=0}^{\infty} \mathbb{E}\{Q_1^{(x_n,z)}(\beta_n)\mathbb{1}_{\overline{D}} \mid B_i\}\mathbb{P}(B_i)+ \\ + \sum_{i=0}^{\infty} \mathbb{E}\{Q_1^{(x_n,z)}(\beta_n)\mathbb{1}_{\overline{D}} \mid \tilde{B}_i\}\mathbb{P}(\tilde{B}_i),\end{aligned}$$

where the events B_i and $\tilde{B}_i$ describe the dynamics of servers within the time interval $[0, \beta_n]$. Namely,

$B_{-1} = \{$ server 2 stays at station 1 all the time ; server 1 switches to station 2 only once and returns to station 1 at time instant $\beta_n\}$;

for $i \ge 0$,

$B_i = \{$ first, both servers switch to station 2 (in any order); second, server 2 switches to station 1 and returs back i times ; finally, both servers switch to station 1 (in any order)$\}$;

$\tilde{B}_{-1} = \{$ server 1 stays at station 1 all the time; server 2 switches to station 2 only once and return back at time instant $\beta_n\}$;

for $i \ge 0$,

$\tilde{B}_i = \{$ first, both servers switch to station 2 (in any order); second, servers 1 and 2 switche to station 1 and returs back several times (alternatively) ; after the last return of server 1 to station 2, server 2 visits station 1 i times ; finally, both servers return to station 1 (in any order)$\}$.

Then

$$\mathbb{E}\{Q_1^{(x_n,z)}(\beta_n)\mathbb{1}_D\} \le 1;$$

and routine (but space-consuming) calculations show that following inequalities are valid :

$$\begin{aligned}\mathbb{E}\{Q_1^{(x_n,z)}(\beta_n)\mathbb{1}_{\overline{D}B_{-1}}\} \le& H(1, \mu_1^{(2)}) < \infty; \\ \mathbb{E}\{Q_1^{(x_n,z)}(\beta_n)\mathbb{1}_{\overline{D}\tilde{B}_{-1}}\} \le& (n-1)c_{11}c_{12} + 1; \\ \mathbb{E}\{Q_1^{(x_n,z)}(\beta_n)\mathbb{1}_{\overline{D}} \mid B_0\} \le& ((n-1)c_{11} + c_{21})max(c_{12}, c_{22}) + c_{12}; \\ \mathbb{E}\{Q_1^{(x_n,z)}(\beta_n)\mathbb{1}_{\overline{D}} \mid B_i\} \le& ((n-1)c_{11} + c_{21})max(c_{12}, c_{22})(c_{21}c_{22})^i + c_{12}; \\ \mathbb{E}\{Q_1^{(x_n,z)}(\beta_n)\mathbb{1}_{\overline{D}} \mid \tilde{B}_i\} \le& H(1, \mu_2^{(2)})(c_{21}c_{22})^i + c_{12}\end{aligned}$$

for all $i \geq 0$.

Thus, $\sup\limits_{n\geq 2} \mathbb{E}\left(\frac{Q_1^{(x_n,z)}(\beta_n)}{n}\right)^{\delta}$ is finite for any $\delta \in (0,1]$ such that

$$(c_{21}c_{22})^{\delta}\frac{\mu_2^{(2)}}{\mu_2^{(1)}+\mu_2^{(2)}} < 1.$$

Therefore, the random variables $\left(\log\left(\frac{Q_1^{(x_n,z)}(\beta_n)}{n}\right)\right)^{+}$ are uniformly integrable. $\square$

Prospects (future works) :

1. We will consider a same polling system but with $\lambda_1 \neq \lambda_2$ and a walking times $W^{(m)}(i_1,i_2) \neq 0$.

2. We will seek the conditions of transience for this new system.

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