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Sadek Bouroubi

Bell Numbers and Engel's Conjecture

ABSTRACT. In this paper, we prove some new properties of the sequence of the Bell numbers and present some results in connection with Engel's conjecture. In addition, using a new approach we state a stronger conjecture.

KEY WORDS. Partition lattice; Bell number; variance; mean; convexity; concavity.

1 Introduction

A partition of the set $[n] = \{1, 2, \ldots, n\}$ is a collection of nonempty, pairwise disjoint subsets of $[n]$ called *blocks* whose union is $[n]$. A partition π_1 is said to refine another partition π_2, denoted by $\pi_1 \leq \pi_2$, if every block of π_1 is contained in some block of π_2. Hence, the refinement relation is a partial ordering of the set $\prod_n$ of all partitions of $[n]$. The number of partitions of $[n]$ having exactly k blocks is the Stirling number of the second kind $S(n,k)$. The total number of partitions of $[n]$ is the n^{th} Bell number B_n. Therefore,

$$B_n = \sum_{k=1}^{n} S(n,k), \ n \geq 1 .$$

Also, recall Dobinski's formula [1]

$$B_n = \frac{1}{e}\sum_{i=0}^{\infty}\frac{i^n}{i!} .$$

Now, for all $x \in \mathbb{R}$, we set

$$B(x) = \frac{1}{e}\sum_{i=0}^{\infty}\frac{i^x}{i!}, \ x \in \mathbb{R} .$$

Note that the series $\sum_{i\geq 0}\frac{i^x}{i!}$ converges for all $x \in \mathbb{R}$ and $B(n) = B_n$ for all $n \in \mathbb{N}$.

2 New properties of the Bell numbers

Theorem 1 *Let $p \in \left]1,+\infty\right[$ and let q be the conjugate exponent of p. Then, for all $x_1, x_2 \in \mathbb{R}$, we have*

$$B\left(x_1+x_2\right) \leq B^{1/p}\left(px_1\right) B^{1/q}\left(qx_2\right).$$

Proof: Let Z be the discrete random variable with distribution function

$$P(Z=i)=\frac{1}{e}\cdot\frac{1}{i!},\ i\in\mathbb{N}.$$

Then

$$E(Z^x)=B(x), \text{ for all } x\in\mathbb{R}. \tag{1}$$

From Hölder's inequality we obtain

$$E\left(Z^{x_1+x_2}\right) \leq E^{1/p}\left(Z^{px_1}\right)\cdot E^{1/q}\left(Z^{qx_2}\right), \text{ for all } x_{1,}x_2\in\mathbb{R}.$$

Hence, if we use (1), the result follows immediately. □

Theorem 2 *For all $x_1,\ x_2 \in \mathbb{R}$, we have*

$$2B\left(x_1+x_2\right) \leq B\left(2x_1\right)+B\left(2x_2\right).$$

Proof: Using the same discrete random variable in the above proof we have

$$E((Z^{x_1}-Z^{x_2})^2)\geq 0.$$

Thus

$$2E(Z^{x_1+x_2})\leq E(Z^{2x_1})+E(X^{2x_2}).$$

Consequently, using (1), we get

$$2B\left(x_1+x_2\right) \leq B\left(2x_1\right)+B\left(2x_2\right).$$

□

As a consequence of Theorem 1, with $x_1,\ x_2\in\mathbb{N}$ and $p=q=2$ we obtain the first following new property of the sequence of the Bell numbers.

Corollary 3 *The inequality*

$$B_{n+m}^2 \leq B_{2n}B_{2m}$$

holds for every $n,\ m\in\mathbb{N}$.

Corollary 4 *The sequence* $(B_{n+1}/B_n)_n$ *is increasing, and, equivalently, the sequence* $(B_n)_n$ *is logarithmically convex, i.e.*

$$B_{n+1}^2 \le B_n B_{n+2}, \ \textit{for all } n \ge 0 \,.$$

Proof: The assertion follows from Theorem 1 with $x_1 = \frac{n}{2}$ $x_2 = \frac{n+2}{2}$, and $p = q = 2$. □

Corollary 5 *The sequence* $(B_n)_n$ *is convex, i.e.*

$$2B_n \le B_{n-1} + B_{n+1}, \ \textit{for all } n \ge 1 \,.$$

Proof: This inequality follows easily from Theorem 2 with $x_1 = \frac{n+1}{2}$ and $x_2 = \frac{n-1}{2}$. □

Henceforth, let τ_n (resp. σ_n^2) denote the average (resp. the variance) of the number of blocks in a partition of the generic n-set $[n]$, i.e.

$$\tau_n = \frac{1}{B_n} \sum_{k=1}^{n} kS\left(n,k\right)$$

and

$$\sigma_n^2 = \frac{1}{B_n} \sum_{k=1}^{n} k^2 S\left(n,k\right) - \left(\frac{1}{B_n} \sum_{k=1}^{n} kS\left(n,k\right) \right)^2 .$$

Using the recurrence relation

$$S\left(n+1,k\right) = S\left(n,k-1\right) + kS\left(n,k\right) \tag{2}$$

we obtain

$$\tau_n = \frac{B_{n+1}}{B_n} - 1$$

and

$$\sigma_n^2 = \frac{B_{n+2}}{B_n} - \left(\frac{B_{n+1}}{B_n} \right)^2 - 1 \,.$$

In studying Alekseev's inequality [4] on the principal ideal of the partition lattice, it was shown in [3] that the inequality is equivalent to

$$\tau_{n_1} + \tau_{n_2} \ge \tau_{n_1+n_2}, \quad \text{for all } n_1, n_2 \in \mathbb{N} \,.$$

Furthermore, K. Engel [5] showed that the inequality above is true if the sequence $(\tau_n)_n$ is concave, and he was led to the conjecture that the sequence $(\tau_n)_n$ is concave, i.e.

$$\tau_n \ge \frac{1}{2} \left(\tau_{n-1} + \tau_{n+1} \right), \text{for all } n \ge 1 \,.$$

We verified the last inequality for $n \leq 1500$ using a computer [3], but no general proof has been found yet. The second purpose of this paper is to contribute to the study of this conjecture by using a new approach.

Let, for $x \in \mathbb{R}$,

$$B_n(x) = \sum_{k=0}^{n} S(n,k)\, x^k\,. \tag{3}$$

It is clear that $(B_n(x))_n$ is a sequence of polynomials, with $B_0(x) \equiv 1$ and $B_n(1) = B_n$ (Bell number). Recall that the polynomial $B_n(x)$ admits n distinct roots, where only one of them is equal to zero and all others are strictly negative. This result is due to L. N. Harper [8] and the detailed proof can be found in [3]. From now on, let $-\alpha_1(n), -\alpha_2(n), \cdots, -\alpha_{n-1}(n)$ denote the $(n-1)$ negative roots of $B_n(x)$, and let $I_n = \{0\} \cup \{-\alpha_i(n), i = 1, ..., n-1\}$. This allows us to write

$$B_n(x) = x \prod_{i=1}^{n-1} (x + \alpha_i(n))\,.$$

In this section we assume that $x \notin I_n$. Setting

$$\tau_n(x) = \frac{B_{n+1}(x)}{B_n(x)} - x$$

and

$$\sigma_n^2(x) = \frac{B_{n+2}(x)}{B_n(x)} - \left(\frac{B_{n+1}(x)}{B_n(x)}\right)^2 - x, \tag{4}$$

we have $\tau_n(1) = \tau_n$ and $\sigma_n^2(1) = \sigma_n^2$.

Theorem 6 *For every $n \in \mathbb{N}^*$,*

i) $\tau_n(x) = 1 + \sum_{j=1}^{n-1} \frac{x}{x+\alpha_j(n)} = n - \sum_{k=1}^{n-1} \frac{\alpha_j(n)}{x+\alpha_j(n)}$,

ii) $\sigma_n^2(x) = x\, d(\tau_n(x))$,

where d is the differential operator $\frac{d}{dx}\cdot$

Proof: Without restriction, we only consider here the case when $x > 0$.

i) It is easy to verify from (2) and (3) that

$$B_{n+1}(x) = x\,(d(B_n(x)) + B_n(x))\,. \tag{5}$$

It follows that

$$\begin{aligned}
\tau_n(x) &= x\frac{d(B_n(x))}{B_n(x)} \\
&= x\ d(\log(B_n(x))) \\
&= x\ d\left(\log x + \sum_{j=1}^{n-1}\log(x+\alpha_j(n))\right) \\
&= 1 + \sum_{j=1}^{n-1}\frac{x}{x+\alpha_j(n)} \\
&= n - \sum_{j=1}^{n-1}\frac{\alpha_j(n)}{x+\alpha_j(n)}\,.
\end{aligned}$$

To prove ii), we have

$$\begin{aligned}
x\ d(\tau_n(x)) &= x\left(\frac{d(B_{n+1}(x))}{B_n(x)} - \frac{B_{n+1}(x)\,d(B_n(x))}{B_n^2(x)} - 1\right) \\
&= \frac{B_{n+2}(x) - xB_{n+1}(x)}{B_n(x)} - \frac{B_{n+1}(x)\,(B_{n+1}(x) - xB_n(x))}{B_n^2(x)} - x \\
&= \frac{B_{n+2}(x)}{B_n(x)} - \left(\frac{B_{n+1}(x)}{B_n(x)}\right)^2 - x \\
&= \sigma_n^2(x)\ .
\end{aligned}$$

Thus, the theorem is proved. □

Corollary 7 *For every $n \geq 2$,*

a) $1 < \ \tau_n(x) < n$, *for each* $x > 0$,

b) $0 < \sigma_n^2(x) < \frac{n-1}{4}$, *for each* $x > 0$,

c) *The sequence* $(B_n(x))_n$ *is logarithmically convex for* $x > 0$ *and logarithmically concave for* $x < 0$,

d) *For every* $n \geq 1$, *the polynomials* $\frac{B_{n+1}(x)}{x}$ *and* $\frac{B_n(x)}{x}$ *are coprime,*

e) $\sigma_{n+1}^2(x) + \sigma_{n-1}^2(x) - 2\sigma_n^2(x) = x\ d(\tau_{n+1}(x) + \tau_{n-1}(x) - 2\tau_n(x))$.

Proof: a) By the fact that $\alpha_j(n)$ is positive for every j, the inequality a) follows immediately from i) of Theorem 6.

b) using i) and ii) of Theorem 6, we obtain

$$\sigma_n^2(x) = x\sum_{j=1}^{n-1}\frac{\alpha_j(n)}{(x+\alpha_j(n))^2}\,. \tag{6}$$

Thus, it is sufficient to notice that the maximum value of the function $x \longrightarrow \frac{\alpha_j(n)x}{(x+\alpha_j(n))^2}$ is $\frac{1}{4}$, for $x > 0$.

c) This result is an immediate consequence of (6). Indeed, if $x > 0$ (*resp.* $x < 0$), then $\sigma_n^2(x) > 0$ (*resp.* $\sigma_n^2(x) < 0$), i.e. using (4)

$$B_{n+2}(x) B_n(x) - B_{n+1}^2(x) - xB_n^2(x) > 0$$

$$(\text{resp.} \;\; B_{n+2}(x) B_n(x) - B_{n+1}^2(x) - xB_n^2(x) < 0)\,.$$

Hence

$$B_{n+2}(x) B_n(x) > B_{n+1}^2(x)\;,$$

$$(\text{resp.} \;\; B_{n+2}(x) B_n(x) < B_{n+1}^2(x))\,.$$

From (5), we get

$$B_{n+1}(-\alpha_j(n)) = \alpha_j^2(n) \prod_{\substack{i=1 \\ i\neq j}}^{n-1} (-\alpha_j(n) + \alpha_i(n)) \neq 0\,. \tag{7}$$

Thus d) is proved.

To prove e), it is sufficient to use ii) of Theorem 6. □

Corollary 8 *We have*

$$2B_n < B_{n+1} < (n+1)B_n\,.$$

Proof: Use a) of Corollary 7 and choose $x = 1$. □

Remark 1 Note that the inequality $2B_n < B_{n+1}$ is stronger than the convexity of the sequence $(B_n)_n$.

Let $u_n(x) = \tau_n(x) + x$. Then we have the following result.

Lemma 9 *For every $n \geq 3$,*

$$\frac{1}{u_{n-1}(x)} = \sum_{j=1}^{n-1} \frac{\beta_j(n)}{x + \alpha_j(n)}\,,$$

where $\beta_j(n) \in]0,1[$.

Proof: We have

$$\frac{1}{u_{n-1}(x)} = \frac{B_{n-1}(x)}{B_n(x)} = \frac{\prod_{i=1}^{n-2}(x+\alpha_i(n-1))}{\prod_{i=1}^{n-1}(x+\alpha_i(n))}.$$

By a decomposition into partial fractions, we get

$$\frac{1}{u_{n-1}(x)} = \sum_{j=1}^{n-1} \frac{\beta_j(n)}{x+\alpha_j(n)},$$

where

$$\beta_j(n) = \frac{B_{n-1}(-\alpha_j(n))}{-\alpha_j(n)\prod_{\substack{i=1\\ i\neq j}}^{n-1}(-\alpha_j(n)+\alpha_i(n))}.$$

Moreover, from (7) we obtain

$$\beta_j(n) = \frac{-\alpha_j(n)\,B_{n-1}(-\alpha_j(n))}{B_{n+1}(-\alpha_j(n))}.$$

On the other hand in view of (6) we have

$$\sigma_{n-1}^2(x) = \frac{B_{n+1}(x)}{B_{n-1}(x)} - \left(\frac{B_n(x)}{B_{n-1}(x)}\right)^2 - x < 0, \text{ for all } x<0 \text{ and } x \notin I_{n-1}.$$

Therefore

$$\frac{B_{n-1}(x)}{B_{n+1}(x)} < \left(\frac{B_n(x)}{B_{n+1}(x)}\right)^2 + x\left(\frac{B_{n-1}(x)}{B_{n+1}(x)}\right)^2, \text{ for all } x<0 \text{ and } x \notin I_{n+1}.$$

If we replace x in the above inequality by $-\alpha_j(n)$, then we obtain

$$\beta_j(n)(1-\beta_j(n)) > 0,$$

thus $\beta_j(n) \in]0,1[\cdot$ □

Theorem 10 *For every $n \geq 2$,*

$$\tau_{n+1}(x) + \tau_{n-1}(x) - 2\tau_n(x) = x\left(\frac{u_{n-1}(x)}{u_n(x)}\right) d\left(\frac{u_n(x)}{u_{n-1}(x)}\right),$$

with $\frac{u_n(x)}{u_{n-1}(x)} = 1 + x\sum_{j=1}^{n-1}\frac{\beta_j(n)}{(x+\alpha_j(n))^2}$ *and* $\beta_j(n) \in]0,1[$.

Proof: From (5) we obtain

$$\tau_n(x) = x\frac{d(B_n(x))}{B_n(x)}.$$

Hence

$$\begin{aligned}\tau_{n+1}(x) + \tau_{n-1}(x) - 2\tau_n(x) &= xd\left(\log(B_{n+1}(x)) + \log(B_{n-1}(x)) - 2\log(B_n(x))\right)\\ &= xd\left(\log\left(\frac{B_{n+1}(x)\,B_{n-1}(x)}{B_n^2(x)}\right)\right)\\ &= xd\left(\log\left(\frac{u_n(x)}{u_{n-1}(x)}\right)\right).\end{aligned}$$

We also have

$$\begin{aligned}d\left(\frac{1}{u_{n-1}(x)}\right) &= d\left(\frac{B_{n-1}(x)}{B_n(x)}\right)\\ &= \frac{d(B_{n-1}(x))}{B_n(x)} - \frac{d(B_n(x))}{B_n(x)}\cdot\frac{B_{n-1}(x)}{B_n(x)}\\ &= \frac{d(B_{n-1}(x))}{B_{n-1}(x)}\cdot\frac{B_{n-1}(x)}{B_n(x)} - \frac{d(B_n(x))}{B_n(x)}\cdot\frac{B_{n-1}(x)}{B_n(x)}\\ &= \frac{1}{x}\frac{1}{u_{n-1}(x)}\left(\tau_{n-1}(x) - \tau_n(x)\right)\\ &= \frac{1}{x}\left(1 - \frac{u_n(x)}{u_{n-1}(x)}\right).\end{aligned}$$

Hence

$$\frac{u_n(x)}{u_{n-1}(x)} = 1 - x\,d\left(\frac{1}{u_{n-1}(x)}\right).$$

Therefore, from Lemma 9 we obtain

$$\frac{u_n(x)}{u_{n-1}(x)} = 1 + x\sum_{j=1}^{n-1}\frac{\beta_j(n)}{(x+\alpha_j(n))^2}.$$

This completes the proof. □

3 Strong Conjecture

Recall that K. Engel conjectured that the sequence $(\tau_n)_n$ is concave in n, i.e.

$$2\tau_n - \tau_{n-1} - \tau_{n+1} \geq 0, \quad \text{for every } n \geq 2.$$

In this section we show that this conjecture is in fact a consequence of the following stronger conjecture.

Theorem 11 *If the positive roots of the equation* $d\left(\frac{u_n(x)}{u_{n-1}(x)}\right) = 0$ *are less than 1, then the sequence* $(\tau_n)_n$ *is concave.*

Proof: Using Theorem 10, we have

$$\frac{u_n\,(x)}{u_{n-1}\,(x)} \geq 1 \text{ when } x \geq 0 \quad \text{and} \quad \lim_{x\to+\infty} \frac{u_n\,(x)}{u_{n-1}\,(x)} = 1\,.$$

Then, assuming that the positive roots of the equation $d\left(\frac{u_n(x)}{u_{n-1}(x)}\right) = 0$ are less than 1, the function $x \mapsto \frac{u_n(x)}{u_{n-1}(x)}$ would be necessarily decreasing in the neighborhood of 1, which means that

$$\tau_{n+1} + \tau_{n-1} - 2\tau_n = \left(\frac{u_{n-1}\,(1)}{u_n\,(1)}\right) d\left(\frac{u_n\,(1)}{u_{n-1}\,(1)}\right) < 0\,.$$

$\square$

Conjecture 1 *For every* $n \geq 1$*, the positive roots of the equation* $d\left(\frac{u_n(x)}{u_{n-1}(x)}\right) = 0$ *are all less than 1.*

Using Maple9, we checked this new conjecture for $2 \leq n \leq 400$.

References

[1] **Aigner, M. :** *Combinatorial theory.* Springer-Verlag, Berlin. Heidelberg, New York, 1979

[2] **Alekseev, V. B. :** *On the number of k-valued functions.* Probl. Kibernet. **28**, 5-24 (1974)

[3] **Bouroubi, S. :** *Thèse de Magister en Mathématiques.* USTHB, Alger, 1991

[4] **Engel, K. :** *On the average rank of an element in a filter of the partition lattice.* J. Combin. Theory Ser. A **64**, 67-78 (1994)

[5] **Engel, K.**, and **Gronau, H.-D. O. F. :** *Sperner Theory in partially Ordered Sets.* BSB B.G. Teubner Verlagsgesellschaft, Leipzig (1985)

[6] **Canfield, E. R. :** *Engel's Inequality for Bell Numbers.* J. Combin. Theory Ser. A **72**, 184-187 (1995)

[7] **Canfield, E. R.**, and **Harper, L. H. :** *Large antichains in the partition lattice.* Random Structures Algorithms **6**, 89-104 (1995)

[8] **Harper, L. N. :** *Stirling behavior is asymptotically normal.* Ann. Math. Stat. **38**, 410-414 (1967)

[9] **Sperner, E. :** *Ein Satz über Untermengen einer endlichen Menge.* Math. Z. **27**, 544-548 (1928)

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Author:

Sadek Bouroubi
USTHB, Faculty of Mathematics,
Department of Operations Research,
Laboratory LAID3, P.Box 32 16111 El-Alia, Bab-Ezzouar, Algiers,
Algeria

e-mail: sbouroubi@usthb.dz or e-mail: bouroubis@yahoo.fr