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Xue Zhiqun

Ishikawa Iterative Process with Errors for Generalized Lipschitz Φ-Accretive Mappings in Uniformly Smooth Banach Spaces[1]

ABSTRACT. Let E be a uniformly smooth real Banach space and $T : E \to E$ be generalized Lipschitz Φ-accretive mapping with $\Phi(r) \to +\infty$ as $r \to +\infty$. Let$\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$ be six real sequences in $[0, 1]$ satisfying the following conditions: (i)$a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$, (ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} b'_n = \lim_{n\to\infty} c'_n = 0$, (iii) $\sum_{n=0}^{\infty} b_n = \infty$, (iv)$c_n = o(b_n)$. For arbitrary $x_0 \in E$, define the Ishikawa iterative process with errors $\{x_n\}_{n=0}^{\infty}$ by (ISE): $y_n = a'_n x_n + b'_n S x_n + c'_n v_n, x_{n+1} = a_n x_n + b_n S y_n + c_n u_n, n \geq 0$. where $S : E \to E$ is defined by $Sx = f + x - Tx, f \in E, \forall x \in E$. Assume that the equation $Tx = f$ has solution and $\{u_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty}$ are arbitrary two bounded sequences in E. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique solution of the equation $Tx = f$. A related result deals with approximation of fixed point of generalized Lipschitz Φ-pseudocontractive mapping.

KEY WORDS. Ishikawa iterative process with errors; generalized Lipschitz; Φ-accretive mapping; Φ- pseudocontractive mapping; uniformly smooth Banach space.

1 Introduction

Let E be real Banach space and E^* be the dual space on E. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$Jx = \{f \in E^* :< x, f >= ||x|| \cdot ||f|| = ||f||^2\} \tag{1.1}$$

for all $x \in E$, where $< \cdot, \cdot >$ denotes the generalized duality pairing. It is well known that if E is an uniformly smooth Banach space, then J is single-valued and such that $J(-x) = -J(x), J(tx) = tJ(x)$ for all $x \in E$ and $t \geq 0$; and J is uniformly continuous on any bounded subset of E. In the sequel we shall denote single-valued normalized duality

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mapping by j. By means of the normalized duality mapping J. In the following we give some concepts.

Definition 1.1 *Let E be real Banach space, and $T : E \supset D(T) \to E$ be a mapping with domain $D(T)$ and range $R(T)$. A mapping T is said to be strongly accretive if for any $x, y \in D(T)$ there exists $j(x-y) \in J(x-y)$ such that*

$$< Tx - Ty, j(x-y) > \geq k\|x-y\|^2 \tag{1.2}$$

for some constant $k \in (0,1)$. A mapping T is called Φ-strongly accretive if for any $x, y \in D(T)$ there exists $j(x-y) \in J(x-y)$ and a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that

$$< Tx - Ty, j(x-y) > \geq \Phi(\|x-y\|)\|x-y\| \tag{1.3}$$

The mapping T is called Φ-accretive if, there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$, and for any $x, y \in D(T)$ there exists $j(x-y) \in J(x-y)$ such that

$$< Tx - Ty, j(x-y) > \geq \Phi(\|x-y\|) \tag{1.4}$$

Recently, Zhou [6] proved the following result: Let X be real uniformly smooth Banach space. Assume that $A : X \to X$ is Lipschitz Φ-strongly accretive mapping with $\Phi(r) \to +\infty$ as $r \to +\infty$. Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be two real sequences in $(0,1)$ satisfying the conditions:

(i) $0 < \alpha_n \leq \dfrac{1}{4(1+L_1)^2}, n \geq 0$, *where $L_1 = 1 + L, L \geq 1$ is Lipschitz constant of A;*

(ii) $b(\alpha_n), \beta_n \to 0$ *as* $n \to \infty$;

(iii) $\sum\limits_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$.

Assume that $\{u_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty}$ are two sequences in X satisfying condition: $\|u_n\| = o(\alpha_n)$, $\|v_n\| \to 0$ as $n \to \infty$, and $\|v_n\| \leq 1, \forall n \geq 0$. Define $S : E \to E$ by $Sx = f + x - Tx, f \in X, \forall x \in X$. Then the Ishikawa iterative process $\{x_n\}_{n=0}^{\infty}$ $x_0 \in X$ by

$$\begin{cases} x_0 \in X, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n + v_n, n \geq 0, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n + u_n, n \geq 0. \end{cases}$$

converges strongly to the unique solution of the equation $Tx = f$. one question arises naturally: If T neither is Lipschitzian nor has the bounded range , whether or not the Ishikawa iterative sequence $\{x_n\}_{n=1}^{\infty}$ generated by converges strongly to the unique solution of the equation $Tx = f$. It is our purpose in this paper to solve the above part question by proving the *following much more general result: If E is an uniformly smooth real Banach space. Assume that $T : E \to E$ is Φ-accretive mapping, and T neither is Lipschizian nor has the bounded range, then the Ishikawa iteration sequence with errors generated by converges strongly to the unique solution of the equation $Tx = f$. For this, we need to give the following concept and Lemma.*

Definition 1.2 *A mapping $T : E \to E$ is called a generalized Lipschitz mapping, if there exists a constant $L > 0$ such that $\|Tx - Ty\| \leq L(1 + \|Tx - Ty\|)$, $\forall x, y \in E$. Clear, every Lipschitz mapping is generalized Lipschitz mapping. However, generalized Lipschitz mapping must not be Lipschitz. See the following example.*

Example Let $E = (-\infty, +\infty)$ and $T : E \to E$ be

$$Tx = \begin{cases} x - 1, & \text{if x} \in (-\infty, 0), \\ x - \sqrt{1 - (x+1)^2}, & \text{if x} \in [-1, 0), \\ x + \sqrt{1 - (x-1)^2}, & \text{if x} \in [0, 1], \\ x + 1, & \text{if x} \in (1, +\infty). \end{cases}$$

Lemma 1.1 ([4]) *Let E be a real Banach space, then for all $x, y \in E$, there exists $j(x + y) \in J(x + y)$ such that $||x + y||^2 \leq ||x||^2 + 2 < y, j(x + y) >$.*

2 Main Results

Now we prove the main the results of this paper, In the sequel, we always assume that E is a uniformly smooth real Banach space.

Theorem 2.1 *Assume that $T : E \to E$ is generalized Lipschitz Φ-accretive mapping with $\Phi(r) \to +\infty$ as $r \to +\infty$. Let$\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$ be real sequences in $[0, 1]$ satisfying the following conditions:*

(i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$;

(ii) $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} b'_n = \lim\limits_{n\to\infty} c'_n = 0$;

(iii) $\sum\limits_{n=0}^{\infty} b_n = \infty$;

(iv) $c_n = o(b_n)$.

For arbitrary $x_0 \in E$, define the Ishikawa iterative process with errors $\{x_n\}_{n=0}^{\infty}$ by (ISE):

$$\begin{cases} y_n = a'_n x_n + b'_n S x_n + c'_n v_n, \\ x_{n+1} = a_n x_n + b_n S y_n + c_n u_n, \end{cases} \tag{2.1}$$

where $S : E \to E$ is defined by $Sx = f + x - Tx, \forall x \in E$. Assume that the equation $Tx = f$ has solution and $\{u_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty}$ are arbitrary two bounded sequences in E. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique solution of the equation $Tx = f$.

Proof: Let q be the solution of the equation $Tx = f$, then q is the unique solution. Since T is generalized Lipschitz Φ-accretive, then there exists $L_0 > 0$ such that $\|Tx-Ty\| \le L_0(1+\|x-y\|)$ and $< Tx-Ty, J(x-y) > \ge \Phi(\|x-y\|)$, for all $x, y \in E$, i.e., $\|Sx-Sy\| \le L(1+\|x-y\|)$, $< Sx - Sy, J(x - y) > \le \|x - y\|^2 - \Phi(\|x - y\|)$, where $L = 1 + L_0$. Especially, for $\forall x \in E$, $< Sx - Sq, J(x - q) > \le \|x - q\|^2 - \Phi(\|x - q\|)$. Observe that (ISE) equivalent form

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n Sx_n + V_n + c_n'(q - x_n) \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n Sy_n + U_n + c_n(q - x_n) \end{cases} \tag{2.2}$$

where $V_n = c_n'(v_n - q)$, $U_n = c_n(u_n - q)$, $\beta_n = b_n'$, $\alpha_n = b_n$. Then $\|V_n\| \to 0$ as $n \to \infty$, $\|U_n\| = o(b_n)$. From form (2.2), we obtain that

$$\begin{aligned} \|y_n - q\| &= \|(1-\beta_n)(x_n - q) + \beta_n(Sx_n - Sq) + V_n + c_n'(q - x_n)\| \\ &\le (1 - \beta_n + \beta_n L + c_n')\|x_n - q\| + \beta_n L + \|V_n\|, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \|x_{n+1} - q\| &= \|(1-\alpha_n)(x_n - q) + \alpha_n(Sy_n - Sq) + U_n + c_n(q - x_n)\| \\ &\le (1 - \alpha_n + \alpha_n L(1 - \beta_n + \beta_n L + c_n') + c_n)\|x_n - q\| \\ &+ \alpha_n(L + \beta_n L^2 + L\|V_n\|) + \|U_n\|. \end{aligned} \tag{2.4}$$

Furthermore, we have the following estimates

$$\begin{aligned} 2c_n\|x_n - q\| \cdot \|x_{n+1} - q\| &\le 2c_n(1 - \alpha_n + \alpha_n L(1 - \beta_n + \beta_n L + c_n') + c_n)\|x_n - q\|^2 \\ &+ 2c_n(\alpha_n(L + \beta_n L^2 + L\|v_n\|) + \|U_n\|)\|x_n - q\| \\ &\le R_n\|x_n - q\|^2 + P_n, \end{aligned} \tag{2.5}$$

where $R_n = 2c_n(1-\alpha_n+\alpha_n L(1-\beta_n+\beta_n L+c_n')+c_n)+c_n(\alpha_n(L+\beta_n L^2+L\|V_n\|)+\|U_n\|)$, $P_n = c_n(\alpha_n(L + \beta_n L^2 + L\|V_n\|) + \|U_n\|)$. And have

$$\begin{aligned} &2\left(\|V_n\| + c_n'\|x_n - q\|\right)\|y_n - q\| \\ &\le 2(\|V_n\| + c_n'\|x_n - q\|)((1 - \beta_n + \beta_n L + c_n')\|x_n - q\| + \beta_n L + \|V_n\|) \\ &\le 2c_n'((1 - \beta_n + \beta_n L + c_n')\|x_n - q\|^2 + 2\|V_n\|(\beta_n L + \|V_n\|)) \\ &+ 2(\|V_n\|(1 - \beta_n + \beta_n L + c_n') + c_n'(\beta_n L + \|v_n\|)\|x_n - q\| \\ &\le 2c_n'((1 - \beta_n + \beta_n L + c_n')\|x_n - q\|^2 + 2\|V_n\|(\beta_n L + \|V_n\|)) \\ &+ (\|V_n\|(1 - \beta_n + \beta_n L + c_n') + c_n'(\beta_n L + \|V_n\|)(1 + \|x_n - q\|^2) \\ &\le ((2c_n' + \|V_n\|)(1 - \beta_n + \beta_n L + c_n') + c_n'(\beta_n L + \|V_n\|))\|x_n - q\|^2) \\ &+ (\|V_n\|(1 - \beta_n + \beta_n L + c_n') + (c_n' + 2\|V_n\|)(\beta_n L + \|V_n\|) \\ &= G_n\|x_n - q\|^2 + H_n \end{aligned} \tag{2.6}$$

where $G_n = (2c'_n + \|V_n\|)(1-\beta_n+\beta_n L + c'_n) + c'_n(\beta_n L + \|V_n\|)$, $H_n = (\|V_n\|(1-\beta_n + \beta_n L + c'_n) + (c'_n + 2\|V_n\|)(\beta_n L + \|V_n\|)$. Set $A_n = \left\| J\left(\frac{x_{n+1}-q}{1+\|x_n-q\|}\right) - J\left(\frac{y_n-q}{1+\|x_n-q\|}\right)\right\|$, $D_n = \left\| J\left(\frac{y_n-q}{1+\|x_n-q\|}\right) - J\left(\frac{x_n-q}{1+\|x_n-q\|}\right)\right\|$, then $A_n \to 0$, $D_n \to 0$ as $n\to\infty$. Indeed $\left\{\frac{y_n-q}{1+\|x_n-q\|}\right\}_{n=0}^{\infty}$, $\left\{\frac{x_n-q}{1+\|x_n-q\|}\right\}_{n=0}^{\infty}$, and $\left\{\frac{x_{n+1}-q}{1+\|x_n-q\|}\right\}_{n=0}^{\infty}$ are all bounded, and $\left\|\frac{x_{n+1}-q}{1+\|x_n-q\|} - \frac{y_n-q}{1+\|x_n-q\|}\right\| \to 0$, $\left\|\frac{y_n-q}{1+\|x_n-q\|} - \frac{x_n-q}{1+\|x_n-q\|}\right\| \to 0$ as $n\to\infty$. Applying uniformly continuity of J on any bounded subset, hence $A_n\to 0$, $D_n \to 0$ as $n\to\infty$. Using Lemma 1.1 and (2.4), (2.5), we may obtain

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \|(1-\alpha_n)(x_n-q) + \alpha_n(Sy_n - Sq) + U_n + c_n(q-x_n)\|^2\\
&\le (1-\alpha_n)^2\|x_n-q\|^2 + 2\alpha_n < Sy_n - Sq, J(x_{n+1}-q) >\\
&+ 2 < U_n, J(x_{n+1}-q) > + 2c_n < q - x_n, J(x_{n+1}-q) >\\
&\le (1-\alpha_n)^2\|x_n-q\|^2 + 2\alpha_n < Sy_n - Sq, J(y_n-q) >\\
&+ 2\alpha_n < Sy_n - Sq, J(x_{n+1}-q) - J(y_n-q) >\\
&+ 2\|U_n\|\cdot\|x_{n+1}-q\| + 2c_n\|x_n-q\|\cdot\|x_{n+1}-q\|\\
&\le (1-\alpha_n)^2\|x_n-q\|^2 + 2\alpha_n(\|y_n-q\|^2 - \Phi(\|y_n-q\|)) \qquad (2.7)\\
&+ 2\alpha_n < Sy_n - Sq, J\left(\frac{x_{n+1}-q}{1+\|x_n-q\|}\right) - J\left(\frac{y_n-q}{1+\|x_n-q\|}\right) > (1+\|x_n-q\|)\\
&+ 2\|U_n\|(1-\alpha_n + \alpha_n L(1-\beta_n+\beta_n L + c'_n) + c_n)\|x_n-q\|\\
&+ 2\|U_n\|(\alpha_n L + \alpha_n\beta_n L^2 + \alpha_n L\|V_n\| + \|U_n\|) + R_n\|x_n-q\|^2 + P_n\\
&\le (1-\alpha_n)^2\|x_n-q\|^2 + 2\alpha_n(\|y_n-q\|^2 - \Phi(\|y_n-q\|))\\
&+ 2\alpha_n A_n L(1+\|y_n-q\|)(1+\|x_n-q\|) + E_n + P_n + R_n\|x_n-q\|^2\\
&+ 2\|U_n\|(1-\alpha_n+\alpha_n L(1-\beta_n+\beta_n L))\|x_n-q\|
\end{aligned}$$

where $E_n = 2\|U_n\|(\alpha_n L + \alpha_n\beta_n L^2 + \alpha_n L\|V_n\| + \|U_n\|)$. Furthermore,

$$\begin{aligned}
&2\|U_n\|(1-\alpha_n+\alpha_n L(1-\beta_n+\beta_n L + c'_n) + c_n)\|x_n-q\|\\
&\quad\le \|U_n\|((1-\alpha_n+\alpha_n L(1-\beta_n+\beta_n L+c'_n)+c_n)^2 + \|x_n-q\|^2) \qquad (2.8)\\
&\quad\le \|U_n\|M_1 + \|U_n\|\|x_n-q\|^2,
\end{aligned}$$

$$\begin{aligned}
&2\alpha_n A_n L(1+\|y_n-q\|)(1+\|x_n-q\|)\\
&\quad\le 2\alpha_n A_n L((1-\beta_n+\beta_n L+c'_n)\|x_n-q\| + 1 + \beta_n L + \|v_n\|)(1+\|x_n-q\|)\\
&\quad\le 4\alpha_n A_n L(1+\beta_n L + c'_n)(1+\|x_n-q\|^2) \qquad (2.9)\\
&\quad= F_n\|x_n-q\|^2 + F_n
\end{aligned}$$

where $M_1 = \sup\{(1-\alpha_n+\alpha_n L(1-\beta_n+\beta_n L+c'_n)+c_n)^2\}$. $F_n = 4\alpha_n A_n L(1+\beta_n L + c'_n)$. Substituting (2.8) and (2.9) in (2.7), we have

$$\begin{aligned}
\|x_{n+1}-q\|^2 \le{}& ((1-\alpha_n)^2 + F_n + R_n + \|U_n\|)\|x_n-q\|^2 + E_n + F_n\\
&+ P_n + \|U_n\|M_1 + 2\alpha_n(\|y_n-q\|^2 - \Phi(\|y_n-q\|)). \qquad (2.10)
\end{aligned}$$

Again using Lemma 1.1 and (2.6), we obtain

$$\begin{aligned}
\|y_n-q\|^2 &\le (1-\beta_n)^2\|x_n-q\|^2+2\beta_n < Sx_n-Sq, J(y_n-q)>\\
&+2(\|V_n\|+c_n'\|x_n-q\|)\|y_n-q\|\\
&\le (1-\beta_n)^2\|x_n-q\|^2+2\beta_n < Sx_n-Sq, J(y_n-q)-J(x_n-q)>\\
&+2\beta_n < Sx_n-Sq, J(x_n-q) > +G_n\|x_n-q\|^2+H_n\\
&\le (1+\beta_n^2+G_n)\|x_n-q\|^2-2\beta_n\Phi(\|x_n-q\|)+H_n\\
&+2\beta_n < Sx_n-Sq, J\left(\frac{y_n-q}{1+\|x_n-q\|}\right)-J\left(\frac{x_n-q}{1+\|x_n-q\|}\right) > (1+\|x_n-q\|)\\
&\le (1+\beta_n^2+G_n)\|x_n-q\|^2-2\beta_n\Phi(\|x_n-q\|)+H_n\\
&+2\beta\|Sx_n-Sq\|D_n(1+\|x_n-q\|\\
&\le (1+\beta_n^2+G_n+4\beta_nD_nL)\|x_n-q\|^2\\
&+H_n+4\beta_nD_nL-2\beta_n\Phi(\|x_n-q\|)\,.
\end{aligned} \tag{2.11}$$

Substituting (2.11) in (2.10), get

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\le (1+\alpha_n^2+F_n+R_n+\|U_n\|+2\alpha_n(\beta_n^2+G_n+4\beta_nD_nL))\\
&\times\|x_n-q\|^2+E_n+F_n+P_n+\|U_n\|M_1+2\alpha_nH_n\\
&+8\alpha_n\beta_nD_nL-2\alpha_n\Phi(\|y_n-q\|))\\
&-4\alpha_n\beta_n\Phi(\|x_n-q\|)\|x_n-q\|)\\
&\le \|x_n-q\|^2+I_n\|x_n-q\|^2+2\alpha_n(O_n-\Phi(\|y_n-q\|))
\end{aligned} \tag{2.12}$$

where $I_n=\alpha_n^2+F_n+R_n+\|U_n\|+2\alpha_n(\beta_n^2+G_n+4\beta_nD_nL)$, $O_n=(E_n+F_n+P_n+\|U_n\|M_1+2\alpha_nH_n+8\alpha_n\beta_nD_nL)/2\alpha_n$. Base on definition of S, for any $\forall x\in E$, $< Sx-Sq-x+q, J(x-q) >\le -\Phi(\|x-q\|)$. Thus, $\Phi(\|x-q\|)\le\|x-Sx\|$. Any choose $x_0\in E$ such that $\|x_0-Sx_0\|\ne 0$, i.e,$x_0\ne q$. If $x_0=q$, then we are done. Suppose this is not the case, then have $\|x_0-q\|\le\Phi^{-1}(\|x_0-Sx_0\|)$. Since $\alpha_n,\beta_n\to 0$ $(n\to\infty)$, so that $I_n=o(\alpha_n), O_n=o(\alpha_n), \|U_n\|=o(\alpha_n)$ and $\|V_n\|\to 0(n\to\infty)$, there exists positive integer N such that $\alpha_n<\frac{\Phi^{-1}(\|x_0-Sx_0\|)}{4(1+L+L^2)\Phi^{-1}(\|x_0-Sx_0\|)+2L^2+4L}$, $\beta_n<\frac{2\Phi^{-1}(\|x_0-Sx_0\|)}{3(1+L)\Phi^{-1}(\|x_0-Sx_0\|)+3L}$, $\|U_n\|<\frac{\Phi^{-1}(\|x_0-Sx_0\|)}{2}$, $\|V_n\|<\min\left\{1,\frac{\Phi^{-1}(\|x_0-Sx_0\|)}{6}\right\}$, $1-\beta_n-\beta_nL-c_n'>\frac{3}{4}$, $\beta_nL+\|V_n\|<\frac{\Phi^{-1}(\|x_0-Sx_0\|)}{6}$, $\frac{I_n(2\Phi^{-1}(\|x_0-Sx_0\|))^2}{2\alpha_n}+O_n<\frac{\Phi\left(\frac{\Phi^{-1}(\|x_0-Sx_0\|)}{2}\right)}{2}$. For all $n\ge N$. Suppose $\|x_N-q\|\le 2\Phi^{-1}(\|x_0-Sx_0\|)$ holds, we prove $\|x_{N+1}-q\|\le 2\Phi^{-1}(\|x_0-Sx_0\|)$. Assume that this is not true, then $\|x_{N+1}-q\|>2\Phi^{-1}(\|x_0-Sx_0\|)$. From (2.2) we may get $(1-\alpha_N)\|x_N-q\|\ge\|x_{N+1}-q\|-\alpha_N\|x_N-Sy_N\|-\|U_N\|-c_N\|x_N-q\|$ (N is enough big, $1-\alpha_N+c_N<1$).

We also obtain the following inequality:

$$\begin{aligned}\|x_N - q\| &\geq \|x_{N+1} - q\| - \alpha_N \|x_N - Sy_N\| - \|U_N\| \\ &\geq 2\Phi^{-1}(\|x_0 - Sx_0\|) \\ &- \alpha_N(2(1 + L + L^2)\Phi^{-1}(\|x_0 - Sx_0\|) + L^2 + 2L) - \|U_N\| \\ &\geq \Phi^{-1}(\|x_0 - Sx_0\|)\end{aligned}$$

and

$$\begin{aligned}\|y_N - q\| &\geq (1 - \beta_N)\|x_N - q\| - \beta_N L\|x_N - q\| - \beta_N L - \|V_N\| - c'_N\|x_N - q\| \\ &= (1 - \beta_N - \beta_N L - c'_N)\|x_N - q\| - \beta_N L - \|V_N\| \\ &\geq \frac{\Phi^{-1}(\|x_0 - Sx_0\|)}{2},\end{aligned}$$

so that $\Phi(\|y_N - q\|) \geq \Phi(\frac{\Phi^{-1}(\|x_0 - Sx_0\|)}{2})$. Using (2.12) and above relevant form, we compute as follows:

$$\begin{aligned}\|x_{N+1} - q\|^2 &\leq \|x_N - q\|^2 + I_N\|x_N - q\|^2 + 2\alpha_N(O_N - \Phi(\|y_N - q\|)) \\ &\leq \|x_N - q\|^2 + 2\alpha_N(\frac{I_N\|x_N - q\|^2}{2\alpha_N} + O_N - \Phi(\|y_N - q\|)) \\ &\leq \|x_N - q\|^2 - \alpha_N\Phi(\frac{\Phi^{-1}(\|x_0 - Sx_0\|)}{2}) \\ &\leq \|x_N - q\|^2 \leq (2\Phi^{-1}(\|x_0 - Sx_0\|))^2 .\end{aligned}$$

contradicting with assumption. By induction, so sequence $\{\|x_n - q\|\}_{n=0}^{\infty}$ is bounded, therefore $\{\|y_n - q\|\}_{n=0}^{\infty}$ is also bounded. Set $W = \sup\{\|x_n - q\|\} + \sup\{\|y_n - q\|\}$, $Q_n = \frac{I_n W^2}{2\alpha_n} + O_n$. Then using (2.12), we have

$$\begin{aligned}\|x_{n+1} - q\|^2 &\leq \|x_n - q\|^2 + 2\alpha_n\left(\frac{I_n\|x_n - q\|^2}{2\alpha_n} + O_n - \Phi(\|y_n - q\|)\right) \\ &\leq \|x_n - q\|^2 + 2\alpha_n(Q_n - \Phi(\|y_n - q\|)) \\ &= \|x_n - q\|^2 + \alpha_n(2Q_n - \Phi(\|y_n - q\|)) - \alpha_n\Phi(\|y_n - q\|)) .\end{aligned} \tag{2.13}$$

In the following we prove that $\liminf_{n\to\infty} \|y_n - q\| = 0$ holds. If not true. Let $\liminf_{n\to\infty} \|y_n - q\| = 2\delta > 0$. Then, there exists an integer N_1 such that $\|y_n - q\| \geq \delta$, $\forall n \geq N_1$, i.e., $\Phi(\|y_n - q\|) \geq \Phi(\delta)$. Since $Q_n \to 0 (n \to \infty)$, there exists positive integer $N_2 > N_1$ such that $Q_n \leq \Phi(\delta), \forall n \geq N_2$. The implies that $Q_n \leq \Phi(\|y_n - q\|)$. Hence, for all $n \geq N_2$, we obtain that

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - \alpha_n\Phi(\|y_n - q\|)) \leq \|x_n - q\|^2 - \alpha_n\Phi(\delta) .$$

This implies that

$$\Phi(\delta)\sum_{n=N_2}^{\infty}\alpha_n \leq \sum_{n=N_2}^{\infty}(\|x_n - q\|^2 - \|x_{n+1} - q\|^2) \leq \|x_{N_2} - q\|^2 < \infty$$

contradicting with condition (iii) of Theorem. So there exists a infty subsequence $\{y_{n_j}-q\}_{j=0}^{\infty}$ of $\{y_n-q\}_{n=0}^{\infty}$. Again from (2.2), get $\|y_{n_j}-q\|\geq(1-\beta_{n_j}-\beta_{n_j}L-c'_{n_j})\|x_{n_j}-q\|-\beta_{n_j}L-\|V_{n_j}\|$, so that $\lim_{j\to\infty}\|x_{n_j}-q\|=0$. Hence, for any $\forall\varepsilon>0$, there exists a positive integer n_{j_0}, such that $\|x_{n_j}-q\|<\varepsilon, \alpha_{n_j}(LW+L)+\|U_{n_j}\|<\frac{\varepsilon}{4}, (\beta_{n_j}+\beta_{n_j}L+c'_{n_j})W+(\beta_{n_j}L+\|V_{n_j}\|)<\frac{\varepsilon}{4}$, for all $n_j>n_{j_0}$. Again choose a positive integer $N_0\geq n_{j_0}$ such that $Q_n<\Phi(\frac{\varepsilon}{2})\frac{1}{2}$, for all $n>N_0$. Next, we want to prove: for arbitrary $\forall m\geq 1, \|x_{n_j+m}-q\|<\varepsilon, n_j>n_{j_0}$. First, we prove that $\|x_{n_j+1}-q\|<\varepsilon$. If it is not the case, then there exists $n_{j_1}>n_{j_0}$ such that $\|x_{n_{j_1}+1}-q\|<\varepsilon$. Using (2.2) again, we have

$$\begin{aligned}\|x_{n_{j_1}+1}-q\| &\leq (1-\alpha_{n_{j_1}})\|x_{n_{j_1}}-q\|+\alpha_{n_{j_1}}\|Sy_{n_{j_1}}-Sq\|+\|U_{n_{j_1}}\|+c_{n_{j_1}}\|x_{n_{j_1}}-q\| \\ &\leq (1-\alpha_{n_{j_1}}+c_{n_{j_1}})\|x_{n_{j_1}}-q\|+\alpha_{n_{j_1}}L(1+\|y_{n_{j_1}}-q\|)+\|U_{n_{j_1}}\| \\ &\leq \|x_{n_{j_1}}-q\|+\alpha_{n_{j_1}}L(1+W)+\|U_{n_{j_1}}\| \\ &\leq \|x_{n_{j_1}}-q\|+\frac{\varepsilon}{4}\end{aligned}$$

lead to $\|x_{n_{j_1}}-q\|>\|x_{n_{j_1}+1}-q\|-\frac{\varepsilon}{4}>\frac{3\varepsilon}{4}$. And we get also

$$\begin{aligned}\|y_{n_{j_1}}-q\| &\geq (1-\beta_{n_{j_1}})\|x_{n_{j_1}}-q\|-\beta_{n_{j_1}}L(\|x_{n_{j_1}}-q\|+1)-\|V_{n_{j_1}}\|-c'_{n_{j_1}}\|x_{n_{j_1}}-q\| \\ &\geq \|x_{n_{j_1}}-q\|-(\beta_{n_{j_1}}+\beta_{n_{j_1}}L+c'_{n_{j_1}})\|x_{n_{j_1}}-q\|-(\beta_{n_{j_1}}L+\|V_{n_{j_1}}\|) \\ &> \frac{3\varepsilon}{4}-(\beta_{n_{j_1}}+\beta_{n_{j_1}}L+c'_{n_{j_1}})W-(\beta_{n_{j_1}}L+\|V_{n_{j_1}}\|) \\ &> \frac{\varepsilon}{2}.\end{aligned}$$

Hence $\Phi(\|y_{n_{j_1}}-q\|)>\Phi(\frac{\varepsilon}{2})$. By (2.12), we obtain that

$$\begin{aligned}\varepsilon^2 &\leq \|x_{n_{j_1}+1}-q\|^2 \\ &\leq \|x_{n_{j_1}}-q\|^2+2\alpha_{n_{j_1}}(Q_{n_{j_1}}-\Phi(\|y_{n_{j_1}}-q\|)) \\ &< \varepsilon^2+2\alpha_{n_{j_1}}(\Phi(\frac{\varepsilon}{2})\frac{1}{2}-\Phi(\frac{\varepsilon}{2})) \\ &= \varepsilon^2-\alpha_{n_{j_1}}\Phi(\frac{\varepsilon}{2})\frac{1}{2} \\ &\leq \varepsilon^2\end{aligned}$$

contradiction. By induction, we obtain that $\|x_{n_j+m}-q\|<\varepsilon$. This show that $x_n\to q$ as $n\to\infty$. About case $\sum_{n=0}^{\infty}\|U_n\|<\infty$, repeating above-mentioned course, we can get the conclusion. Completing proof of Theorem 2.1. □

Remark 1 Theorem 2.1 contains a good number of the known results as its special cases. In particular, if the mapping T considered here satisfies one of the following assumptions: (i) $T:K\to K$ is a Lipschitzian. (ii) T has the bounded range. Then T satisfied the conditions of Theorem 2.1.

Remark 2 It is well known that T is strongly pseudocontractive (Φ-strongly pseudocontractive, Φ-pseudocontractive)if and only if $(I-T)$ is strongly accertive (Φ-strongly accretive, Φ-accretive), where I denotes the identity operator. In the following we give about the results of Φ-pseudocontractive.

Theorem 2.2 *Let K be nonempty closed convex subset of E and $T : K \to K$ be generalized Lipschitz Φ-pseudocontractive mapping. Assume that $\Phi(r) \to +\infty$ as $r \to +\infty$ and $F(T) \neq \emptyset$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$ be six real sequences in $[0, 1]$ satisfying the following conditions:*

(i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$;

(ii) $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} b'_n = \lim\limits_{n\to\infty} c'_n = 0$;

(iii) $\sum\limits_{n=0}^{\infty} b_n = \infty$;

(iv) $c_n = o(b_n)$ *or* $\sum\limits_{n=0}^{\infty} c_n < +\infty$.

For arbitrary $x_0 \in K$, define the Ishikawa iterative process with errors $\{x_n\}_{n=0}^{\infty}$ by (ISE):

$$\begin{cases} y_n = a'_n x_n + b'_n T x_n + c'_n v_n, \\ x_{n+1} = a_n x_n + b_n T y_n + c_n u_n, \end{cases} \tag{2.14}$$

Suppose $\{u_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty}$ are arbitrary two bounded sequences in K. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.

Proof: Applying Theorem 2.1, we obtain directly conclusion of Theorem 2.2. □

Remark 3 Our two Theorems extend the main known results from Lipschitzian or the boundedness range to more general class of neithe Lipschitzian nor the range boundedness mappings, and also from strongly pseudocontractive (accretive) to Φ- pseudocontractive (accretive).

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Author:

Xue Zhiqun
Department of Mathematics,
Shijiazhuang Railway College,
Shijiazhuang 050043
China

e-mail: xuezhiqun@126.com