Rostock. Math. Kolloq. **61**, 21–30 (2006)

Subject Classification (AMS)
**39A10**

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# A Note on Periodic Character of a Higher Order Difference Equation

ABSTRACT. In this note we prove that every positive solution of the difference equation

$$x_n = 1 + \frac{x_{n-k}}{x_{n-m}}, \quad n = 0, 1, \dots$$

where $k, m \in \mathbf{N}$ are so that $k < m$, and $2m = k(L+1)$ for some $L \in \mathbf{N}$, converges to a $k$-periodic solution. A similar result is proved for a corresponding symmetric system of difference equations. We also consider the systems of difference equations whose all solutions are periodic with the same period. It is generalized and solved Open Problem 2.9.1 in M. R. S. Kulenović and G. Ladas, *Dynamics of Second Order Rational Difference Equations. With open problems and conjectures.* Chapman and Hall/CRC, 2002.

KEY WORDS. $k$-periodic solution, difference equation, positive solution, system of difference equations

## 1 Introduction

In this note we study the difference equation

$$x_n = 1 + \frac{x_{n-k}}{x_{n-m}}, \quad n = 0, 1, \dots \tag{1}$$

where $k, m \in \mathbf{N}$ are so that $k < m$, and $2m = k(L+1)$ for some $L \in \mathbf{N}$ and initial conditions $x_{-m}, \dots, x_{-2}, x_{-1}$ are positive real numbers.

In [5, Theorem 4.1] was proved that every positive solution of the difference equation $x_n = 1 + \frac{x_{n-2}}{x_{n-3}}$ converges to a period two solution. This motivated us to generalize the result in [10]. In [6] we prove the following result:

**Theorem A** *Let $I$ be an open interval of the real line, $\varphi : I^k \to I$ be a continuous function which is nondecreasing in each variable and increasing in the first one and*

*$\varphi(x, x, \ldots, x) \le x$, for every $x \in I$. If $(a_n)$ is a bounded sequence which satisfies the inequality*

$$a_{n+k} \le \varphi(a_{n+k-1}, a_{n+k-2}, \ldots, a_n) \quad \textit{for} \quad n \in \mathbf{N} \cup \{0\},$$

*then it converges.*

Other useful globally convergence results can be found, for example, in [7, 8].

It is easy to prove that every positive solution of Eq. (1) is bounded, moreover, in [9] we prove that if $p \ge 1$ and $m, k \in \mathbf{N}$, then every positive solution of the difference equation

$$x_n = p + \frac{x_{n-k}}{x_{n-m}}, \tag{2}$$

is bounded. By a slight modification of the proof of Theorem 3 in [10] it follows that if $p > 1$, then every positive solution of Eq. (2) converges, see, also [9]. Unlike the case $p > 1$, Theorem 4.1 in [5] shows that positive solutions of equation $x_n = 1 + \frac{x_{n-2}}{x_{n-3}}$ need not converge. Hence, Eq. (2) is more interesting in the case $p = 1$. The case $p \in (0, 1)$ was considered in paper [3]. Our aim is to generalize the main results in [2], [5] and [10]. In Section 2 we generalize the main result in [10] developing the main idea from the same paper. In Section 3 we show that the main result in [2] is an easy consequence of known results, also a generalization of the result is given. Section 4 is devoted to the systems of difference equations which all solutions are periodic with the same period. In the section we generalize and solve Open Problem 2.9.1 from [4].

## 2 Asymptotic periodicity of solutions of Eq. (1)

In this section we consider the positive solutions of Eq. (1). We prove the following result:

**Theorem 1** *Let $k, m \in \mathbf{N}$ be such that $k < m$ and $2m = k(L+1)$ for some $L \in \mathbf{N}$. Then every positive solution of Eq. (1) converges to a not necessarily prime $k$-periodic solution of Eq. (1). If $L$ is odd, then every positive solution of Eq. (1) converges to the equilibrium $x^* = 2$.*

**Proof:** We have

$$\begin{aligned}
x_n &= 1+\frac{x_{n-k}}{1+\dfrac{x_{n-m-k}}{x_{n-2m}}} = 1+\frac{x_{n-k}}{1+\dfrac{1}{x_{n-2m}}\left(1+\dfrac{x_{n-m-2k}}{x_{n-2m-k}}\right)} \\
&= 1+\frac{x_{n-k}}{1+\dfrac{1}{x_{n-2m}}+\dfrac{1}{x_{n-2m}x_{n-2m-k}}\left(1+\dfrac{x_{n-m-3k}}{x_{n-2m-2k}}\right)} \\
&= \cdots \\
&= 1+\frac{x_{n-k}}{1+\sum\limits_{i=0}^{l-2}\prod\limits_{j=0}^{i}\dfrac{1}{x_{n-2m-jk}}+\dfrac{x_{n-m-lk}}{\prod\limits_{j=0}^{l-1}x_{n-2m-jk}}}\,, \qquad (3)
\end{aligned}$$

for every $n \geq lk+m-k$.

Let $l, t \in \mathbf{N}$ are chosen such that $t < l$ and $l-t = L$. Since $2m = k(L+1)$ we have that

$$[n-m-lk]+m-k = n-2m-tk\,. \qquad (4)$$

For such chosen $l$ and $t$ it follows from (1) that

$$\frac{x_{n-m-lk}}{x_{n-2m-tk}} = \frac{1}{x_{n-2m-(t-1)k}-1}\,, \quad \text{for} \quad n \geq lk\,. \qquad (5)$$

From (3) and (5) it follows that

$$x_n = 1+\frac{x_{n-k}}{1+\sum\limits_{i=0}^{l-2}\prod\limits_{j=0}^{i}\dfrac{1}{x_{n-2m-jk}}+\dfrac{(x_{n-2m-(t-1)k}-1)^{-1}}{\prod\limits_{j=0,j\neq t}^{l-1}x_{n-2m-jk}}}\,,$$

that is

$$x_n = 1+\frac{x_{n-k}}{1+\sum\limits_{i=0}^{l-2}\prod\limits_{j=0}^{i}\dfrac{1}{x_{n-k(L+1+j)}}+\dfrac{(x_{n-k(L+t)}-1)^{-1}}{\prod\limits_{j=0,j\neq t}^{l-1}x_{n-k(L+1+j)}}}\,, \qquad (6)$$

Using the changes $y_m^{(i)} = x_{km+i}$, $i = 0, 1, \ldots, k-1$, Eq. (6) separates into the following $k$ equations

$$y_m^{(i)} = 1+\frac{y_{m-1}^{(i)}}{1+\sum\limits_{i=0}^{l-2}\prod\limits_{j=0}^{i}\dfrac{1}{y_{m-(L+1+j)}^{(i)}}+\dfrac{1}{\left(y_{m-(L+t)}^{(i)}-1\right)}\dfrac{1}{\prod\limits_{j=0,j\neq t}^{l-1}y_{m-(L+1+j)}^{(i)}}}\,, \qquad (7)$$

$i = 0, 1, \ldots, k-1$.

Eq. (7) can be written in the following form

$$y_m^{(i)} = F\big(y_{m-1}^{(i)}, \ldots, y_{m-(L+l)}^{(i)}\big), \quad i = 0, 1, \ldots, k-1\,.$$

Since

$$F(x, \ldots, x) = 1 + \frac{x}{\displaystyle\sum_{i=0}^{l-1} \frac{1}{x^i} + \frac{1}{x^{l-1}(x-1)}} = x, \qquad \text{for} \quad 0 \neq x \neq 1\,,$$

and since $F$ is nondecreasing in each variable and increasing in the first one, we see that all conditions in Theorem A are satisfied on the interval $(1, \infty)$, which implies that the sequence $y_m^{(i)}$, that is, $x_{km+i}$ converges to $y_i^*$, for each $i = 0, 1, \ldots, k-1$. It is clear that $(y_0^*, \ldots, y_{k-1}^*)$ is a $k$-cycle of Eq. (1), from which the first statement follows.

Assume now that $L$ is odd. Then $L = 2s+1$ for some $s \in \mathbf{N} \cup \{0\}$. Hence Eq. (1) can be written as follows

$$x_n = 1 + \frac{x_{n-k}}{x_{n-(s+1)k}}\,. \tag{8}$$

Using the changes $y_m^{(i)} = x_{km+i}$, $i = 0, 1, \ldots, k-1$, Eq. (8) can be separated into the following $k$ equations

$$y_m^{(i)} = 1 + \frac{y_{m-1}^{(i)}}{y_{m-(s+1)}^{(i)}}, \quad i \in \{0, 1, \ldots, k-1\}\,. \tag{9}$$

Each of equations in (9) is a special case of Eq. (1) with $k = 1$, $m = s+1$ and $L = 2m-1$. According to the first part of the theorem it follows that every positive solution of each of equations in (9) converges to a periodic solution of period one, that is, to $y^* = 2$, from which it follows that every positive solution of Eq. (1) in this case, converges to the equilibrium $x^* = 2$, as desired. $\square$

For $k = 1$ we obtain the following global stability result.

**Corollary 1** *Let $m \in \mathbf{N}$. Then every positive solution of the difference equation*

$$x_n = 1 + \frac{x_{n-1}}{x_{n-m}}, \quad n = 0, 1, \ldots$$

*converges to the positive equilibrium $x^* = 2$.*

## 3 A symmetric system of difference equations

In this section we consider the following symmetric system of difference equations

$$x_n = 1 + \frac{x_{n-k}}{y_{n-m}} \quad \text{and} \quad y_n = 1 + \frac{y_{n-k}}{x_{n-m}}, \quad n = 0, 1, \dots , \tag{10}$$

which corresponds to Eq. (1). A little surprising fact is that the method in Theorem 1 can be used also in studying of system (10). As a by-product we obtain a very short proof of the main result in [2]. The main result in this section is the following:

**Theorem 2** *Let $k, m \in \mathbf{N}$ be such that $k < m$ and $2m = k(L+1)$ for some $L \in \mathbf{N}$. Then every positive solution of system (10) converges to a $k$-periodic solution of the system.*

**Proof:** We have

$$\begin{aligned}
x_n &= 1 + \cfrac{x_{n-k}}{1 + \cfrac{y_{n-m-k}}{x_{n-2m}}} \\
&= 1 + \cfrac{x_{n-k}}{1 + \cfrac{1}{x_{n-2m}}\left(1 + \cfrac{y_{n-m-2k}}{x_{n-2m-k}}\right)} \\
&= 1 + \cfrac{x_{n-k}}{1 + \cfrac{1}{x_{n-2m}} + \cfrac{1}{x_{n-2m}x_{n-2m-k}}\left(1 + \cfrac{y_{n-m-3k}}{x_{n-2m-2k}}\right)} \\
&= \cdots \\
&= 1 + \cfrac{x_{n-k}}{1 + \sum_{i=0}^{l-2} \prod_{j=0}^{i} \cfrac{1}{x_{n-2m-jk}} + \cfrac{y_{n-m-lk}}{\prod_{j=0}^{l-1} x_{n-2m-jk}}},
\end{aligned} \tag{11}$$

for every $n \geq lk + m - k$.

Let $l, t \in \mathbf{N}$ are chosen such that $t < l$ and $l - t = L$. Since $2m = k(L+1)$ we have that

$$\frac{y_{n-m-lk}}{x_{n-2m-tk}} = \frac{1}{x_{n-2m-(t-1)k} - 1}, \quad \text{for} \quad n \geq lk. \tag{12}$$

From (11) and (12) it follows (6). As in the proof of Theorem 1 we have that $x_n$ converges to a $k$-cycle, say $(x_0^*, \dots, x_{k-1}^*)$.

Similarly, it can be proved that $y_n$ satisfies Eq. (6) and that it converges to a $k$-cycle, say $(y_0^*, \dots, y_{k-1}^*)$. It is easy to see that $(x_0^*, \dots, x_{k-1}^*), (y_0^*, \dots, y_{k-1}^*)$, is a $k$-periodic solution of system (10), from which the result follows. □

**Remark 1** Note that unlike the scalar Eq. (1) with $k = 3$ and $m = 6$, the corresponding system

$$x_n = 1 + \frac{x_{n-3}}{y_{n-6}} \quad \text{and} \quad y_n = 1 + \frac{y_{n-3}}{x_{n-6}}, \quad n = 0, 1, \dots, \tag{13}$$

has prime three-periodic solutions of the form

$$(x_n) = (a, b, c, a, b, c, \dots), \qquad (y_n) = \left(\frac{a}{a-1}, \frac{b}{b-1}, \frac{c}{c-1}, \dots\right).$$

## 4 Periodic solutions of Eq. (1)

In this section we find a subclass of Eq. (1) which have periodic solutions. Before we formulate and prove the main result of this section say that $GCD(m, k)$ denotes the greatest common divisor of integers $m$ and $k$.

**Theorem 3** *Let $m = 2^i m_1$ where $m_1$ is odd, and $2^{i+1} \mid k$. Then Eq. (1) has infinitely many periodic solutions with period $2GCD(m, k)$.*

**Proof:** . First note that $k = 2^{i+1}k_1$, for some $k_1 \in \mathbf{N}$. Then $m$ and $k$ can be written in the following forms

$$m = 2^i GCD(m_1, k_1)m_2 = GCD(m, k)m_2$$

and

$$k = 2^{i+1}GCD(m_1, k_1)k_2 = 2GCD(m, k)k_2\,.$$

Hence Eq. (1) can be written

$$x_n = 1 + \frac{x_{n-2GCD(m,k)k_2}}{x_{n-GCD(m,k)m_2}}\,. \tag{14}$$

Since every natural number $n$ can be written in the following form $n+1 = GCD(m, k)l + r$, where $l \in \mathbf{N} \cup \{0\}$ and $r = 0, 1, \dots, GCD(m, k) - 1$, it follows that Eq. (14) is separated into $GCD(m, k)$ independent equations of the form

$$x_l^{(i)} = 1 + \frac{x_{l-2k_2}^{(i)}}{x_{l-m_2}^{(i)}}\,, \tag{15}$$

$i \in \{0, 1, \dots, GCD(m, k) - 1\}$.

Now note that $m_2$ is odd. This means that the numbers $l$ and $l - 2k_2$ have the same parity, but $l - m_2$ has different one. Hence, each equation in (15) has a 2-periodic solution

$$\phi, \psi, \dots, \phi, \psi, \dots$$

with
$$\phi = 1 + \frac{\phi}{\psi} \quad \text{and} \quad \psi = 1 + \frac{\psi}{\phi},$$
which is equivalent to $\phi + \psi = \psi\phi$. It means that each equation in (15) has infinitely many periodic solutions with period two of the form
$$\phi, \frac{\phi}{\phi - 1}, \ldots, \phi, \frac{\phi}{\phi - 1}, \ldots .$$
From all above mentioned the result follows, that is, Eq. (1) has infinitely many periodic solutions with period $2GCD(m, k)$. □

For the readers interested in this topic we leave the following interesting open problem:

**Open Problem 1** *Investigate the behavior of the positive solutions of system (10) when $k, m \in \mathbf{N}$ are so that $k \neq m$, and $2m \neq k(L + 1)$ for every $L \in \mathbf{N}$.*

In view of Theorem 1 and Theorem 3 we also believe that the following conjecture holds:

**Conjecture 1** *Assume that $k, m \in \mathbf{N}$ such that $k < m$ and $2^i$ is the largest power of 2 which divides $m$. Show that the following statements are true:*

(a) *If $2^{i+1} \nmid k$, then every positive solution of Eq. (1) converges to the equilibrium $x^* = 2$.*

(b) *If $2^{i+1} | k$, then every positive solution of Eq. (1) converges to a $2GCD(m, k)$-periodic solution.*

## 5 On systems which have only periodic solutions

In [4, p. 43] the authors claim that for a linear equation, every solution is periodic with period $p \geq 2$, if and only if every root of the characteristic equation is a $p$th root of unity. However, this is only true if we add the condition that all these roots are simple and the equation is homogeneous, or the right-hand side constant and no resonance case. Motivated by this observation they posed the following open problem.

**Open Problem 2.9.1** *Assume that $f \in C^1[(0, \infty)^2, (0, \infty)]$ is such that every positive solution of the equation*
$$x_{n+1} = f(x_n, x_{n-1}), \qquad n = 0, 1, \ldots \tag{16}$$
*is periodic with period $p \geq 2$.*

*Is it true that the linearized equation about a positive equilibrium of Eq. (16) has the property that every one of its solutions is also periodic with the same period $p$?*

In [1] L. Berg shows that for ten nonlinear difference equations, whose all solutions are periodic with the same period $p$, all solutions of the corresponding linearized equations are periodic with the same period. This Berg's paper motivated us to believe that not only Open Problem 2.9.1 is true but that a more general result holds.

In order to solve Open Problem 2.9.1 we need a useful result contained in the following lemma. Before formulating it we say that for a mapping $f : X \to X$, $(f^{[p]})_{p\in\mathbf{N}\cup\{0\}}$ denotes the sequence of iterates of $f$, that is, $f^{[0]} = I$, the identity function on $X$, $f^{[1]} = f$ and generally $f^{[p+1]} = f \circ f^{[p]}$ for any $p \in \mathbf{N}$.

**Lemma 1** *Let $I \subset \mathbf{R}$ be an interval. Consider the system of difference equations*

$$\vec{x}_{n+1} = f(\vec{x}_n)\,, \tag{17}$$

*where $f \in C^1[I^k, I^k]$, and $x^*$ is an equilibrium of Eq. (17). If all solutions of Eq. (17) are periodic with period p, then Jacobi's matrix $Df(x^*)$ is diagonalizable and all its eigenvalues are pth roots of unity (here $Df$ denotes Jacobi's matrix of the function $f$).*

**Proof:** Since all solutions of Eq. (17) are periodic with period $p$, then we have

$$\vec{x}_p = f^{[p]}(\vec{x}_0) = \vec{x}_0\,, \tag{18}$$

for every $\vec{x}_0 \in I^k$. Differentiating (18) we have that

$$D(f^{[p]}(\vec{x})) = D\vec{x} = Id\,, \tag{19}$$

for every $\vec{x} \in I^k$, where $Id$ denotes the identity operator on $\mathbf{R}^n$.

Now note that every stationary point $x^*$ of system (17) is a fixed point of the equation $f(\vec{x}) = \vec{x}$.

Using this fact, chain rule and taking $\vec{x} = x^*$ in (19), we have that

$$[Df(x^*)]^p = Id\,, \tag{20}$$

that is, $p$th power of Jacobi's matrix $[Df(x^*)]$ is equal to $Id$. Using Jordan's decomposition of the matrix and (20), it follows that the matrix $[Df(x^*)]$ is diagonalizable and that all roots of the characteristic polynomial of the matrix are $p$th roots of unity, as desired. □

Notice that the linearized system of (17) at $x^*$ is

$$\vec{y}_{n+1} = [Df(x^*)]\vec{y}_n.$$

From all above mentioned it follows that the characteristic polynomial of the matrix in the corresponding linearized equation about an equilibrium has only zeros which are $p$th roots of unity.

As a corollary of Lemma 1 we obtain the next result which among other things solves Open Problem 2.9.1.

**Corollary 2** *Let $I \subset \mathbf{R}$ be an interval. Consider the difference equation*

$$x_{n+1} = f(x_n, \dots, x_{n-k+1}), \tag{21}$$

*where $f \in C^1[I^k, I]$, and $x^*$ is an equilibrium of Eq. (21). If all solutions of Eq. (21) are periodic with period $p$, then the zeros of the characteristic polynomial of the linearized equation*

$$y_{n+1} = \frac{\partial f}{\partial x_1}(x^*)y_n + \cdots + \frac{\partial f}{\partial x_k}(x^*)y_{n-k+1} \tag{22}$$

*about the equilibrium $x^*$ of Eq. (21) are simple $p$th roots of unity and consequently all solutions of Eq. (22) are periodic with period $p$.*

**Proof:** By standard transformation Eq. (21) can be written as a $k \times k$ system of difference equations of first order. The corresponding linearized system have the following matrix

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \dfrac{\partial f}{\partial x_k}(x^*) & \dfrac{\partial f}{\partial x_{k-1}}(x^*) & \cdot & \cdots & \dfrac{\partial f}{\partial x_1}(x^*) \end{bmatrix}. \tag{23}$$

By Lemma 1 the characteristic polynomial

$$\lambda^k - \frac{\partial f}{\partial x_1}(x^*)\lambda^{k-1} - \cdots - \frac{\partial f}{\partial x_k}(x^*) = 0 \tag{24}$$

of the system has only zeros which are $p$th roots of unity.

By a well known result (see [11, No. 9.67 point 4]) if the polynomial (24) has multiple zeros, then matrix (23) cannot be diagonalizable. By Lemma 1 matrix (23) is diagonalizable, which is a contradiction. Hence all zeros of polynomials (24) are simple $p$th roots of unity, which implies that all solutions of Eq. (22) are periodic with period $p$, as claimed. □

**Acknowledgement.** I would like to express my sincere thanks to Professor Lothar Berg for his helpful suggestions and comments during the preparation of this paper.

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**received:** September 15, 2005

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