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The Compactons and Generalized Kink Waves to a generalized CAMASSA-Holm Equation[2]

ABSTRACT. In this paper, the bifurcation method of planar systems and simulation method of differential equations are employed to investigate the bounded travelling waves of a generalized Camassa-Holm equation. The bounded travelling waves defined on finite core regions are found and their integral or implicit expressions are obtained. Their planar simulation graphs show that they possess the properties of compactons or generalized kink waves.

KEY WORDS. Camassa-Holm equation; compactons; generalized kink waves.

1 Introduction

In recent years the so-called Camassa-Holm [1] equation has caught a great deal of attention. It is a nonlinear dispersive wave equation that takes the form

$$u_t + 2ku_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}\,. \tag{1.1}$$

When $k > 0$ this equation models the propagation of unidirectional shallow water waves on a flat bottom, and $u(t, x)$ represents the fluid velocity at time t in the horizontal direction x [1,2]. The Camassa-Holm equation possesses a bi-Hamiltonian structure [1,3] and is completely integrable [1,4,5]. Moreover, when $k = 0$ it has an infinite number of solitary wave solutions, called peakons due to the discontinuity of their first derivatives at the wave peak, interacting like solitons:

$$u(x, t) = c\ \exp(-|x - ct|)\,. \tag{1.2}$$

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Liu and Qian [6] investigated the peakons of the following generalized Camassa-Holm equation

$$u_t + 2ku_x - u_{xxt} + au^m u_x = 2u_x u_{xx} + uu_{xxx} \,. \tag{1.3}$$

with $a > 0, k \in R, m \in N$ and the integral taken as zero. In the case of $m = 1, 2, 3$ and $k \neq 0$, they gave the explicit expressions for the peakons. The concept of compacton: soliton with compact support, or strict localization of solitary waves, appeared in the work of Rosenau and Hyman [7] where a genuinely nonlinear dispersive equation $K(n, n)$ defined by

$$u_t + a(u^n)_x + (u^n)_{xxx} = 0 \,, \tag{1.4}$$

was subjected to experimental and analytical studies. They found certain solitary wave solutions which vanish identically outside a finite core region. These solutions have been called compactons. Several studies have been conducted in [8]-[12]. The aim was to examine if other nonlinear dispersive equation may generate compacton structures.

In fact, When $a = 3$ and $m = 2$, the Eq. (1.3) has another kind of bounded travelling waves which possess some properties of kink waves. We call them generalized kink waves. Therefore, in this paper, we shall consider the compactons and generalized kink waves of the Eq. (1.3) when $a = 3$ and $m = 2$. In the conditions of $a = 3$ and $m = 2$, the Eq. (1.3) can be rewritten as:

$$u_t + 2ku_x - u_{xxt} + 3u^2 u_x = 2u_x u_{xx} + uu_{xxx} \,, \tag{1.5}$$

where the constant $k \in R$ is given.

The rest of this paper is organized as follows. In Section 2, we firstly derive travelling wave equation and travelling wave system. Then we study the bifurcations of phase portrait of the travelling wave system. In Section 3, using the information of phase portrait, we make the numerical simulation for bounded integral curves of travelling wave equation. In Section 4, we obtain the integral representations of compactons and the implicit or integral representations of the generalized kink waves from the bifurcations of phase portrait and the bounded integral curves. Finally, a short conclusion is given in Section 5.

2 Travelling Waves System and its Bifurcation Phase Portrait

In this section we derive travelling wave system and study its bifurcation phase portrait. Substituting $u(x, t) = \phi(\xi)$ with $\xi = x - ct$ into (1.5), we have

$$-c\phi' + 2k\phi' + c\phi''' + 3\phi^2\phi' = 2\phi'\phi'' + \phi\phi''' \,, \tag{2.1}$$

where c is the wave speed. Integrating it once gives

$$(\phi - c)\phi'' = \phi^3 + (2k - c)\phi - \frac{1}{2}(\phi')^2 , \tag{2.2}$$

where the integral constant is taken as 0. Letting $\phi' = y$, we obtain a planar system

$$\frac{d\phi}{d\xi} = y , \qquad \frac{dy}{d\xi} = \frac{\phi^3 + (2k - c)\phi - \frac{1}{2}y^2}{\phi - c} , \tag{2.3}$$

which is called travelling wave system. Our aim is to study the phase portrait of system (2.3). But system (2.3) has a singular line $\phi = c$ which is inconvenient to our study. So we make the transformation $d\xi = (\phi - c)d\tau$. Thus system (2.3) becomes Hamiltonian system

$$\frac{d\phi}{d\tau} = (\phi - c)y, \qquad \frac{dy}{d\tau} = \phi^3 + (2k - c)\phi - \frac{1}{2}y^2 , \tag{2.4}$$

Thus system (2.3) and (2.4) have the same first integral

$$H(\phi, y) = (\phi - c)y^2 - \frac{1}{2}\phi^4 + (c - 2k)\phi^2 = h . \tag{2.5}$$

Therefore both systems (2.3) and (2.4) have same topological phase portraits except the straight line $\phi = c$.

Now we consider the singular points of system (2.4) and their properties. Let

$$y^0_{\pm} = \pm\sqrt{2(c^2 - c + 2k)c} \quad \text{for} \quad 2(c^2 - c + 2k)c > 0 , \tag{2.6}$$

$$\phi^0_{\pm} = \pm\sqrt{c - 2k} \quad \text{for} \quad 2k < c , \tag{2.7}$$

$$\phi^*_{\pm} = \pm\sqrt{-c^2 + 2c - 4k} \quad \text{for} \quad -c^2 + 2c - 4k \geq 0 , \tag{2.8}$$

$$\phi^1_{\pm} = \pm\sqrt{2(c - 2k)} \quad \text{for} \quad 2k \leq c , \tag{2.9}$$

$$k_1(c) = \frac{c}{2} , \tag{2.10}$$

$$k_2(c) = \frac{2c - c^2}{4} \quad \text{for} \quad 0 < c , \tag{2.11}$$

$$k_3(c) = \frac{c - c^2}{2} , \tag{2.12}$$

Thus, the $k = k_i(c)$ $(i = 1, 2, 3)$ have a unique intersection point $(0, 0)$, and

$$k_3(c) < k_2(c) < k_1(c) \quad \text{for} \quad 0 < c , \tag{2.13}$$

and

$$k_3(c) < k_1(c) \quad \text{for} \quad 0 < c . \tag{2.14}$$

By the theory of planar dynamical system and (2.4)-(2.14), we derive the following proposition for the equilibrium points of the system (2.4):

Proposition 2.1 1). *When $c < 0$ and $k < k_3(c)$ or $0 < c$ and $k_3(c) < k$, the (c, y_-^0) and (c, y_+^0) are two singular points of the system (2.4). They are saddle points and $H(c, y_-^0) = H(c, y_+^0)$.*

2). *When $0 < c$ and $k_1(c) \leq k$, the system (2.4) has three singular points $(0,0), (c, y_-^0)$ and (c, y_+^0). The (0, 0) is a center point.*

3). *When $c = 0$ and $0 \leq k$, the system (2.4) has only one singular point $(0,0)$ and this point is a degenerate saddle point.*

4). *When $c < 0$ and $k_1(c) \leq k$, the system (2.4) has only one singular point $(0,0)$ and this point is a saddle point.*

5). *When $c < 0$ and $k_3(c) < k < k_1(c)$, the system (2.4) has three singular points $(0,0), (\phi_-^0, 0)$ and $(\phi_+^0, 0)$ and $c < \phi_-^0 < 0 < \phi_+^0$. The $(0,0)$ is a center point, $(\phi_-^0, 0)$ and $(\phi_+^0, 0)$ are saddle points and $H(\phi_-^0, 0) = H(\phi_+^0, 0)$.*

6). *When $c < 0$ and $k = k_3(c)$, the system (2.4) has three singular points $(0,0), (c, 0)$ and $(-c, 0)$. The $(0,0)$ is a center point, $(c, 0)$ is a degenerate saddle point, $(-c, 0)$ is a saddle point and $H(c, 0) = H(-c, 0)$.*

7). *When $c < 0$ and $k < k_3(c)$, the system (2.4) has five singular points $(0,0)$, $(\phi_-^0, 0)$, $(\phi_+^0, 0)$, (c, y_-^0) and (c, y_+^0), and $\phi_-^0 < c < 0 < \phi_+^0$. The $(0,0)$ and $(\phi_-^0, 0)$ are center points, $(\phi_+^0, 0)$ is a saddle point.*

8). *When $c = 0$ and $k < 0$, the system (2.4) has three singular points $(0,0), (\phi_-^0, 0)$ and $(\phi_+^0, 0)$, and $\phi_-^0 < 0 < \phi_+^0$. The $(0,0)$ is a degenerate saddle point, $(\phi_-^0, 0)$ is center point and $(\phi_+^0, 0)$ is a saddle point.*

9). *When $c > 0$ and $k < k_3(c)$, the system (2.4) has three singular points $(0,0), (\phi_-^0, 0)$ and $(\phi_+^0, 0)$ and $\phi_-^0 < 0 < c < \phi_+^0$. The $(0,0)$ and $(\phi_+^0, 0)$ are saddle points, $(\phi_-^0, 0)$ is center point.*

10). *When $c > 0$ and $k = k_3(c)$, the system (2.4) has three singular points $(0,0), (-c, 0)$ and $(c, 0)$. The $(0,0)$ is a saddle point, $(c, 0)$ is a degenerate saddle point and $(-c, 0)$ is a center point.*

11). *When $c > 0$ and $k_3(c) < k < k_2(c)$, the system (2.4) has five singular points $(0,0)$, $(\phi_-^0, 0)$, $(\phi_+^0, 0)$, (c, y_-^0) and (c, y_+^0), and $\phi_-^0 < 0 < \phi_+^0 < c$. The $(0,0)$ is a saddle point, $(\phi_-^0, 0)$ and $(\phi_+^0, 0)$ are center points.*

12). *When* $c > 0$ *and* $k = k_2(c)$, *the system* (2.4) *has five singular points* $(0,0)$, $(\phi_-^0, 0)$, $(\phi_+^0, 0)$, (c, y_-^0) *and* (c, y_+^0), *and* $\phi_-^0 < 0 < \phi_+^0 < c$. *The* $(0,0)$ *is a saddle point,* $(\phi_-^0, 0)$ *and* $(\phi_+^0, 0)$ *are center points, and* $H(0,0) = H(c, y_-^0) = H(0, y_+^0)$.

13). *When* $c > 0$ *and* $k_2(c) < k < k_1(c)$, *the system* (2.4) *has five singular points* $(0,0)$, $(\phi_-^0, 0)$, $(\phi_+^0, 0)$, (c, y_-^0) *and* (c, y_+^0), *and* $\phi_-^0 < 0 < \phi_+^0 < c$. *The* $(0,0)$ *is a saddle point,* $(\phi_-^0, 0)$ *and* $(\phi_+^0, 0)$ *are center points.*

Proof: It is easy to see that all of the singular points of (2.4) are only distributed on ϕ-axis or the line $\phi = c$. Firstly we consider system (2.4) on the line $\phi = c$. From (2.6), on the line $\phi = c$, (2.4) has two singular points (c, y_-^0) and (c, y_+^0) when $c < 0$ and $k < k_3(c)$ or $0 < c$ and $k_3(c) < k$, has one singular point $(c, 0)$ when $k = k_3(c)$, and has not singular point when $c < 0$ and $k > k_3(c)$ or $0 < c$ and $k_3(c) > k$. Assume that $\lambda(\phi, y)$ is an eigenvalue of the linearized system of (2.4) at point (ϕ, y). Then we have

$$\lambda^2(c, y_-^0) = \lambda^2(c, y_+^0) = 2c(c^2 - c + 2k) > 0\,, \tag{2.15}$$

for $c < 0$ and $k < k_3(c)$ or $0 < c$ and $k_3(c) < k$, and

$$\lambda^2(c, 0) = 0, \quad \text{for} \quad k = k_3(c)\,. \tag{2.16}$$

Now we consider system (2.4) on ϕ-axis. Let

$$f(\phi) = \phi^3 + (2k - c)\phi\,, \tag{2.17}$$

then the $(\phi_0, 0)$ is singular point of system (2.4) if and only if $f(\phi_0) = 0$. It is easy to see that we obtain the following facts:

(1^0) When $k_1(c) \le k$, the system (2.4) has one zero point $(0,0)$. Thus the $(0,0)$ is singular point of system (2.4) on ϕ-axis. From (2.7) and (2.17) we have $f'(0) > 0$ and $f'(0) = 0$ when $k_1(c) < k$ and $k = k_1(c)$ respectively.

(2^0) When $k_1(c) > k$, the system (2.4) has three zero points $(\phi_-^0, 0)$, $(0,0)$ and $(\phi_+^0, 0)$. Thus the $(\phi_-^0, 0)$, $(0,0)$ and $(\phi_+^0, 0)$ are singular points of system (2.4) on ϕ-axis. From (2.7) and (2.17) we have $f'(\phi_-^0) > 0$, $f'(0) < 0$ and $f'(\phi_+^0) > 0$.

On the other hand, we have

$$\lambda^2(\phi_-^0, c) = f'(\phi_-^0)(\phi_-^0 - c)\,, \tag{2.18}$$

$$\lambda^2(0, 0) = -cf'(0)\,, \tag{2.19}$$

$$\lambda^2(\phi_+^0, c) = f'(\phi_+^0)(\phi_+^0 - c)\,, \tag{2.20}$$

From (2.5) and (2.15) - (2.20) the proof is completed.

According to the above analysis, we draw the bifurcation phase portrait of (2.3) and (2.4), shown in Fig. 1.

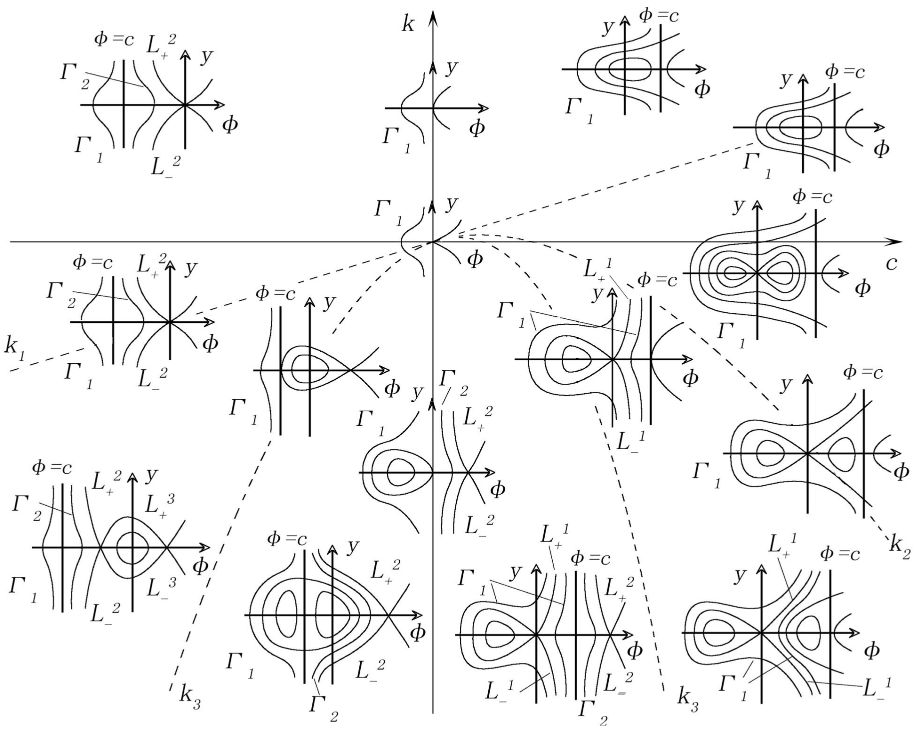


Fig. 1 The bifurcation phase portrait of systems (2.3) and (2.4)

3 Numerical Simulations of Bounded Integral Curves of Travelling Wave Equation

From the derivation in Sec. 2 we see that the bounded travelling waves of Eq. (1.5) correspond to the bounded integral curves of Eq. (2.2), and the bounded integral curves of Eq. (2.2) correspond to the orbits of systems (2.3) in which $\phi = \phi(\xi)$ is bounded. Therefore we can simulate the bounded integral curves of Eq. (2.2) by using the information of the phase portrait of systems (2.3).

From Fig. 1 it is seen that $\phi = \phi(\xi)$ is bounded in the following orbits of system (2.3):

(1). The homoclinic orbits, (2). The periodic orbits, (3). The orbits Γ_1 and Γ_2, (4). The heteroclinic orbits $L^1_\pm$, $L^2_\pm$and $L^3_\pm$.

When (i). $c > 0$ and $k < k_3(c)$, (ii). $c < 0$ and $k_3(c) < k < k_1(c)$, according to the above analysis we will simulate the bounded integral curves of Eq. (2.2) by using the mathematical software *Maple*. In the other case we can use a similar argument. We assume that $(\phi_0, 0)$ is the initial point of an orbit of system (2.3) in the following cases.

Case 1. $c > 0$ and $k < k_3(c)$. For this case, system (2.3) has an orbit Γ_1 on which ϕ is bounded when $\phi_0 < \phi_-^1$ or $0 < \phi_0 < c$, has a homoclinic orbit when $\phi_0 = \phi_-^1$, has a periodic orbit when $\phi_-^1 < \phi_0 < \phi_-^0$, two heteroclinic orbits $L_\pm^1$ on which ϕ are bounded when $\phi_0 = 0$, has an orbit Γ_2 on which is bounded when $c < \phi_0 < \phi_+^0$, and has two heteroclinic orbits $L_\pm^2$ which lie on the left side of the line $\phi = \phi_+^0$ on which ϕ are bounded when $\phi_0 = \phi_+^0$. For example, choosing $c = 2$ and $k = -4$, we have $\phi_-^1 = -4.472135955$ and $\phi_\pm^0 = \pm 3.16227766$.

(i). We respectively take $\phi_0 = -4.48, -4.472135955, -4.4721, 0.01, 3.1622$ and 3.1622776, letting $\phi(0) = \phi_0$ and $\phi'(0) = 0$, we simulate the integral curves of Eq. (2.2) as (a), (b), (c), (f) , (g) and (h) in Fig. 2.

(ii). The two heteroclinic orbits $L_\pm^1$ respectively have expressions

$$y_1^\pm(\phi) = \pm\sqrt{\frac{\phi^4 + 2(2k-c)\phi^2}{2(\phi - c)}}, \text{ for } 0 \le \phi < c\,. \tag{3.1}$$

If $0 < \phi_1^0 < c$, then from the first equation of system (2.3) we have $\frac{d\phi}{d\xi}|_{\xi=\xi_0} = y_1^\pm(\phi_1^0)$ at $\phi = \phi_1^0$. For example, when $c = 2$ and $k = -4$, taking $\phi_1^0 = 0.2$, we have $y_1^\pm(\phi_1^0) = \pm 0.4709328804$. Letting $\phi(0) = 0.2$ and $\phi'(0) = \pm 0.4709328804$, respectively we simulate the integral curves of Eq. (2.2) as (d) and (e) in Fig. 2.

(iii). The two heteroclinic orbits $L_\pm^2$ respectively have expressions

$$y_2^\pm(\phi_2^0) = \pm\sqrt{\frac{\phi^4 + 2(2k-c)\phi^2 + 2h(\phi_+^0)}{2(\phi - c)}}, \text{ for } c < \phi \le \phi_+^0\,, \tag{3.2}$$

$$h(\phi_+^0) = -\frac{1}{2}(\phi_+^0)^4 + (c - 2k)(\phi_+^0)^2\,. \tag{3.3}$$

If $c < \phi_2^0 \le \phi_+^0$, then from the first equation of system (2.3) we have $\frac{d\phi}{d\xi}|_{\xi=\xi_0} = y_2^\pm(\phi_2^0)$ at $\phi = \phi_2^0$. For example, when $c = 2$ and $k = -4$, taking $\phi_2^0 = 3$, we have $y_2^\pm(\phi_2^0) = \pm 0.7071067812$. Letting $\phi(0) = 3$ and $\phi'(0) = \pm 0.7071067812$, respectively we simulate the integral curves of Eq. (2.2) as (i) and (j) in Fig. 2.

Remark 1 Under the conditions of Case 1 the following facts can be seen from Fig.2:

(1) The integral curve is only defined on $[-\xi_0^1, \xi_0^1]$ or $[-\xi_0^2, \xi_0^2]$ and it is of peak form [see

(a), (f), (g) and (h) in Fig. 2] when $\phi_0 < \phi_-^1$ or $0 < \phi_0 < c$ or $c < \phi_0 < \phi_+^0$, where

$$\xi_0^1 = \int_{\phi_0}^{c} \sqrt{\frac{2(s-c)}{(s^2-\phi_0^2)(s^2-\alpha)}}ds, \quad \text{for} \quad \phi_0 < \phi_-^1 \quad \text{or} \quad 0 < \phi_0 < c\,, \tag{3.4}$$

$$\xi_0^2 = \int_{c}^{\phi_0} \sqrt{\frac{2(s-c)}{(s^2-\phi_0^2)(s^2-\alpha)}}ds, \quad \text{for} \quad c < \phi_0 < \phi_0^+\,, \tag{3.5}$$

$$\alpha = -\phi_0^2 - 4k + 2c\,. \tag{3.6}$$

The point $(0, \phi_0)$ is the peak of the integral curve $\phi = \phi(\xi)$ which tends to c when $|\xi|$ tends to ξ_0, where $\xi_0 = \xi_0^1$ or ξ_0^2. Following Rosenau and Hyman [7] we call a compacton. For example, when $c = 2$ and $k = -4$, we respectively take $\phi_0 = 0.01$ and 3.1622776, from (3.4) and (3.5) we obtain $\xi_0^1 = 2.417690442$ and $\xi_0^2 = 4.1086580$ which is identical with the simulation [see Figs. 2 (f) and (h)].

(2) When $\phi_0 = 0$, Eq. (2.2) has two bounded integral curves $\phi_1(\xi)$ and $\phi_2(\xi)$ [see Figs. 2 (d) and (e)]. $\phi_1(\xi)$ is defined on $(-\infty, \xi_1]$ and tends to c when ξ tends to ξ_1, to 0 when ξ tends to $-\infty$. $\phi_2(\xi)$ is defined on $[-\xi_1, +\infty)$ and tends to 0 when ξ tends to $+\infty$, to c when ξ tends to $-\xi_1$, where

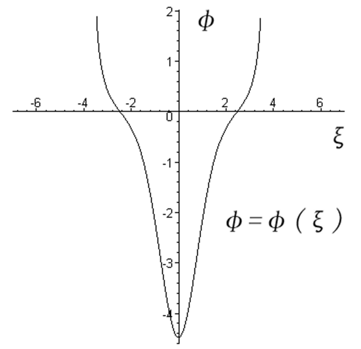
$$\xi_1 = \int_{\phi_1^0}^{c} \frac{1}{s}\sqrt{\frac{2(s-c)}{s^2-2(c-2k)}}ds, \quad \text{for} \quad 0 < \phi_1^0 < c\,. \tag{3.7}$$

For example, for the above $c = 2, k = -4$, taking $\phi_1^0 = 0.2$, from (3.7) we obtain $\xi_1 = 0.7904027914$ which is identical with the simulation [see Figs. 2 (d) and (e)].

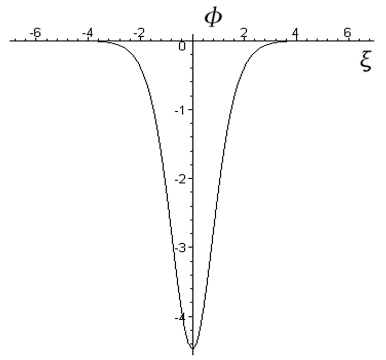
(3) When $\phi_0 = \phi_+^0$, Eq. (2.2) has two bounded integral curves $\phi_3(\xi)$ and $\phi_4(\xi)$ [see Figs. 2 (i) and (j)]. $\phi_3(\xi)$ is defined on $[-\xi_2, +\infty)$ and tends to c when ξ tends to $-\xi_2$, to ϕ_+^0 when ξ tends to $+\infty$. $\phi_4(\xi)$ is defined on $(-\infty, \xi_2]$ and tends to ϕ_+^0 when ξ tends to $-\infty$, to c when ξ tends to ξ_2, where

$$\xi_2 = \int_{c}^{\phi_2^0} \frac{\sqrt{2(s-c)}}{(\phi_+^0 - s)(\phi_+^0 + s)}ds, \quad \text{for} \quad c < \phi_2^0 < \phi_+^0\,. \tag{3.8}$$

For example, for the above $c = 2, k = -4$, taking $\phi_2^0 = 3$, from (3.8) we obtain $\xi_2 = 0.3697389765$ which is identical with the simulation [see Figs. 2 (i) and (j)].


ϕ
ξ
$\phi = \phi\ (\ \xi\ )$

(a)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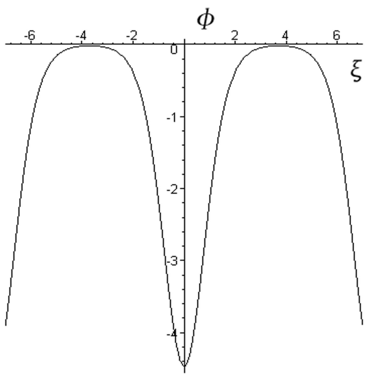
(b)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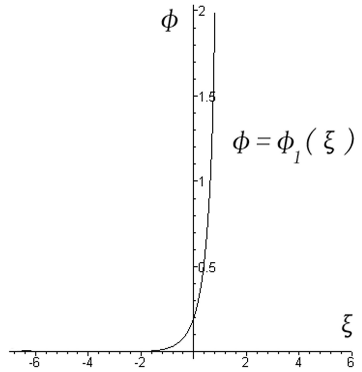
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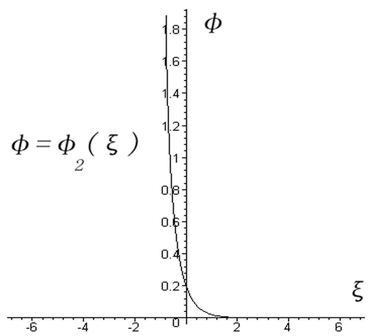
(d)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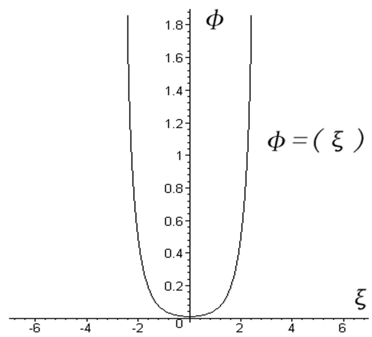
(e)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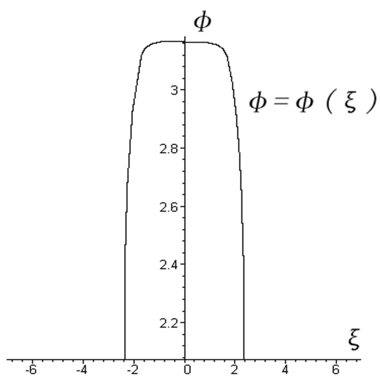
(f)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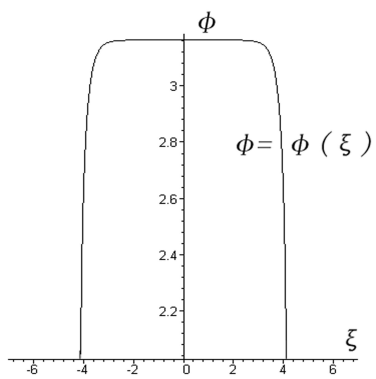
(g)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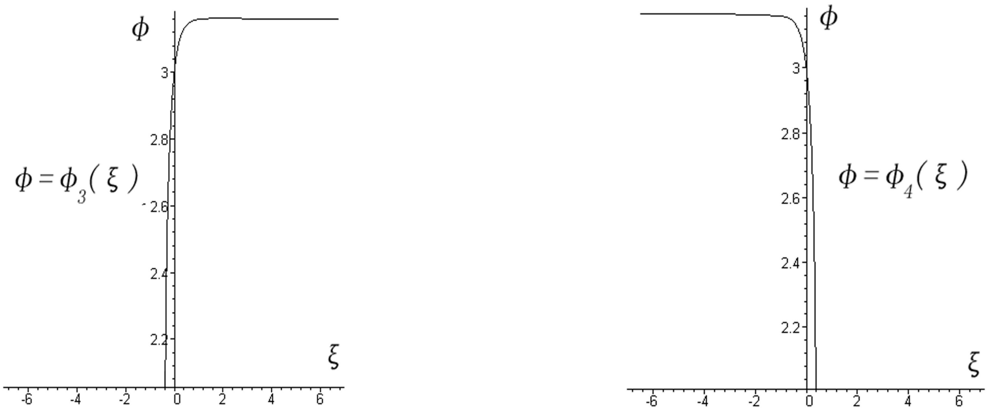
(h)



(i) (j)

Fig. 2 The simulation of the integral curves of Eq. (2.2) when $c = 2$ and $k = -4$.

(a) $\phi(0) = -4.48$ and $\phi(0)' = 0$, (b) $\phi(0) = -4.472135955$ and $\phi(0)' = 0$, (c) $\phi(0) = -4.4721$ and $\phi(0)' = 0$, (d) $\phi(0) = 0.2$ and $\phi(0)' = 0.4709328804$, (e) $\phi(0) = 0.2$ and $\phi(0)' = -0.4709328804$, (f) $\phi(0) = 0.01$ and $\phi(0)' = 0$, (g) $\phi(0) = 3.1622$ and $\phi(0)' = 0$, (h) $\phi(0) = 3.1622776$ and $\phi(0)' = 0$, (i) $\phi(0) = 3$ and $\phi(0)' = 0.7071067812$, (j) $\phi(0) = 3$ and $\phi(0)' = -0.7071067812$.

Case 2. $c < 0$ and $k_3(c) < k < k_1(c)$. For this case, system (2.3) has an orbit Γ_1 on which ϕ is bounded when $\phi_0 < c$, has an orbit Γ_2 on which ϕ is bounded when $c < \phi_0 < \phi_-^0$ and four heteroclinic orbits $L_\pm^2$ and $L_\pm^3$ are bounded when $\phi_0 = \phi_-^0$, has a periodic orbit when $\phi_-^0 < \phi_0 < 0$. For example, choosing $c = -2$ and $k = -2$, we have $\phi_\pm^0 = \pm 1.414213562$.

(i) We respectively take $\phi_0 = -1.4$ and -1.4133, letting $\phi(0) = \phi_0$ and $\phi'(0) = 0$, the simulation integral curves of Eq. (2.2) are (a) and (b) in Fig. 3.

(ii) The two heteroclinic orbits $L_\pm^3$ respectively have expressions

$$y_3^\pm = \pm\sqrt{\frac{\phi^4 + 2(2k-c)\phi^2 + 2h(\phi_-^0)}{2(\phi - c)}}, \text{ for } \phi_-^0 \le \phi \le \phi_+^0\,, \tag{3.9}$$

where

$$h(\phi_-^0) = -\frac{1}{2}(\phi_-^0)^4 + (c - 2k)(\phi_-^0)^2\,. \tag{3.10}$$

If $\phi_-^0 \le \phi_3^0 \le \phi_+^0$, then from the first equation of system (2.3) we have $\frac{d\phi}{d\xi}|_{\xi=\xi_0} = y_3^\pm(\phi_3^0)$ at $\phi = \phi_3^0$. For example, taking $\phi_3^0 = 0$, we have $y_3^\pm(\phi_3^0) = \pm 1$. Letting $\phi(0) = 0$ and $\phi'(0) = \pm 1$ respectively, we simulate the integral curves of Eq. (2.2) as (c) and (d) in Fig. 3.

(iii) When $\phi_0 = \phi_-^0$, $L_\pm^2$ lie on the left side of the line $\phi = \phi_-^0$, the simulation integral curve of Eq. (2.2) is similar to Figs. 2 (i) - (j), when $\phi_0 < c$, to Fig. 2 (a) or (f), when $c < \phi_0 < \phi_-^0$, to Fig. 2 (g) or (h).

Remark 2 The simulation in Fig. 3 imply that under of case 2, the integral curve $\phi = \phi_5(\xi)$ and $\phi = \phi_6(\xi)$ are defined on $(-\infty, +\infty)$, $\phi_5(\xi)$ tends to ϕ_-^0 when ξ tends to $-\infty$ or tends to ϕ_+^0 when ξ tends to $+\infty$ and $\phi_6(\xi)$ tends to ϕ_+^0 when ξ tends to $-\infty$ or tends to ϕ_-^0 when ξ tends to $+\infty$.

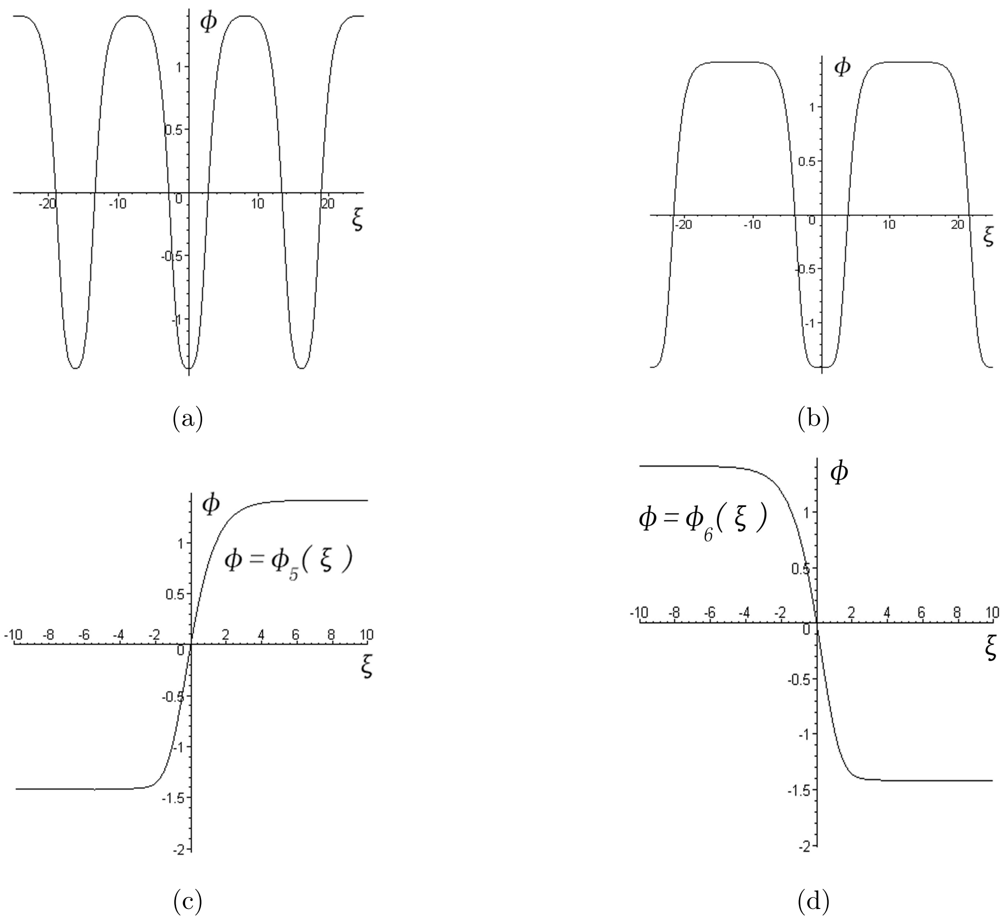


Fig. 3 The simulation of the integral curves of Eq. (2.2) when $c = -2$ and $k = -2$.

(a) $\phi(0) = -1.4$ and $\phi'(0) = 0$, (b) $\phi(0) = -1.4133$ and $\phi'(0) = 0$, (c) $\phi^{(}0) = 0$ and $\phi'(0) = 1$, (d) $\phi^{(}0) = 0$ and $\phi'(0) = -1$.

4 The Expressions of Compactons and Generalized Kink Waves

In this section, we derive the expressions of compactons and generalized kink waves by using the information obtained from above sections.

4.1 Integral Expressions of Compactons

For given c and k, we give hypotheses as follows:

(H1) $c < 0$, $k < k_3(c)$ and ϕ_0 satisfies $\phi_0 < \phi_-^*$.

(H2) $c = 0$, $k < 0$ and ϕ_0 satisfies $\phi_0 < \phi_-^1$.

(H3) $c > 0$, $k \le k_3(c)$ and ϕ_0 satisfies $\phi_0 < \phi_-^1$ *or* $0 < \phi_0 < c$.

(H4) $c > 0$, $k_3(c) < k < k_2(c)$ and ϕ_0 satisfies $\phi_0 < \phi_-^1$ *or* $0 < \phi_0 < \phi_+^*$.

(H5) $c \ge 0$, $k_2(c) \le k$ and ϕ_0 satisfies $\phi_0 < -c$.

(H6) $c < 0$, $k \ge k_3(c)$ and ϕ_0 satisfies $\phi_0 < c$.

(H7) $c < 0$, $k < k_3(c)$ and ϕ_0 satisfies $-c < \phi_0 < \phi_+^0$.

(H8) $c \ge 0$, $k < k_3(c)$ and ϕ_0 satisfies $c < \phi_0 < \phi_+^0$.

(H9) $c < 0$, $k \ge k_1(c)$ and ϕ_0 satisfies $c < \phi_0 < 0$.

(H10) $c < 0$, $k_3(c) < k < k_1(c)$ and ϕ_0 satisfies $c < \phi_0 < \phi_-^0$.

Proposition 4.1 (i) *If one of hypotheses (H1) – (H6) holds, then Eq. (1.5) has a concave compacton $u = \phi(\xi)$ which satisfies integral equation*

$$\xi_0 - |\xi| = \int_\phi^c \sqrt{\frac{2(s-c)}{(s^2-\phi_0^2)(s^2-\alpha)}}ds, \ \ for\ |\xi| \le \xi_0\,, \tag{4.1}$$

where

$$\xi_0 = \int_{\phi_0}^c \sqrt{\frac{2(s-c)}{(s^2-\phi_0^2)(s^2-\alpha)}}ds\,. \tag{4.2}$$

(ii) *If one of hypotheses (H7) – (H10) holds, then Eq. (1.5) has a convex compacton $u = \phi(\xi)$ which satisfies integral equation*

$$\xi_0 - |\xi| = \int_c^\phi \sqrt{\frac{2(s-c)}{(s^2-\phi_0^2)(s^2-\alpha)}}ds, \ \ for\ |\xi| \le \xi_0\,, \tag{4.3}$$

where

$$\xi_0 = \int_c^{\phi_0} \sqrt{\frac{2(s-c)}{(s^2-\phi_0^2)(s^2-\alpha)}}ds\,. \tag{4.4}$$

Proof: From Fig. 1 it is seen that the unique orbit Γ_1 or Γ_2 of system (2.3) passes the point $(\phi_0, 0)$ when one of above hypotheses holds. From (2.5) the Γ_1 and Γ_2 have expression

$$2(\phi - c)y^2(\phi) = (\phi^2 - \phi_0^2)(\phi^2 - \alpha)\,. \tag{4.5}$$

Substituting $y = \frac{d\phi}{d\xi}$ into (4.5), we have

$$\pm\sqrt{\frac{2(\phi - c)}{(\phi^2 - \phi_0^2)(\phi^2 - \alpha)}}d\phi = d\xi\,. \tag{4.6}$$

Thus along Γ_1 and Γ_2 respectively integrate (4.6), the (4.1) and (4.3) are obtained.

4.2 Implicit or Integral Expressions of Generalized Kink Waves

For given c and k, we give hypotheses as follows:

(H11) $c > 0, k < k_2(c)$ and ϕ_1^0 satisfies $0 < \phi_1^0 < c < \phi_+^0$.

(H12) $k < k_3(c)$ and ϕ_2^0 satisfies $\phi_-^0 < c < \phi_2^0 < \phi_+^0$.

(H13) $c < 0, k \geq k_1(c)$ and ϕ_2^0 satisfies $c < \phi_2^0 < 0$.

(H14) $c < 0, k_3(c) < k < k_1(c)$ and ϕ_2^0 satisfies $c < \phi_2^0 < \phi_-^0$.

(H15) $c < 0, k_3(c) < k < k_1(c)$ and ϕ_3^0 satisfies $c < \phi_-^0 < \phi_3^0 < \phi_+^0$.

Proposition 4.2 (i) *If hypothesis (H11) holds, then Eq. (1.5) has two generalized kink waves $u = \phi_1(\xi)$ and $u = \phi_2(\xi)$ which satisfy integral equation*

$$\int_{\phi_1^0}^{\phi_1} \frac{1}{s}\sqrt{\frac{2(s-c)}{s^2 + 2(2k - c)}}ds = \xi, \; \textit{for} \; -\infty < \xi < \xi_1 \tag{4.7}$$

and

$$\int_{\phi_1^0}^{\phi_2} -\frac{1}{s}\sqrt{\frac{2(s-c)}{s^2 + 2(2k - c)}}ds = \xi, \; \textit{for} \; -\xi_1 < \xi < +\infty \tag{4.8}$$

where

$$\xi_1 = \int_{c}^{\phi_1^0} \frac{1}{s}\sqrt{\frac{2(s-c)}{s^2 + 2(2k - c)}}ds\,. \tag{4.9}$$

(ii) *If hypothesis (H12) holds, then Eq. (1.5) has two generalized kink waves* $u = \phi_3(\xi)$ *and* $u = \phi_4(\xi)$ *which respectively satisfy equation*

$$\frac{\sqrt{\phi_+^0 - c}}{2}\ln\left(\frac{\sqrt{\phi_+^0 - c} + \sqrt{\phi_3 - c}}{\sqrt{\phi_+^0 - c} - \sqrt{\phi_3 - c}}\right) - \sqrt{\phi_+^0 + c}\,\arctan\frac{\sqrt{\phi_3 - c}}{\sqrt{\phi_+^0 + c}} = \frac{\phi_+^0}{\sqrt{2}}(\xi + \xi_2)\,, \tag{4.10}$$

for $-\xi_2 < \xi < +\infty$*. And*

$$\frac{\sqrt{\phi_+^0 - c}}{2}\ln\left(\frac{\sqrt{\phi_+^0 - c} + \sqrt{\phi_4 - c}}{\sqrt{\phi_+^0 - c} - \sqrt{\phi_4 - c}}\right) - \sqrt{\phi_+^0 + c}\,\arctan\frac{\sqrt{\phi_4 - c}}{\sqrt{\phi_+^0 + c}} = \frac{\phi_+^0}{\sqrt{2}}(-\xi + \xi_2)\,, \tag{4.11}$$

for $-\infty < \xi < \xi_2$*. Where*

$$\xi_2 = \frac{\sqrt{2}}{\phi_+^0}\left[\frac{\sqrt{\phi_+^0 - c}}{2}\ln\left(\frac{\sqrt{\phi_+^0 - c} + \sqrt{\phi_2^0 - c}}{\sqrt{\phi_+^0 - c} - \sqrt{\phi_2^0 - c}}\right) - \sqrt{\phi_+^0 + c}\,\arctan\frac{\sqrt{\phi_2^0 - c}}{\sqrt{\phi_+^0 + c}}\right]. \tag{4.12}$$

(iii) *If hypothesis (H13) holds, then Eq. (1.5) has two generalized kink waves* $u = \phi_3(\xi)$ *and* $u = \phi_4(\xi)$ *which respectively satisfy integral equation*

$$\int_{\phi_2^0}^{\phi_3} -\frac{1}{s}\sqrt{\frac{2(s-c)}{s^2 + 2(2k - c)}}ds = \xi, \ \textit{for} \ -\xi_2 < \xi < +\infty \tag{4.13}$$

and

$$\int_{\phi_2^0}^{\phi_4} \frac{1}{s}\sqrt{\frac{2(s-c)}{s^2 + 2(2k - c)}}ds = \xi, \ \textit{for} \ -\infty < \xi < \xi_2 \tag{4.14}$$

where

$$\xi_2 = \int_c^{\phi_2^0} -\frac{1}{s}\sqrt{\frac{2(s-c)}{s^2 + 2(2k - c)}}ds\,. \tag{4.15}$$

(iv) *If hypothesis (H14) holds, then Eq. (1.5) has two generalized kink waves* $u = \phi_3(\xi)$ *and* $u = \phi_4(\xi)$ *which satisfy equation*

$$\left(\frac{\sqrt{\phi_-^0 - c} - \sqrt{\phi_3 - c}}{\sqrt{\phi_-^0 - c} + \sqrt{\phi_3 - c}}\right)^{\sqrt{\phi_-^0 - c}}\left(\frac{\sqrt{-\phi_-^0 - c} + \sqrt{\phi_3 - c}}{\sqrt{-\phi_-^0 - c} - \sqrt{\phi_3 - c}}\right)^{\sqrt{-\phi_-^0 - c}} = \beta_1 e^{(\sqrt{2}\phi_-^0\xi)}\,, \tag{4.16}$$

for $-\xi_2 < \xi < +\infty$, *and*

$$\left(\frac{\sqrt{\phi_-^0 - c} - \sqrt{\phi_4 - c}}{\sqrt{\phi_-^0 - c} + \sqrt{\phi_4 - c}}\right)^{\sqrt{\phi_-^0 - c}} \left(\frac{\sqrt{-\phi_-^0 - c} + \sqrt{\phi_4 - c}}{\sqrt{-\phi_-^0 - c} - \sqrt{\phi_4 - c}}\right)^{\sqrt{-\phi_-^0 - c}} = \beta_1 e^{-\sqrt{2}\phi_-^0 \xi}, \tag{4.17}$$

for $-\infty < \xi < \xi_2$, *where*

$$\beta_1 = \left(\frac{\sqrt{\phi_-^0 - c} - \sqrt{\phi_2^0 - c}}{\sqrt{\phi_-^0 - c} + \sqrt{\phi_2^0 - c}}\right)^{\sqrt{\phi_-^0 - c}} \left(\frac{\sqrt{-\phi_-^0 - c} + \sqrt{\phi_2^0 - c}}{\sqrt{-\phi_-^0 - c} - \sqrt{\phi_2^0 - c}}\right)^{\sqrt{-\phi_-^0 - c}}, \tag{4.18}$$

and

$$\xi_2 = \ln \beta_1 . \tag{4.19}$$

(v) *If hypotheses* (H15) *holds, then Eq.* (1.5) *has two generalized kink waves* $u = \phi_5(\xi)$ *and* $u = \phi_6(\xi)$ *which satisfies equation*

$$\left(\frac{\sqrt{\phi_+^0 - c} + \sqrt{\phi_5 - c}}{\sqrt{\phi_+^0 - c} - \sqrt{\phi_5 - c}}\right)^{\sqrt{\phi_+^0 - c}} \left(\frac{\sqrt{\phi_5 - c} - \sqrt{-\phi_+^0 - c}}{\sqrt{\phi_5 - c} + \sqrt{-\phi_+^0 - c}}\right)^{\sqrt{-\phi_+^0 - c}} = \beta_2 e^{(\sqrt{2}\phi_+^0 \xi)}, \tag{4.20}$$

for $-\infty < \xi < +\infty$ *and*

$$\left(\frac{\sqrt{\phi_+^0 - c} + \sqrt{\phi_6 - c}}{\sqrt{\phi_+^0 - c} - \sqrt{\phi_6 - c}}\right)^{\sqrt{\phi_+^0 - c}} \left(\frac{\sqrt{\phi_6 - c} - \sqrt{-\phi_+^0 - c}}{\sqrt{\phi_6 - c} + \sqrt{-\phi_+^0 - c}}\right)^{\sqrt{-\phi_+^0 - c}} = \beta_2 e^{-(\sqrt{2}\phi_+^0 \xi)}, \tag{4.21}$$

for $-\infty < \xi < +\infty$. *Where*

$$\beta_2 = \left(\frac{\sqrt{\phi_+^0 - c} + \sqrt{\phi_3^0 - c}}{\sqrt{\phi_+^0 - c} - \sqrt{\phi_3^0 - c}}\right)^{\sqrt{\phi_+^0 - c}} \left(\frac{\sqrt{\phi_3^0 - c} - \sqrt{-\phi_+^0 - c}}{\sqrt{\phi_3^0 - c} + \sqrt{-\phi_+^0 - c}}\right)^{\sqrt{-\phi_+^0 - c}}. \tag{4.22}$$

Proof: Here we only proof (ii), in the other cases one can use a similar arguments. If hypotheses (H12) holds, then there are two heteroclinic orbits L_+^2 and L_-^2 of system (2.3) passes the point $(\phi_+^0, 0)$, From (2.5) they have expressions

$$2(\phi - c)y^2(\phi) = [(\phi_+^0)^2 - \phi^2]^2, \text{ for } \phi_-^0 < c < \phi < \phi_+^0 . \tag{4.23}$$

Substituting $y = \frac{d\phi}{d\xi}$ into (4.23) and letting $\phi(0) = \phi_2^0$, we have

$$\frac{\sqrt{2(\phi - c)}}{(\phi_+^0)^2 - \phi^2} d\phi = d\xi, \ -\xi_2 < \xi < +\infty \text{ and } \phi_-^0 < c < \phi < \phi_+^0 , \tag{4.24}$$

and

$$-\frac{\sqrt{2(\phi-c)}}{(\phi_+^0)^2-\phi^2}d\phi = d\xi,\ -\infty < \xi < \xi_2 \text{ and } \phi_-^0 < c < \phi < \phi_+^0\,. \tag{4.25}$$

Integrating (4.24) and (4.25) along L_+^2 and L_-^2 respectively, we have

$$\int_{\phi_2^0}^{\phi_3} \frac{\sqrt{2(s-c)}}{(\phi_+^0)^2-s^2}ds = \int_0^{\xi} ds,\ -\xi_2 < \xi < +\infty \text{ and } \phi_-^0 < c < \phi < \phi_+^0\,, \tag{4.26}$$

and

$$-\int_{\phi_2^0}^{\phi_4} \frac{\sqrt{2(s-c)}}{(\phi_+^0)^2-s^2}ds = \int_0^{\xi} ds,\ -\infty < \xi < \xi_2 \text{ and } \phi_-^0 < c < \phi < \phi_+^0\,. \tag{4.27}$$

From (4.26) and (4.27) we obtain (4.10) and (4.11).

5 Conclusion

In this paper, we have employed both the bifurcation method of planar dynamical systems and numerical simulation method of differential equations to investigate the bounded traveling waves of a generalized Camassa-Holm equation. We have found another kind of bounded traveling waves which have the properties of compactons or generalized kink waves. Their planar graphs are simulated (see Figs. 2 (a), (f), (g) and (h) for compactons; Figs. 2 (d), (e), (i) and (j) and Figs. 3 (c) and (d) for generalized kink waves). Their integral or implicit representations are obtained (see Proposition 4.1 for compactons; Proposition 4.2 for generalized kink waves).

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