Rostock. Math. Kolloq. **61**, 95–105 (2006)

Subject Classification (AMS)
Primary 05C38, 15A15;
Secondary 05A15, 15A18

Arif Rafiq

Convergence of an iterative scheme due to Agarwal et al.

ABSTRACT. In this paper, we are concerned with the study of an iterative scheme with errors due to Agarwal et al [1] associated with two mappings satisfying certain condition. We approximate the common fixed points of these two mappings by weak and strong convergence of the scheme in a uniformly convex Banach space under a certain condition.

KEY WORDS AND PHRASES. Iterative Scheme with Errors, Common Fixed Point, Condition (AU-N), Condition (AR), Weak and Strong Convergence

1 Introduction

Let C be a nonempty convex subset of a normed space E and $S, T : C \to C$ be two mappings. Xu [15] introduced the following iterative schemes known as Mann iterative scheme with errors and Ishikawa iterative scheme with errors:

(1) The sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 \in C, \\ x_{n+1} = a_n x_n + b_n T x_n + c_n u_n, \ n \geq 1, \end{cases} \tag{1.1}$$

where $\{a_n\}, \{b_n\}, \{c_n\}$ are sequences in $[0,1]$ such that $a_n + b_n + c_n = 1$ and $\{u_n\}$ is a bounded sequence in C, is known as Mann iterative scheme with errors. This scheme reduces to Mann iterative scheme [8] if $c_n = 0$, i.e.,

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1 - b_n) x_n + b_n T x_n, \ n \geq 1, \end{cases} \tag{M}$$

where $\{b_n\}$ is a sequence in $[0,1]$.

(2) The sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 \in C, \\ x_{n+1} = a_n x_n + b_n T y_n + c_n u_n, \\ y_n = a'_n x_n + b'_n T x_n + c'_n v_n, n \geq 1, \end{cases} \tag{1.2}$$

where $\{a_n\}, \{b_n\}, \{c_n\}\{a'_n\}, \{b'_n\}, \{c'_n\}$ are sequences in $[0,1]$ satisfying $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C, is known as Ishikawa iterative scheme with errors. This scheme becomes Ishikawa iterative scheme [5] if $c_n = 0 = c'_n$, i.e.,

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1 - b_n) x_n + b_n T y_n, \\ y_n = (1 - b'_n) x_n + b'_n T x_n, \ n \geq 1, \end{cases} \tag{I}$$

where $\{b_n\}, \{b'_n\}$ are sequences in $[0,1]$.

A generalization of Mann and Ishikawa iterative schemes [5, 8] was given by Das and Debata [4] and Takahashi and Tamura [13]. This scheme dealt with two mappings:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1 - b_n)\, x_n + b_n S y_n \\ y_n = (1 - b_n)\, x_n + b_n T x_n, \ n \geq 1. \end{cases} \tag{1.3}$$

In [1] Agarwal et al introduced the following scheme for quasi-contractive mappings as follows.

(3) The sequence $\{x_n\}$, in this case, is defined by

$$\begin{cases} x_1 \in C, \\ x_{n+1} = a_n x_n + b_n S y_n + c_n u_n, \\ y_n = a'_n x_n + b'_n T x_n + c'_n v_n, \ n \geq 1, \end{cases} \tag{1.4}$$

where $\{a_n\}, \{b_n\}, \{c_n\}\{a'_n\}, \{b'_n\}, \{c'_n\}$ are sequences in $[0,1]$ with $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C.

A Banach space E is said to satisfy Opial's condition [9] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that $\|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$.

A mapping $T : C \to E$ is called demiclosed with respect to $y \in E$ if for each sequence $\{x_n\}$ in C and each $x \in E$, $x_n \rightharpoonup x$ and $T x_n \to y$ imply that $x \in C$ and $Tx = y$.

Next we state the following useful lemmas.

Lemma 1 [11] *Suppose that E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all positive integers n. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2 [14] *Let $\{s_n\}, \{t_n\}$ be two nonnegative sequences satisfying*

$$s_{n+1} \leq s_n + t_n \quad \text{for all } n \geq 1.$$

If $\sum_{n=1}^{\infty} t_n < \infty$ then $\lim_{n\to\infty} s_n$ exists.

Lemma 3 [2] *Let E be a uniformly convex Banach space satisfying Opial's condition and let C be a nonempty closed convex subset of E. Let T be a nonexpansive mapping of C into itself. Then $I - T$ is demiclosed with respect to zero.*

Nonexpansive mappings since their introduction have been extensively studied by many authors in different frames of work. One is the convergence of iteration schemes constructed through nonexpansive mappings.

Recently Khan et al presented the following results in [6].

Definition 1 *Two mappings $S, T : C \to C$ where C a subset of E, are said to satisfy condition (A') if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that $\frac{1}{2}(\|x - Tx\| + \|x - Sx\|) \geq f(d(x,F))$ for all $x \in C$ where $d(x,F) = \inf\{\|x - x^*\| : x^* \in F = F(S) \cap F(T)\}$.*

Lemma 4 *Let E be a normed space and C its nonempty bounded convex subset. Let $S, T : C \to C$ be nonexpansive mappings. Let $\{x_n\}$ be the sequence as defined in (1.4) with $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} c'_n < \infty$. If $F(S) \cap F(T) \neq \phi$, then $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F(S) \cap F(T)$.*

Lemma 5 *Let E be a uniformly convex Banach space and C its nonempty bounded closed convex subset. Let $S, T : C \to C$ be nonexpansive mappings and $\{x_n\}$ be the sequence as defined in (1.4) with $0 < \delta \leq b_n,\ b'_n \leq 1 - \delta < 1$, $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} c'_n < \infty$. If $F(S) \cap F(T) \neq \phi$, then $\lim_{n\to\infty} \|Sx_n - x_n\| = 0 = \lim_{n\to\infty} \|Tx_n - x_n\|$.*

Lemma 6 *Let E be a uniformly convex Banach space satisfying the Opial's condition and C, S, T and $\{x_n\}$ be as taken in Lemma 5. If $F(S) \cap F(T) \neq \phi$, then $\{x_n\}$ converges weakly to a common fixed point of S and T.*

Lemma 7 *Let E be a uniformly convex Banach space and C, $\{x_n\}$ be as taken in Lemma 5. Let $S, T : C \to C$ be two nonexpansive mappings satisfying condition (A'). If $F(S) \cap F(T) \neq \phi$, then $\{x_n\}$ converges strongly to a common fixed point of S and T.*

The following observations about the results of Khan et al [6] have been made.

1: In [6] the authors claimed that, the iterative scheme (1.4) is new. Infect it is studied by Agarwal et al in [1].

2: Unfortunately, just as in [10], one cannot directly deduce the Mann type convergence theorems for one mapping due to the condition $0 < \delta \leq b'_n \leq 1 - \delta < 1$ (similar to the condition $1 - \beta_n < 1 - \epsilon$, $\epsilon > 0$ in [10]).

3: To say that S and T are nonexpansive (separately) is meaningless, the classical definition of two nonexpansive mappings is stated as follows: Two mappings $S, T : C \to C$ are said to be nonexpansive, if

$$\|Sx - Ty\| \leq \|x - y\|, \tag{AU-N}$$

for all $x, y \in C$. For $S = T$, we get the usual definition of nonexpansive mappings.

4: In [6] the authors stated that, $\{u_n\}, \{v_n\}$ are bounded sequences in C, while they are taking C as bounded. Hence $\{u_n\}, \{v_n\}$ should be arbitrary sequences in C (just as in [3]).

In this paper, we study the iterative scheme given in (1.4) for weak and strong convergence for two mappings satisfying (AU-N) in a uniformly convex Banach space. In order to prove our results, we do not need C to be bounded. We also remove the condition $0 < \delta \leq b'_n \leq 1 - \delta < 1$. Similar results for usual Ishikawa iterations for one mapping can be obtained, and consequently results including of Schu [11] can be recapture.

2 Main Results

In this section, we shall prove the weak and strong convergence of the iteration scheme (1.4) to a common fixed point of two mappings S and T satisfying (AU-N). Let $F(T)$ denote the set of all fixed points of T. We first prove the following lemmas.

Lemma 8 *Let E be a normed space and C its nonempty convex subset. Let $S, T : C \to C$ be two mappings satisfying (AU-N). Let $\{x_n\}$ be the sequence as defined in (1.4) with $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} c'_n < \infty$. If $F(S) \cap F(T) \neq \phi$, then $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F(S) \cap F(T)$.*

Proof: Assume that

$$M = \max\{\sup_{n\geq 1} \|u_n - x^*\|, \sup_{n\geq 1} \|v_n - x^*\|\},$$

and $F(S)\cap F(T)\neq\phi$. Let $x^*\in F(S)\cap F(T)$. Then

$$\begin{aligned}\|x_{n+1}-x^*\| &= \|a_nx_n+b_nSy_n+c_nu_n-x^*\|\\ &= \|a_n(x_n-x^*)+b_n(Sy_n-x^*)+c_n(u_n-x^*)\|\\ &\leq a_n\|x_n-x^*\|+b_n\|Sy_n-x^*\|+c_n\|u_n-x^*\|\\ &\leq (1-b_n)\|x_n-x^*\|+b_n\|Sy_n-Tx^*\|+Mc_n\\ &\leq (1-b_n)\|x_n-x^*\|+b_n\|y_n-x^*\|+Mc_n. \end{aligned} \tag{2.1}$$

$$\begin{aligned}\|y_n-x^*\| &= \|a_n'x_n+b_n'Tx_n+c_n'v_n-x^*\|\\ &= \|a_n'(x_n-x^*)+b_n'(Tx_n-x^*)+c_n'(v_n-x^*)\|\\ &\leq a_n'\|x_n-x^*\|+b_n'\|Tx_n-x^*\|+c_n'\|v_n-x^*\|\\ &\leq (1-b_n')\|x_n-x^*\|+b_n'\|Tx_n-Sx^*\|+Mc_n'\\ &\leq (1-b_n')\|x_n-x^*\|+b_n'\|x_n-x^*\|+Mc_n'\\ &= \|x_n-x^*\|+Mc_n'. \end{aligned} \tag{2.2}$$

Substituting (2.2) in (2.1) yields

$$\|x_{n+1}-x^*\|\leq\|x_n-x^*\|+M(b_nc_n'+c_n).$$

Using Lemma 2, $\lim_{n\to\infty}\|x_n-x^*\|$ exists for each $x^*\in F(S)\cap F(T)$, and the sequence $\{x_n\}$ is bounded. □

Lemma 9 *Let E be a uniformly convex Banach space and C its nonempty closed convex subset. Let $S,T: C\to C$ be two mappings satisfying (AU-N) and $\{x_n\}$ be the sequence as defined in (1.4) with $\{b_n\}\subset[\delta,1-\delta]$ for some $\delta\in(0,1)$, $\limsup_{n\to\infty} b_n'<1$, $\sum_{n=1}^{\infty}c_n<\infty$ and $\sum_{n=1}^{\infty}c_n'<\infty$. If $F(S)\cap F(T)\neq\phi$, then $\lim_{n\to\infty}\|Sx_n-x_n\|=0=\lim_{n\to\infty}\|Tx_n-x_n\|$.*

Proof: Assume that

$$M_1=\max\{\sup_{n\geq1}\|x_n-x^*\|,\sup_{n\geq1}\|u_n-x^*\|,\sup_{n\geq1}\|v_n-x^*\|\}.$$

By Lemma 8, $\lim_{n\to\infty}\|x_n-x^*\|$ exists. Suppose $\lim_{n\to\infty}\|x_n-x^*\|=c$ for some $c\geq0$.

Taking limsup on both the sides of (2.2), we have

$$\limsup_{n\to\infty}\|y_n-x^*\|\leq c. \tag{2.3}$$

Next consider

$$\begin{aligned}\|Sy_n-x^*+c_n(u_n-x_n)\| &\leq \|Sy_n-x^*\|+c_n\|u_n-x_n\|\\ &\leq \|Sy_n-Tx^*\|+2M_1c_n\\ &\leq \|y_n-x^*\|+2M_1c_n.\end{aligned}$$

Taking limsup on both the sides in the above inequality and then using (2.3), we get that

$$\limsup_{n\to\infty} \|Sy_n - x^* + c_n(u_n - x_n)\| \leq c.$$

Also

$$\begin{aligned} \|x_n - x^* + c_n(u_n - x_n)\| &\leq \|x_n - x^*\| + c_n \|u_n - x_n\| \\ &\leq \|x_n - x^*\| + 2M_1 c_n, \end{aligned}$$

gives that

$$\limsup_{n\to\infty} \|x_n - x^* + c_n(u_n - x_n)\| \leq c.$$

Further, $\lim_{n\to\infty} \|x_{n+1} - x^*\| = c$ means that

$$\lim_{n\to\infty} \|(1-b_n)(x_n - x^* + c_n(u_n - x_n)) + b_n(Sy_n - x^* + c_n(u_n - x_n))\| = c.$$

Hence applying Lemma 1, we obtain that

$$\lim_{n\to\infty} \|x_n - Sy_n\| = 0. \tag{2.4}$$

Next consider

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|x_n - Sy_n\| + \|Sy_n - Tx_n\| \\ &\leq \|x_n - Sy_n\| + \|y_n - x_n\|. \end{aligned} \tag{2.5}$$

$$\begin{aligned} \|y_n - x_n\| &= \|a'_n x_n + b'_n Tx_n + c'_n v_n - x_n\| \\ &\leq b'_n \|x_n - Tx_n\| + c'_n \|v_n - x_n\| \\ &\leq b'_n \|x_n - Tx_n\| + 2M_1 c'_n. \end{aligned} \tag{2.6}$$

Substituting (2.6) in (2.5), we get

$$\|x_n - Tx_n\| \leq \|x_n - Sy_n\| + b'_n \|x_n - Tx_n\| + 2M_1 c'_n,$$

implies

$$\|x_n - Tx_n\| \leq \frac{1}{1-b'_n} \|x_n - Sy_n\| + 2M_1 \frac{c'_n}{1-b'_n},$$

gives us with the help of condition $\lim\limits_{n\to\infty} \sup b'_n < 1$,

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{2.7}$$

Now observe that

$$\begin{aligned}
\|x_{n+1}-Sx_{n+1}\| &= \|a_nx_n+b_nSy_n+c_nu_n-Sx_{n+1}\| \\
&= \|(1-b_n)x_n+b_nSy_n+c_n(u_n-x_n)-Sx_{n+1}\| \\
&= \|(1-b_n)(x_n-Sx_{n+1})+b_n(Sy_n-Sx_{n+1})+c_n(u_n-x_n)\| \\
&\leq (1-b_n)\|x_n-Sx_{n+1}\|+b_n\|Sy_n-Sx_{n+1}\|+c_n\|u_n-x_n\| \\
&\leq (1-b_n)(\|x_n-x_{n+1}\|+\|x_{n+1}-Sx_{n+1}\|) \\
&\quad +b_n(\|Sy_n-Tx_n\|+\|Tx_n-Sx_{n+1}\|)+2M_1c_n \\
&\leq (1-b_n)(\|x_n-x_{n+1}\|+\|x_{n+1}-Sx_{n+1}\|) \\
&\quad +b_n(\|y_n-x_n\|+\|x_n-x_{n+1}\|)+2M_1c_n \\
&= \|x_n-x_{n+1}\|+(1-b_n)\|x_{n+1}-Sx_{n+1}\|+b_n\|y_n-x_n\| \\
&\quad +2M_1c_n,
\end{aligned}$$

implies

$$\begin{aligned}
\|x_{n+1}-Sx_{n+1}\| &\leq \frac{1}{b_n}\|x_n-x_{n+1}\|+\|y_n-x_n\|+2M_1\frac{c_n}{b_n} \\
&\leq \frac{1}{\delta}\|x_n-x_{n+1}\|+\|y_n-x_n\|+2M_1\frac{c_n}{\delta}. \qquad (2.8)
\end{aligned}$$

Also

$$\begin{aligned}
\|x_n-x_{n+1}\| &= \|x_n-a_nx_n+b_nSy_n+c_nu_n\| \\
&\leq b_n\|x_n-Sy_n\|+c_n\|u_n-x_n\| \\
&\leq (1-\delta)\|x_n-Sy_n\|+2M_1c_n. \qquad (2.9)
\end{aligned}$$

Substituting (2.6) and (2.9) in (2.8) yields

$$\|x_{n+1}-Sx_{n+1}\| \leq \frac{1-\delta}{\delta}\|x_n-Sy_n\|+b'_n\|x_n-Tx_n\|+4M_1\frac{c_n}{\delta}+2M_1c'_n,$$

implies

$$\lim_{n\to\infty}\|x_{n+1}-Sx_{n+1}\|=0.$$

Thus

$$\lim_{n\to\infty}\|x_n-Sx_n\|=0.$$

Hence

$$\lim_{n\to\infty}\|Sx_n-x_n\|=0=\lim_{n\to\infty}\|Tx_n-x_n\|.$$

This completes the proof of the lemma. □

Theorem 1 *Let E be a uniformly convex Banach space satisfying the Opial's condition and C,S,T and $\{x_n\}$ be as taken in Lemma 9. If $F(S)\cap F(T)\neq\phi$, then $\{x_n\}$ converges weakly to a common fixed point of S and T.*

Proof: Let $x^* \in F(S) \cap F(T)$. Then as proved in Lemma 8, $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Now we prove that $\{x_n\}$ has a unique weak subsequential limit in $F(S) \cap F(T)$. To prove this, let z_1 and z_2 be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By Lemma 9, $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$ and $I - S$ is demiclosed with respect to zero by Lemma 3, therefore we obtain $Sz_1 = z_1$. Similarly, $Tz_1 = z_1$. Again in the same way, we can prove that $z_2 \in F(S) \cap F(T)$. Next, we prove the uniqueness. For this suppose that $z_1 \neq z_2$, then by the Opial's condition

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - z_1\| &= \lim_{n_i\to\infty} \|x_{n_i} - z_1\| \\
&< \lim_{n_i\to\infty} \|x_{n_i} - z_2\| \\
&= \lim_{n\to\infty} \|x_n - z_2\| \\
&= \lim_{n_j\to\infty} \|x_{n_j} - z_2\| \\
&< \lim_{n_j\to\infty} \|x_{n_j} - z_1\| \\
&= \lim_{n\to\infty} \|x_n - z_1\|.
\end{aligned}$$

This is a contradiction. Hence $\{x_n\}$ converges weakly to a point in $F(S) \cap F(T)$. □

The following condition is due to Senter and Dotson [12].

Definition 2 *A mapping $T : C \to C$ where C is a subset of E, is said to satisfy condition (A) if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in C$ where $d(x, F(T)) = \inf\{\|x - x^*\| : x^* \in F(T)\}$.*

Senter and Dotson [12] approximated fixed points of a nonexpansive mapping T by Mann iterates. Later on, Maiti and Ghosh [7] and Tan and Xu [14] studied the approximation of fixed points of a nonexpansive mapping T by Ishikawa iterates under the same condition (A) which is weaker than the the requirement that T is demicompact.

We modify the condition (A) and (A') for two mappings $S, T : C \to C$ as follows:

Definition 3 *Two mappings $S, T : C \to C$ where C a subset of E, are said to satisfy condition (AR) if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ and $\lambda \in [0,1]$ such that $\lambda \|x - Tx\| + (1-\lambda) \|x - Sx\| \geq f(d(x, F))$ for all $x \in C$ where $d(x, F) = \inf\{\|x - x^*\| : x^* \in F = F(S) \cap F(T)\}$.*

Note that condition (AR) reduces to condition (A) when $S = T$ and (A') if we take $\lambda = \frac{1}{2}$. We shall use condition (AR) instead of compactness of C to study the strong convergence of $\{x_n\}$ defined in (1.4). It is worth noting that in case of two mappings $S, T : C \to C$ satisfying (AU-N), condition (AR) is weaker than the compactness of C.

Theorem 2 *Let E be a uniformly convex Banach space and C, S, T and $\{x_n\}$ be as taken in Lemma 9. Further let $S, T : C \to C$ be two mappings satisfying condition (AR). If $F(S) \cap F(T) \neq \phi$, then $\{x_n\}$ converges strongly to a common fixed point of S and T.*

Proof: By Lemma 8, $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F(S)\cap F(T)$. Let it be c for some $c \geq 0$. If $c = 0$, there is nothing to prove. Suppose $c > 0$. By Lemma 9, $\lim_{n\to\infty} \|Sx_n - x_n\| = 0 = \lim_{n\to\infty} \|Tx_n - x_n\|$. Moreover, $\|x_{n+1} - x^*\| \leq \|x_n - x^*\| + M(b_n c'_n + c_n)$ for all $x^* \in F(S)\cap F(T)$. This implies that $d(x_{n+1}, F) \leq d(x_n, F) + (b_n c'_n + c_n)$ gives that $\lim_{n\to\infty} d(x_n, F)$ exists by virtue of Lemma 2. Now by condition (AR), $\lim_{n\to\infty} f(d(x_n, F) = 0$. Since f is a nondecreasing function and $f(0) = 0$, therefore $\lim_{n\to\infty} d(x_n, F) = 0$. The rest of proof is the same as provided by Tan and Xu [14]. □

Lemma 10 *Let E be a normed space and C its nonempty convex subset. Let $S, T : C \to C$ be two mappings satisfying (AU-N) and $\{x_n\}$ be the sequence as defined in (1.3). If $F(S) \cap F(T) \neq \phi$, then $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F(S) \cap F(T)$.*

Lemma 11 *Let E be a uniformly convex Banach space and C its nonempty closed convex subset. Let $S, T : C \to C$ be two mappings satisfying (AU-N) and $\{x_n\}$ be the sequence as defined in (1.3) with $\{b_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 1)$ and $\limsup_{n\to\infty} b'_n < 1$. If $F(S) \cap F(T) \neq \phi$, then $\lim_{n\to\infty} \|Sx_n - x_n\| = 0 = \lim_{n\to\infty} \|Tx_n - x_n\|$.*

Theorem 3 *Let E be a uniformly convex Banach space satisfying the Opial's condition and C, S, T and $\{x_n\}$ be as taken in Lemma 11. If $F(S) \cap F(T) \neq \phi$, then $\{x_n\}$ converges weakly to a common fixed point of S and T.*

Theorem 4 *Let E be a uniformly convex Banach space and C, S, T and $\{x_n\}$ be as taken in Lemma 11. Further let $S, T : C \to C$ be two mappings satisfying condition (AR). If $F(S) \cap F(T) \neq \phi$, then $\{x_n\}$ converges strongly to a common fixed point of S and T.*

Lemma 12 *Let E be a normed space and C its nonempty convex subset. Let $T : C \to C$ be a nonexpansive mapping and $\{x_n\}$ be the sequence as defined in (M) with $\{b_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 1)$. If $F(T) \neq \phi$, then $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F(T)$.*

Lemma 13 *Let E be a uniformly convex Banach space and C its nonempty closed convex subset. Let $T : C \to C$ be a nonexpansive mapping and $\{x_n\}$ be the sequence as defined in (M) with $\{b_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 1)$. If $F(T) \neq \phi$, then $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$.*

Theorem 5 *Let E be a uniformly convex Banach space satisfying the Opial's condition and C, T and $\{x_n\}$ be as taken in Lemma 13. If $F(T) \neq \phi$, then $\{x_n\}$ converges weakly to a fixed point of T.*

Theorem 6 *Let E be a uniformly convex Banach space and C, T and $\{x_n\}$ be as taken in Lemma 13. Furthere let $T : C \to C$ be a nonexpansive mapping satisfying condition (A). If $F(T) \neq \phi$, then $\{x_n\}$ converges strongly to a fixed point of T.*

References

[1] **Agarwal, R. P., Cho, Y. J., Li, J.**, and **Huang, N. J. :** *Stability of iterative procedures with errors approximating common fixed points for a couple of quasi-contractive mappings in q-uniformly smooth Banach spaces.* J. Math. Anal. Appl. **272**, 435-447, (2002)

[2] **Browder, F. E. :** *Nonlinear operators and nolinear equations of evolution in Banach spaces.* "Proc. Symp. Pure Math.", Vol. 18, Proc. Amer. Math. Soc., Providence, RI, 1976

[3] **Chidume, C. E.**, and **Moore, Chika :** *Fixed points iteration for pseudocontractive maps.* Proc. Amer. Math. Soc. **127(4)**, 1163-1170, (1999)

[4] **Das, G.**, and **Debata, J. P. :** *Fixed points of Quasi-nonexpansive mappings.* Indian J. Pure. Appl. Math. **17**, 1263-1269, (1986)

[5] **Ishikawa, S. :** *Fixed points by a new iteration method.* Proc. Amer. Math. Soc. 44, 147-150, (1974)

[6] **Khan, S. H. et al. :** *Weak and strong convergence of a scheme with errors for two nonexpansive mappings.* Nonlinear Anal. **61**, 1295-1301, (2005)

[7] **Maiti, M.**, and **Gosh, M. K. :** *Approximatig fixed points by Ishikawa iterates.* Bull. Austral. Math. Soc. **40**, 113-117, (1989)

[8] **Mann, W. R. :** *Mean value methods in iteration.* Proc. Amer. Math. Soc. 4, 506-510, (1953)

[9] **Opial, Z. :** *Weak convergence of the sequence of successive approximations for nonexpansive mappings.* Bull. Amer. Math. Soc. **73**, 591-597, (1967)

[10] **Rhoades, B. E. :** *Fixed point iterations for certain nonlinear mappings.* J. Math. Anal. Appl. **183**, 118-120, (1994)

[11] **Schu, J. :** *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings.* Bull. Austral. Math. Soc. **43**, 153-159, (1991)

[12] **Senter, H. F.**, and **Dotson, W. G. :** *Approximatig fixed points of nonexpansive mappings.* Proc. Amer. Math. Soc. **44(2)**, 375-380, (1974)

[13] **Takahashi, W.**, and **Tamura, T. :** *Convergence theorems for a pair of nonexpansive mappings.* J. Convex Analysis **5(1)**, 45-58, (1998)

[14] **Tan, K. K.**, and **Xu, H. K. :** *Approximatig fixed points of nonexpansive mappings by the Ishikawa iteration process.* J. Math. Anal. Appl. **178**, 301–308, (1993)

[15] **Xu, Y. :** *Ishikawa and Mann Iteration process with errors for nonlinear strongly accretive operator equations.* J. Math. Anal. Appl. **224**, 91-101, (1998)

received: May 9, 2006

Author:

Arif Rafiq
COMSATS
Institute of Information Technology,
Islamabad,
Pakistan

e-mail: arafiq@comsats.edu.pk