Rostock. Math. Kolloq. **61**, 13–20 (2006)

Subject Classification (AMS)
39A11,41A60

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Nonlinear Difference Equations with Periodic Solutions

ABSTRACT. For ten nonlinear difference equations with only p-periodic solutions it is shown that the characteristic polynomials of the corresponding linearized equations about the equilibria have only zeros which are p-th roots of unity. An analogous result is shown concerning two systems of such equations. Five counterexamples show that the reverse is not true. Some remarks are made concerning equations with asymptotically p-periodic solutions.

KEY WORDS. Nonlinear difference equations, systems of such equations, p-periodic solutions, asymptotically p-periodic solutions

In their new book [5] the authors E. A. Grove and G. Ladas present a series of examples of nonlinear difference equations

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k})\,, \tag{1}$$

with fixed $k \in \mathbb{N}_0$ and variable $n \in \mathbb{N}_0$, such that all solutions are periodic with the same (not necessarily prime) period p. On p. 25 they put the following two questions:

"*What is it that makes every solution of a difference equation periodic with the same period?*"

"*Is there an easily verifiable test that we can apply to determine whether or not this is true?*"

We deal with these questions under the following conditions:

(i) Assume that $f : G^{k+1} \mapsto G$ for some non-empty complex open set G, and let (1) have an equilibrium $\overline{x} \in G$ defined by

$$\overline{x} = f(\overline{x}, \overline{x}, \dots, \overline{x})\,. \tag{2}$$

(ii) Assume that $f(u_0, u_1, \dots, u_k)$ is holomorphic in some neighbourhood of $(\overline{x}, \overline{x}, \dots, \overline{x})$.

Then there exists the linearized equation of (1) about the equilibrium $\overline{x}$

$$z_{n+1} = f_0 z_n + f_1 z_{n-1} + \cdots + f_k z_{n-k} \tag{3}$$

with

$$f_j = \frac{\partial f}{\partial u_j}(\overline{x}, \overline{x}, \dots, \overline{x})$$

and with the corresponding characteristic polynomial

$$\lambda^{k+1} - f_0\lambda^k - \cdots - f_{k-1}\lambda - f_k\,. \tag{4}$$

We call a solution *admissible* if the initial values belong to G.

Conjecture *If all admissible solutions of the difference equation* (1) *are p-periodic, and if the conditions* (*i*)-(*ii*) *are satisfied concerning all equilibria $\overline{x} \in G$ then:*

(iii) All zeros λ of (4) *are simple p-th roots of unity (*simple *means that all zeros have multiplicity* 1*).*

Note that the book [5] contains also examples (1) with only periodic solutions, where f is not differentiable at the positive equilibrium. Such examples contain the maximum function, cf. [5, p. 27] or the function $|\cdot|$. We cannot prove this conjecture, but we check assertion (iii) for eight examples of [5] and two further ones, all with periodic solutions only, and we give five counterexamples with nonperiodic solutions, where (iii) is fulfilled nevertheless. In particular, we discuss the case $\lambda = 1$ in (iii). It follows an analogous check concerning two systems of [6], and we make some remarks concerning equations with asymptotically periodic solutions.

Single equations. We begin the following seven cases of [5, Section 2.2]:

(1)	$\overline{x}$	(4)	p
$x_{n+1} = \dfrac{1}{x_n}$	± 1	$\lambda + 1$	2
$x_{n+1} = \dfrac{1}{x_n x_{n-1}}$	$\sqrt[3]{1}$	$\lambda^2 + \lambda + 1$	3
$x_{n+1} = \dfrac{1}{x_{n-1}}$	± 1	$\lambda^2 + 1$	4
$x_{n+1} = \dfrac{1 + x_n}{x_{n-1}}$	$\dfrac{1}{2}(1 \pm \sqrt{5})$	$\lambda^2 - \frac{1}{\overline{x}}\lambda + 1$	5
$x_{n+1} = \dfrac{x_n}{x_{n-1}}$	1	$\lambda^2 - \lambda + 1$	6
$x_{n+1} = \dfrac{1 + x_n + x_{n-1}}{x_{n-2}}$	$1 \pm \sqrt{2}$	$\lambda^3 - \dfrac{1}{\overline{x}}(\lambda^2 + \lambda) + 1$	8
$x_{n+1} = \dfrac{x_n x_{n-2}}{x_{n-1} x_{n-3}}$	1	$\lambda^4 - \lambda^3 + \lambda^2 - \lambda + 1$	10

Here, the validity of assertion (iii) follows immediately from

$$\begin{aligned}
\lambda^2 - 1 &= (\lambda+1)(\lambda-1) \\
\lambda^3 - 1 &= (\lambda^2+\lambda+1)(\lambda-1) \\
\lambda^4 - 1 &= (\lambda^2+1)(\lambda^2-1) \\
\lambda^5 - 1 &= \left(\lambda^2 - \frac{1}{\overline{x}}\lambda + 1\right)(\lambda^2+\overline{x}\lambda+1)(\lambda-1) \\
\lambda^6 - 1 &= (\lambda^2-\lambda+1)(\lambda^2+\lambda+1)(\lambda^2-1) \\
\lambda^8 - 1 &= \left(\lambda^3 - \frac{1}{\overline{x}}(\lambda^2+\lambda)+1\right)\left(\lambda^3 + \frac{1}{\overline{x}}(\lambda^2-\lambda)-1\right)(\lambda^2+1) \\
\lambda^{10} - 1 &= (\lambda^4-\lambda^3+\lambda^2-\lambda+1)(\lambda^4+\lambda^3+\lambda^2+\lambda+1)(\lambda^2-1)\,.
\end{aligned}$$

This assertion also comes true for the *equation* [5, (2.66)]

$$x_{n+1} = \frac{x_n + x_{n-1} + \cdots + x_{n-k}}{x_n x_{n-1} \ldots x_{n-k} - 1} \tag{5}$$

with $k+1$ equilibria $\overline{x} = (k+2)^{\frac{1}{k+1}}$, the further equilibrium $\overline{x} = 0$, the characteristic polynomial $\lambda^{k+1} + \lambda^k + \cdots + 1$, which is a factor of $\lambda^{k+2} - 1$, and the period $p = k+2$. But it does not come true for equation [5, (2.26)] $x_{n+1} = |x_n| - x_{n-1}$ with only 9-periodic solutions, since its equilibrium is $\overline{x} = 0$ and the corresponding linearized equation does not exist.

A *funny example* for an equation with only 3-periodic solutions is

$$x_{n+1} = \frac{5(x_n + x_{n-1}) - 4x_n x_{n-1} - 3}{4(x_n + x_{n-1}) - 5} \tag{6}$$

with the unique equilibrium $\overline{x} = 1$ and the corresponding characteristic polynomial $\lambda^2+\lambda+1$.

Recall that a p-periodic sequence x_n can be represented as discrete Fourier series

$$x_n = \sum_{m=0}^{p-1} b_m z^{nm} \tag{7}$$

with $z = \exp\left\{\frac{2\pi i}{p}\right\}$ and the inversion

$$b_m = \frac{1}{p}\sum_{k=0}^{p-1} x_k z^{-mk}$$

where we want to point out that the last formula with $p = 5$ was misprinted in [1, p. 1073].

Counterexamples. Next we show by means of two symmetric examples with nonperiodic solutions that nevertheless assertion (iii) comes true at least for one equilibrium. Here we call an equation *symmetric*, if it is uniquely solvable with respect to x_{n-k}, and if this solution reads

$$x_{n-k} = f(x_{n-k+1}, \ldots, x_n, x_{n+1})$$

with the same function f as before. It can easily be proved:

Lemma 1 *If equation (1) is symmetric and all solutions of it are p-periodic with $p \geq k+3$, then the solutions x_n with equal initial values $x_0 = x_{-1} = \cdots = x_{-k}$ have the property*

$$x_j = x_{p-k-j} \quad (j = 1, \ldots, p-k-1)\,. \tag{8}$$

A third example with nonperiodic solutions concerns the case where (iii) is satisfied completely.

As *first example* we choose the symmetric equation

$$x_{n+1}x_{n-1} = 2(1-\sqrt{2}) + 2x_n \tag{9}$$

with the equilibria $\overline{x}_1 = \sqrt{2}$, $\overline{x}_2 = 2 - \sqrt{2}$. The linearized equation

$$\overline{x}(x_{n+1} + x_{n-1}) - 2x_n = 0$$

has the characteristic polynomial

$$\lambda^2 - \frac{2}{\overline{x}}\lambda + 1\,, \tag{10}$$

and for the first equilibrium its zeros are simple 8th roots of unity in view of

$$\lambda^8 - 1 = (\lambda^2 - \sqrt{2}x + 1)(\lambda^2 + \sqrt{2}x + 1)(\lambda^4 - 1)\,.$$

For the initial values $x_{-1} = x_0 = 1$ (cf. [5, p. 26]) we find

$$x_1 = 4 - 2\sqrt{2}\,, \quad x_2 = 10 - 6\sqrt{2}\,, \quad x_3 = 4 - \frac{3}{2}\sqrt{2}\,, \quad x_4 = \frac{1}{14}(20 + 5\sqrt{2})\,,$$

so that $x_3 \neq x_4$, i.e. (8) with $j = 3$, $p = 8$, $k = 1$ is not fulfilled. According to Lemma 1 Equation (9) cannot have only 8-periodic solutions. Here, the zeros of (10) concerning the second equilibrium are $\lambda = 1 + \frac{1}{2}\sqrt{2} \pm \sqrt{\frac{1}{2} + \sqrt{2}}$, and these real numbers cannot be roots of unity.

As *second example* we choose

$$x_{n+1}x_{n-1} = 2 - x_n \tag{11}$$

with the equilibria $\overline{x}_1 = 1$ and $\overline{x}_2 = -2$. The linearized equation

$$\overline{x}(x_{n+1} + x_{n-1}) + x_n = 0$$

has the characteristic polynomial

$$\lambda^2 + \frac{1}{\overline{x}}\lambda + 1\,, \tag{12}$$

and for the first equilibrium its zeros are simple third roots of unity. For the initial values $x_{-1} = x_0 = 3$ we find $x_1 = -\frac{1}{3}$, $x_2 = \frac{7}{9}$, so that this solution is not 3-periodic. Here, the zeros of (12) concerning the second equilibrium are $\lambda = \frac{1}{4}(1 \pm i\sqrt{15})$, and according to $\lambda^3 = -\frac{1}{32}(22 \pm 6i\sqrt{15})$ no third roots of unity.

The *third example* reads

$$x_{n+1} = 3 - \frac{1}{2}\left(3x_n + \frac{x_{n-1}^2}{x_n}\right) \tag{13}$$

with the single equilibrium $\overline{x} = 1$ and the single characteristic polynomial (12). For the initial values $x_{-1} = x_0 = 2$ we find $x_1 = -1$, $x_2 = \frac{13}{2}$, so that this solution is not 3-periodic.

The case $\boldsymbol{\lambda = 1}$. In the foregoing examples all zeros of (4) are different from 1. But the case $\lambda = 1$ is likewise possible. By differentiation of (2) with respect to $\overline{x}$ we easily see:

Lemma 2 *Let (2) be satisfied for all $\overline{x} \in G$, and let (ii) be satisfied in G. Then the characteristic polynomial (4) has the zero $\lambda = 1$.*

This case can appear by linear homogeneous difference equations with constant coefficients. A nonlinear *example* is

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}} \tag{14}$$

with arbitrary $\overline{x}$, the single characteristic polynomial $\lambda^3 - \lambda^2 + \lambda - 1 = (\lambda - 1)(\lambda^2 + 1)$, and the 4-periodic solutions

$$a, b, c, \frac{ac}{b}, a, \ldots$$

with arbitrary nonvanishing constants a, b, c.

A *fourth counterexample* is

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{x_{n-1}^2}{x_n}\right) \tag{15}$$

with arbitrary $\overline{x}$, the single characteristic polynomial $\lambda^2 - 1$ with the roots ± 1, but $x_2 = x_0$ if and only if $x_0 = x_1$. Hence, disregarding the constant solutions, no further solution is 2-periodic.

As *last counterexample* we consider the equation

$$x_n x_{n-k} = 2x_n - 1 \tag{16}$$

with $k \in \mathbb{N}$, the single equilibrium $\overline{x} = 1$, the linearized equation

$$x_n = x_{n-k}\,,$$

and the single characteristic polynomial $\lambda^k - 1$. All zeros are simple k-th roots of unity, but the solutions of (16) are not k-periodic (disregarding $x_n = 1$). In this case the first hypothesis of Lemma 2 is not fulfilled.

Systems of equations. The foregoing conjecture can also be transferred to systems of nonlinear difference equations. We shall show this for the two systems from [6]

$$x_{n+2}^{(m)}\, x_n^{(m+2)} = 1 + x_{n+1}^{(m+1)} \tag{17}$$

and

$$x_{n+3}^{(m)}\, x_n^{(m+3)} = 1 + x_{n+1}^{(m+2)} + x_{n+2}^{(m+1)} \tag{18}$$

with variable $m, n \in \mathbb{Z}$, where the solutions in both cases shall be k-periodic in m with a fixed $k \in \mathbb{N}$. For $k = 1$ the systems reduce to special cases from before.

In [6] it was proved: *Every admissible solution of* (17) *is p-periodic with $p = 5k$ when $5 \nmid k$ and $p = k$ else. Every admissible solution of* (18) *is p-periodic with $p = 8q$ when $k = 2^j q$ $(0 \le j \le 2)$, $2 \nmid q$ and $p = k$ else.*

In both cases we shall show that *every zero of the corresponding characteristic polynomials about the common equilibria is a p-th root of unity.*

We begin with (17). The common equilibria are the solutions of

$$\overline{x}^2 = 1 + \overline{x}\,. \tag{19}$$

Fixing one of it, the corresponding linearized system reads

$$\overline{x}\left(x_{n+2}^{(m)} + x_n^{(m+2)}\right) = x_{n+1}^{(m+1)}\,,$$

where as before $x_n^{(m)}$ is k-periodic in m. With the ansatz $x_n^{(m)} = \xi_m \lambda^n$ we get the cyclic system

$$\overline{x}\lambda^2 \xi_m - \lambda \xi_{m+1} + \overline{x}\xi_{m+2} = 0$$

with k-periodic ξ_m. The matrix is the circulant matrix

$$\mathrm{Circ}\,(\overline{x}\lambda^2, -\lambda, \overline{x}, 0, \ldots, 0)$$

where the eigenvalues are the discrete Fourier transform (7) of the first line

$$\overline{x}\lambda^2 - \lambda\varepsilon^m + \overline{x}\varepsilon^{2m}\,, \tag{20}$$

$m = 0, \ldots, k-1$, with $\varepsilon = \exp\left\{\frac{2\pi i}{k}\right\}$, cf. [4]. The characteristic polynomial is the determinant of the circulant matrix, and its zeros are the zeros of (20), i.e. the solutions of

$$\overline{x}\lambda^2 = \lambda\varepsilon^m - \overline{x}\varepsilon^{2m}\,.$$

From this it follows by means of (19) that

$$\overline{x}\lambda^4 = -\lambda\overline{x}\varepsilon^{3m} + \varepsilon^{4m}$$

and $\lambda^5 = \varepsilon^{5m}$, i.e. $\lambda^{5q} = 1$ and therefore $p = 5q$ when either $k = q$ and $5 \nmid q$ or when $k = 5q$ with an integer q. The result is independent of the chosen equilibrium.

In the case (18) the common equilibria are the solutions of

$$\overline{x}^2 = 1 + 2\overline{x}\,. \tag{21}$$

Fixing one of it, the corresponding linearized system reads

$$\overline{x}\left(x_{n+3}^{(m)} + x_n^{(m+3)}\right) = x_{n+1}^{(m+2)} + x_{n+2}^{(m+1)}\,,$$

and the ansatz $x_n^{(m)} = \xi_m\lambda^n$ with k-periodic ξ_m yields the cyclic system

$$\overline{x}\lambda^3\xi_m - \lambda^2\xi_{m+1} - \lambda\xi_{m+2} + \overline{x}\xi_{m+3} = 0$$

with the circulant matrix

$$\mathrm{Circ}\,(\overline{x}\lambda^3, -\lambda^2, -\lambda, \overline{x}, 0, \ldots, 0)\,.$$

Again, the eigenvalues must vanish, so that

$$\overline{x}\lambda^3 - \lambda^2\varepsilon^m - \lambda\varepsilon^{2m} + \overline{x}\varepsilon^{3m} = 0\,, \tag{22}$$

$m = 0, \ldots, k-1$, with the same ε as before. The left-hand side of (22) can be factorized as

$$(\overline{x}(\lambda^2 - \lambda\varepsilon^m + \varepsilon^{2m}) - \lambda\varepsilon^m)(\lambda + \varepsilon^m)\,,$$

so that one solution of (22) is $\lambda = -\varepsilon^m$. For the zeros of the other factor it follows by means of (21) that $\lambda^4 = -\varepsilon^{4m}$, hence in both cases $\lambda^8 = \varepsilon^{8m}$. This implies $\lambda^{8q} = 1$ and therefore $p = 8q$, when either $k = 2^j q$ $(0 \le j \le 2)$ and q odd or when $k = 8q$ with an integer q. The result is independent of the chosen equilibrium.

Asymptotically periodic solutions. In [5, p. 61] there is contained a further question: "*What is it that makes all the solutions of a difference equation be eventually periodic with the same period?*"

Under the assumptions (i) and (ii) the last assertion comes true, *if some* (*but not all*) *zeros of the characteristic polynomial* (4) *about an equilibrium are simple p-th roots of unity, all other zeros* λ_i *satisfy* $0 < |\lambda_i| < 1$, *and if the general solution of* (1) *can be represented in the form* (7), *where the coefficients are convergent power series in* λ_i^n *for large* n (with polynomial coefficients in case of need), cf. the special cases in [1, (7.12)] and [2, Propositions 3.3 and 3.4]. But the situation can be more complicated, cf. [3, Example 2], in particular in the case $\lambda = 1$ for one zero of (4), cf. [1, (1.7)], i.e. [5, (5.2) with $\alpha = \beta = 0$ and $\gamma = A = 1$].

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received: September 15, 2005

revised: October 14, 2005

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