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Periodic wave solutions and solitary cusp wave solutions for a higher order wave equation of KdV type[2]

ABSTRACT. This paper is the continuation of Ref. [1]. Both the bifurcation theory of planar dynamical system and elliptic function integral method are applied to study a higher order wave equation of KdV type. And the parametric space is redivided when the integral constant $g \neq 0$. Many explicit and implicit solutions of periodic wave and solitary cusp wave are obtained.

KEY WORDS. higher order wave equation of KdV type, solitary cusp wave solution, periodic wave solution, elliptic function integral method.

1 Introduction

In this paper, we will seek periodic wave and solitary cusp wave solutions for the following higher order wave equation of KdV type (see [1, 2]):

$$u_t + u_x + \alpha u u_x + \beta u_{xxx} + \alpha^2 \rho_1 u^2 u_x + \alpha\beta(\rho_2 u u_{xxx} + \rho_3 u_x u_{xx}) = 0\,, \tag{1.1}$$

where $\rho_i (i = 1, 2, 3)$ are free parameters and α, β are positive real constants which characterize, respectively, the long wavelength and short amplitude of the waves. Just as Tzirtzilakis, E. [2] said, the equation (1.1) is a water wave equation of KdV type which is more physically and practically meaningful. By the local coordinate transformation

$$u = v - \alpha\rho_1 v^2 - \beta\left(3\rho_1 + \frac{7}{4}\rho_2 - \frac{1}{2}\rho_3\right) v_{xx}\,, \tag{1.2}$$

Eq. (1.1) can be transformed into the following simple equation, see [3, 4]:

$$v_t - \frac{3}{2}\beta\rho_2 v_{xxt} + \beta(1 - \frac{3}{2}\rho_2) v_{xxx} + \alpha v v_x - \frac{1}{2}\alpha\beta\rho_2(v v_{xxx} + 2 v_x v_{xx}) = 0\,, \tag{1.3}$$

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where $\rho_2 \neq 0$. In Ref. [1], we obtained two explicit parametric representations of periodic solutions of equation (1.3) when integral constant $g = 0$. In this case, we also proved the existence of all travelling wave solutions. However, when $g \neq 0$, the bifurcation of travelling solutions had not been studied. In fact, when the integral constant $g \neq 0$, the dynamical behaviors of the equation (1.3) are better than the case of $g = 0$. Therefore, we shall use bifurcation method of planar dynamical system [5]-[8] and elliptic function integral method [9, 10] to investigate the explicit and implicit travelling wave solutions of (1.3) when $g \neq 0$.

Let $v(x,t) = \psi(x - ct) = \psi(\xi)$, where c is the wave speed, then the equation (1.3) becomes the following ordinary differential equation

$$\frac{1}{2}\alpha(\psi^2)_\xi - c\psi_\xi + \left(\frac{3}{2}c\beta\rho_2 + \beta(1 - \frac{3}{2}\rho_2)\right)\psi_{\xi\xi\xi} - \frac{1}{2}\alpha\beta\rho_2\left(\psi\psi_{\xi\xi} + \frac{1}{2}\psi_\xi^2\right)_\xi = 0\,, \tag{1.4}$$

Integrating once with respect to ξ, we obtain the following wave equation of (1.3)

$$\beta(3c\rho_2 + 2 - 3\rho_2 - \alpha\rho_2\psi)\psi_{\xi\xi} - \frac{1}{2}\alpha\beta\rho_2\psi_\xi^2 + \alpha\psi^2 - 2c\psi + g = 0\,, \tag{1.5}$$

where g is the integral constant and $g \neq 0$.

Clearly, (1.5) is equivalent to the following two-dimensional systems:

$$\frac{d\psi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{\frac{1}{2}\alpha\beta\rho_2 y^2 - \alpha\psi^2 + 2c\psi - g}{\beta(3\rho_2(c-1) + 2 - \alpha\rho_2\psi)}\,. \tag{1.6}$$

System (1.6) is a planar dynamical system defined by the 5-parameter space $(\alpha, \beta, c, \rho_2, g)$. Because the phase orbits defined by the vector field of (1.6) determine all travelling wave solutions, we will investigate bifurcations of phase portraits of the system, when the parameters vary. Since (1.3) is a physical model where only the bounded travelling waves are meaningful, so we only consider their bounded travelling wave solutions.

Suppose that $\psi(x - ct) = \psi(\xi)$ is a continuous solution of (1.6) for $\xi \in (-\infty, \infty)$ and $\lim\limits_{\xi\to\infty}\psi(\xi) = a$, $\lim\limits_{\xi\to-\infty}\psi(\xi) = b$. It is well known that (i) $\psi(x,t)$ is called a solitary wave solution if $a = b$; (ii) $\psi(x,t)$ is called a kink or anti-kink solution if $a \neq b$. Usually, a solitary wave solution of (1.3) corresponds to a homoclinic orbit of (1.6); a kink (or anti-kink) wave solution of (1.3) corresponds to a heteroclinic orbit (or the so-called connecting orbit) of (1.6). Similarly, a periodic orbit of (1.3) corresponds to a periodically travelling wave solution of (1.6). Therefore, we must find all periodic annuli, heterclinic and homoclinic orbits of (1.6) in order to investigate the bifurcations of periodic waves and solitary cusp waves of (1.3). Thus, the bifurcation theory of dynamical systems and some computational method of travelling wave solutions are very important and useful, see [5]-[11].

We notice that the right-hand side of the second equation in (1.6) is not continuous when $\psi = \psi_s = \frac{\eta}{\alpha\rho_2}$, where $\eta = 3\rho_2(c-1) + 2$. In other words, on such straight line $\psi = \psi_s$ in the phase plane (ψ, y), the function $\psi_{\xi\xi}$ is not defined. It implies that the smooth system (1.3) sometimes has non-smooth travelling wave solutions. The similar phenomenon has been considered before, see [1, 7, 8, 10].

In Section 2, we discuss bifurcations of phase portraits of (1.6), where explicit parametric conditions will be derived. In Section 3, we derive the explicit parameter representations of the smooth periodic wave and non-smooth solitary cusp wave solutions of (1.3). In Sections 4, we derive the implicit parameter representations of the smooth periodic wave solutions.

2 Bifurcations of phase portraits of system (1.6)

Because the function $\psi_{\xi\xi}$ is not defined on the singular straight line $\psi = \frac{\eta}{\alpha\rho_2}$, we make a transformation $d\zeta = 2\beta(\eta - \alpha\rho_2\psi)d\xi$, $\eta = 3\rho_2(c-1)+2$. Then the system (1.6) becomes the following system:

$$\frac{d\psi}{d\zeta} = 2\beta(\eta - \alpha\rho_2\psi)y, \quad \frac{dy}{d\zeta} = -(-\alpha\beta\rho_2 y^2 + 2\alpha\psi^2 - 4c\psi + 2g)\,. \tag{2.1}$$

It is easy to see that (1.6) and (2.1) have the same first integral

$$H(\psi, y) = \beta(\eta - \alpha\rho_2\psi)y^2 + \frac{2}{3}\alpha\psi^3 - 2c\psi^2 + 2g\psi = h\,, \tag{2.2}$$

where h is integral constant.

By system (2.1), we define the $\psi = \psi_s = \frac{\eta}{\alpha\rho_2}$ is a singular straight line L and write

$$f(\psi) = \alpha\psi^2 - 2c\psi + g, \quad \Delta = c^2 - \alpha g, \quad \psi_{1,2} = \frac{c \pm \sqrt{\Delta}}{\alpha}, \quad Y_\pm = \pm\sqrt{\frac{2f(\psi_s)}{\alpha\beta\rho_2}}\,. \tag{2.3}$$

Thus, we obtain the following conclusion for equilibrium points of system (2.1):

(1) when $\Delta > 0$, (2.1) has two equilibrium points at $A_{1,2}(\psi_{1,2}, 0)$ in the ψ-axis;

(2) when $\Delta = 0$ and $c \neq 0$, (2.1) has only one equilibrium point at $A_0(\frac{c}{\alpha}, 0)$ in the ψ-axis;

(3) When $\rho_2 f(\psi_s) > 0$, there exist two equilibrium points of (2.1) at $S_\pm(\psi_s, Y_\pm)$ in L;

(4) When $f(\psi_s) = 0$, there exist only one equilibrium point of (2.1) at $S_0(\psi_s, 0)$ which is the intersection point of the line L and the ψ-axis.

Let $M(\psi_i, y_j)$ be the coefficient matrix of the linearized system of (2.1) at an equilibrium point, (ψ_i, y_j). Then we have $Trace(M(\psi_{1,2}, 0)) = 0$ and

$$J(\psi_{1,2}, 0) = detM(\psi_{1,2}, 0) = -8\beta[\rho_2\sqrt{\Delta} \pm (3\rho_2 - 2\rho_2 c - 2)]\sqrt{\Delta}, \; J\left(\frac{c}{\alpha}, 0\right) = 0\,. \tag{2.4}$$

$$J(\psi_s, Y_\pm) = detM(\psi_s, Y_\pm) = -4\alpha^2\beta^2\rho_2^2 Y_\pm^2 < 0, \quad J(\psi_s, 0) = 0\,. \tag{2.5}$$

By the theory of planar dynamical systems, we know that for an equilibrium point of a planar integrable system, if $J < 0$ then the equilibrium point is a saddle point; if $J > 0$ and $Trace(M(\psi_i, 0)) = 0$ then it is a center point; if $J > 0$ and $(Trace(M(\psi_i, 0)))^2 - 4J(\psi_i, 0) > 0$ then it is a node; if $J = 0$ and the index of the equilibrium point is zero then it is a cusp; if $J = 0$ and the index of the equilibrium point is'nt zero then it is a high order singular point.

Notice that for $H(\psi, y) = h$ defined by (2.2), we have

$$h_{1,2} = H(\psi_{1,2}, 0) = -\frac{2(c \pm \sqrt{\Delta})[(2c^2 \pm c\sqrt{\Delta}) - (\Delta + 3\alpha g)]}{3\alpha^2}\,, \tag{2.6}$$

$$h_0 = H(\frac{c}{\alpha}, 0) = -\frac{2c(2c^2 - 3\alpha g)}{3\alpha^2}\,, \tag{2.7}$$

$$h_s = H(\psi_s, Y_\pm) = \frac{2\eta(\eta^2 - 3c\rho_2\eta + 3g\alpha\rho_2^2)}{3\alpha^2\rho_2^3}\,, \tag{2.8}$$

$$h_{s0} = H(\phi_s, 0) = \frac{2\eta^2(3c\rho_2 - 2\eta)}{3\alpha^2\rho_2^3}\,. \tag{2.9}$$

From $\Delta = 0$, we have

$$(\Gamma_1): \qquad g = g_1(c) = \frac{c^2}{\alpha}\,. \tag{2.10}$$

For a fixed ρ_2, the case of $h_1 = h_s$ or $h_2 = h_s$ imply

$$(\Gamma_2): \qquad g = g_2(c) = \frac{(12\rho_2^2 - 8\rho_2)c - 9\rho_2^2 + 12\rho_2 - 4}{4\alpha\rho_2^2}\,, \tag{2.11}$$

and

$$(\Gamma_3): \qquad g = g_3(c) = -\frac{3\rho_2^2c^2 - 12\rho_2^2 c + 8\rho_2 c + 9\rho_2^2 - 12\rho_2 + 4}{\alpha\rho_2^2}\,. \tag{2.12}$$

It is easy to see that $\psi_s = \psi_1(\psi_s = \psi_2)$ corresponds to $J(\psi_1, 0) = 0(J(\psi_2, 0) = 0)$ when the parameter $(c, g) \in \Gamma_3$. In this case, $f(\psi_s) = 0$ corresponds to $\rho_2\sqrt{\Delta} \pm (3\rho_2 - 2\rho_2 c - 2) = 0$ when the parameter $(c, g) \in \Gamma_3$.

Write

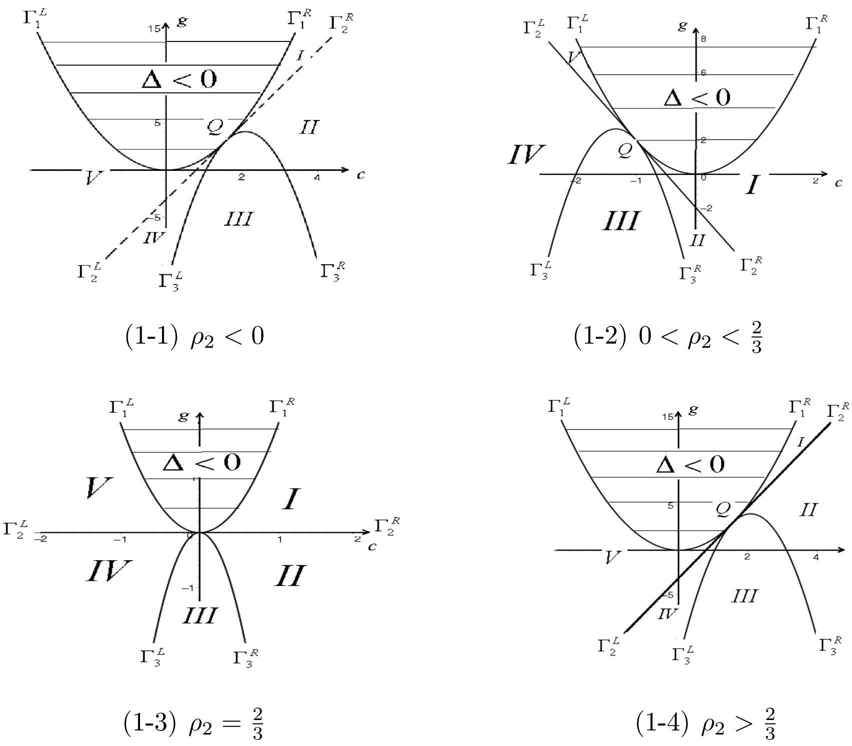
$$\Gamma_1 \bigcap \Gamma_2 \bigcap \Gamma_3 = Q\,, \tag{2.13}$$

where $Q\left(\frac{3\rho_2-2}{2\rho_2}, \frac{(3\rho_2-2)^2}{4\alpha\rho_2^2}\right)$ is intersection point of $\Gamma_1,\ \Gamma_2,\ \Gamma_3$.

Here, we express the part of $c > \frac{3\rho_2-2}{2\rho_2}$ on curve Γ with Γ^R; We express the part of $c < \frac{3\rho_2-2}{2\rho_2}$ on curve Γ with Γ^L; Similarly, we express the part of $g > 0$ at the regional $I - V$ with

$I^+ - V^+$; we express the part of $g < 0$ at the regional $I - V$ with $I^- - V^-$; we express the part of $g > 0$ on curve Γ with Γ^+; We express the part of $g < 0$ on curve Γ with Γ^-.

Thus, the bifurcation curves Γ_1^R, Γ_2^R, Γ_3^R, Γ_1^L, Γ_2^L, Γ_3^L which are defined by (2.10), (2.11), (2.12) divided the plane (c, g) into six regions, i.e. $(I) - (V)$ and the region of $\Delta < 0$, shown in Fig. 1.



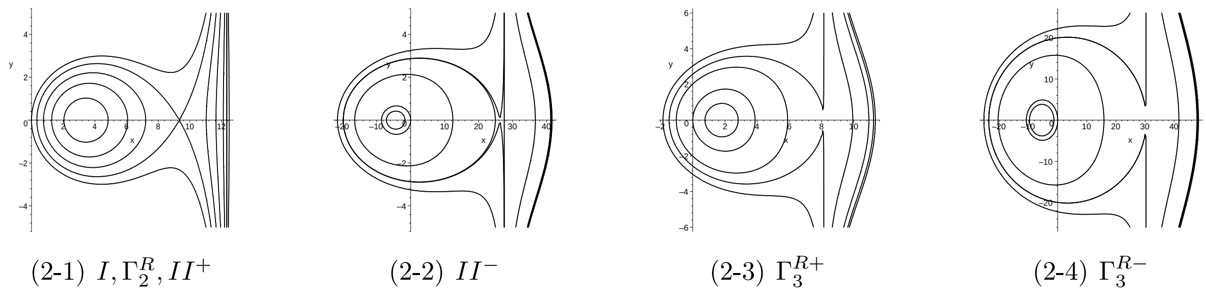
(1-1) $\rho_2 < 0$ (1-2) $0 < \rho_2 < \frac{2}{3}$

(1-3) $\rho_2 = \frac{2}{3}$ (1-4) $\rho_2 > \frac{2}{3}$

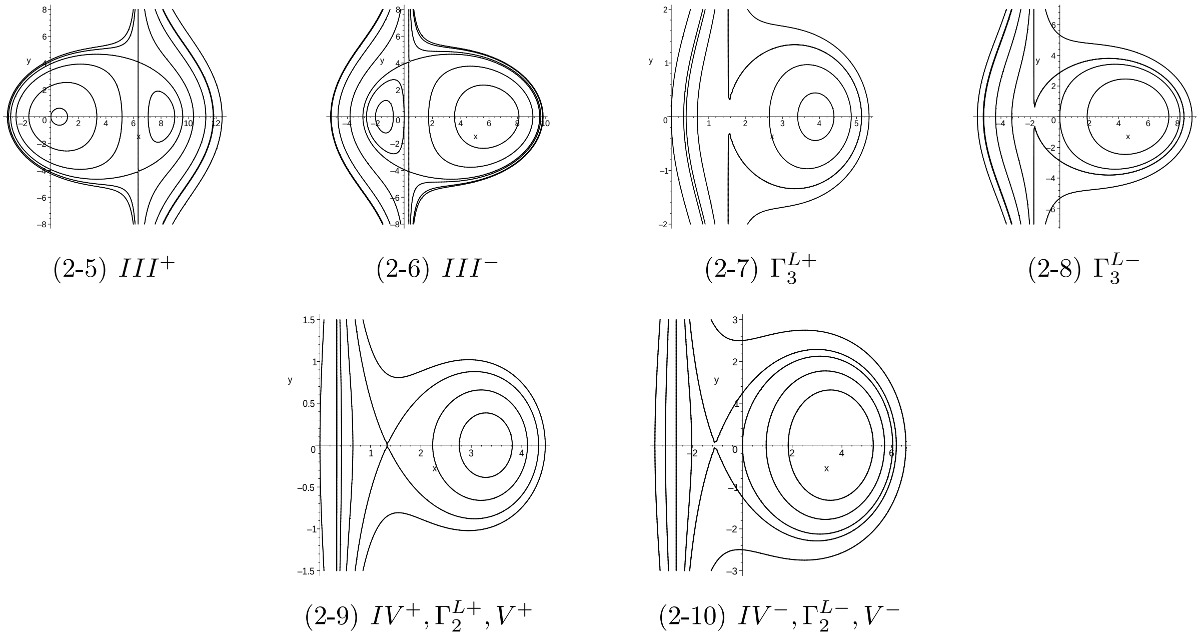
Fig. 1 The bifurcation curves and the six regions of (2.1)

For a fixed h, the level curve $H(\phi, y) = h$ defined by (2.2) determines a set of invariant curves of (2.1), which contains different branches of curves. As h vary, it defines different families of orbits of (2.1), with different dynamical behaviors.

Corresponding to the bifurcation curves $\Gamma_{1,2,3}$ and regions $I - V$ of the plane (c, g) in the Fig. 1 (1-1), we obtain the following different phase portraits



(2-1) I, Γ_2^R, II^+ (2-2) II^- (2-3) Γ_3^{R+} (2-4) Γ_3^{R-}

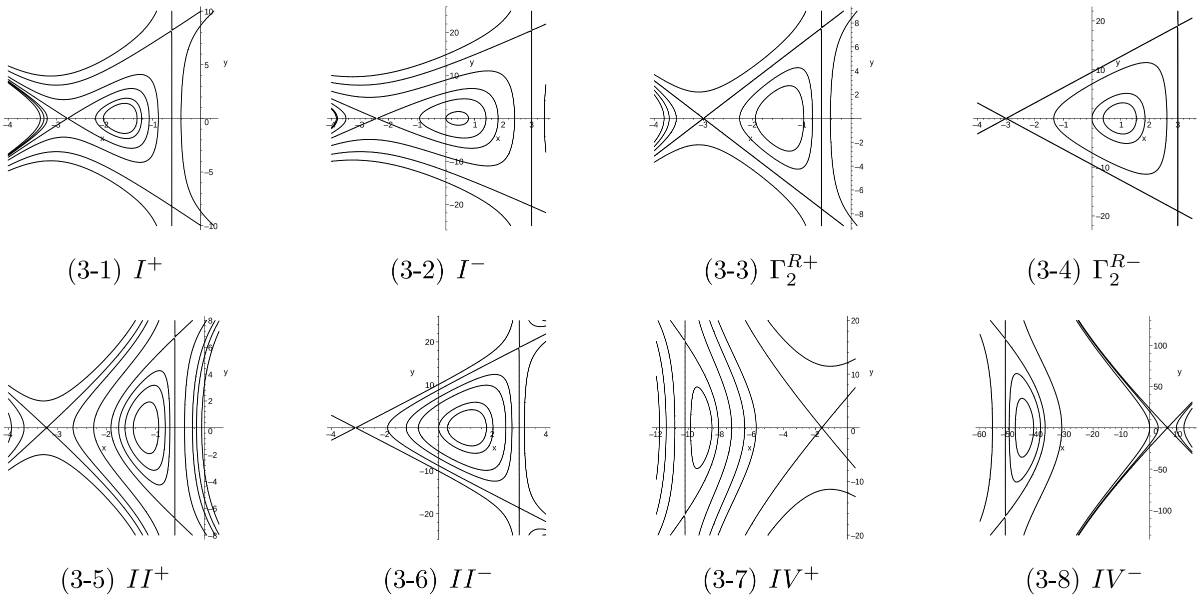


(2-5) III^{+} (2-6) III^{-} (2-7) Γ_3^{L+} (2-8) Γ_3^{L-}

(2-9) $IV^{+}, \Gamma_2^{L+}, V^{+}$ (2-10) $IV^{-}, \Gamma_2^{L-}, V^{-}$

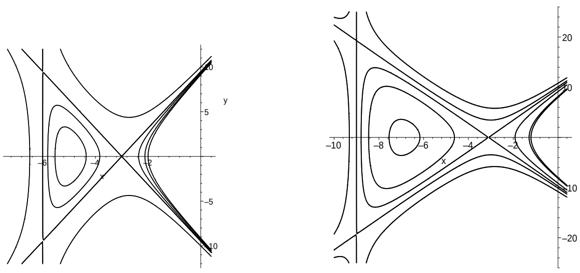
Fig. 2 The phase portraits of (2.1) for $\rho_2 < 0,\ g \neq 0$

Corresponding to the bifurcation curves $\Gamma_{1,2,3}$ and regions $I - V$ of the plane (c, g) in the Fig. 1 (1-2), we obtain the following different phase portraits



(3-1) I^{+} (3-2) I^{-} (3-3) Γ_2^{R+} (3-4) Γ_2^{R-}

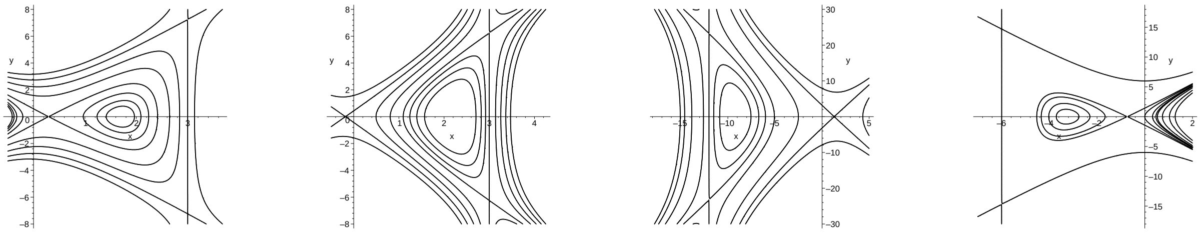
(3-5) II^{+} (3-6) II^{-} (3-7) IV^{+} (3-8) IV^{-}



(3-9) Γ_2^L (3-10) V

Fig. 3 The phase portraits of (2.1) for $0 < \rho_2 < \frac{2}{3},\ g \neq 0$.

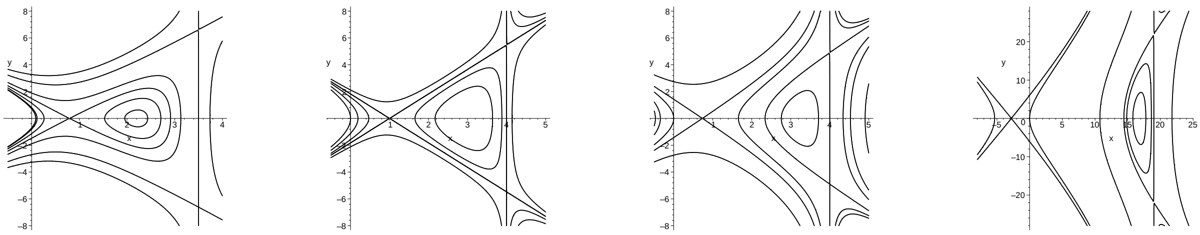
Corresponding to the bifurcation curves $\Gamma_{1,2,3}$ and regions $I - V$ of the plane (c, g) in the Fig. 1 (1-3), we obtain the following different phase portraits



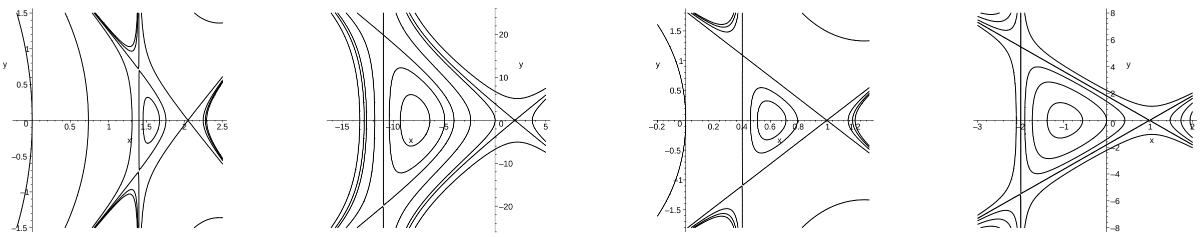
(4-1) I (4-2) II (4-3) IV (4-4) V

Fig. 4 The phase portraits of (2.1) for $\rho_2 = \frac{2}{3},\ g \neq 0$.

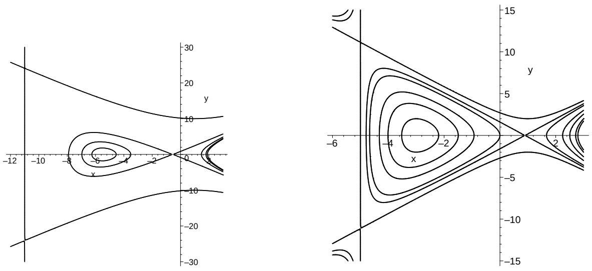
Corresponding to the bifurcation curves $\Gamma_{1,2,3}$ and regions $I - V$ of the plane (c, g) in the Fig. 1 (1-4), we obtain the following different phase portraits



(5-1) I (5-2) Γ_2^R (5-3) II^+ (5-4) II^-



(5-5) IV^+ (5-6) IV^- (5-7) Γ_2^{L+} (5-8) Γ_2^{L-}



(5-9) V^+ (5-10) V^-

Fig. 5 The phase portraits of (2.1) for $\rho_2 > \frac{2}{3}$, $g \neq 0$.

Note: When $\Delta < 0$ and $\rho_2 > 0, (c,g) \in \Gamma_1, \Gamma_3, III$, system (2.1) has not closed orbit. Here we omit their phase portraits.

3 Explicit expressions of periodic wave solutions and solitary cusp wave solutions of (1.3)

According to the analysis in the section 2, we derive the explicit expressions of periodic wave solutions and solitary cusp wave solutions of (1.3). See the computational process and results below.

3.1 Suppose that $\rho_2 < 0$, $(c,g) \in \Gamma_3^R$ i.e. $\rho_2 < 0$, $c > \frac{3\rho_2 - 2}{2\rho_2}$, $g = g_3(c)$. In this case, we get $\psi_1 = \psi_s$, $h_1 = h_{s0} = \frac{2\eta^2(3c\rho_2 - 2\eta)}{3\alpha^2\rho_2^3}$. When $h = h_{s0}$, system (2.1) has a periodic orbit to the point $S_0(\psi_s, 0)$ and around the center point $A_2(\psi_2, 0)$, see Fig. 2 (2-3), (2-4). Substituting $h = h_{s0}$ into (2.2) yields the following algebraic equations for this periodic orbit

$$y = \pm\sqrt{\frac{2}{3\beta(-\rho_2)}}\sqrt{(\psi_s - \psi)(\psi - \psi_0)}\,, \tag{3.1}$$

where $\psi_s = \frac{\eta}{\alpha\rho_2} = \frac{3\rho_2(c-1)+2}{\alpha\rho_2}$, $\psi_0 = \psi(0) = \frac{-4+6\rho_2-3\rho_2 c}{\alpha\rho_2}$ and $\psi_0 < \psi_s$.

Substituting (3.1) into the first equation of (2.1) yields the following equation

$$\pm\frac{d\psi}{\sqrt{(\psi_s - \psi)(\psi - \psi_0)}} = \sqrt{\frac{2}{3\beta(-\rho_2)}}d\xi\,. \tag{3.2}$$

Integrating (3.2) along this periodic orbit yields

$$\int_{\psi_s}^{\psi}\frac{d\psi}{\sqrt{(\psi_s - \psi)(\psi - \psi_0)}} = \sqrt{\frac{2}{3\beta(-\rho_2)}}\int_0^{\xi} d\xi, \qquad \xi > 0 \tag{3.3}$$

and

$$-\int_{\psi}^{\psi_s} \frac{d\psi}{\sqrt{(\psi_s-\psi)(\psi-\psi_0)}} = \sqrt{\frac{2}{3\beta(-\rho_2)}} \int_{\xi}^{0} d\xi, \qquad \xi \le 0 \tag{3.4}$$

By (3.3) and (3.4), we obtain a smooth periodic wave solution of (1.3):

$$v(x-ct) = \psi(x-ct) = \frac{1}{2}\left[(\psi_s+\psi_0)+(\psi_s-\psi_0)\cos\omega(x-ct)\right], \tag{3.5}$$

where $\omega = \sqrt{\frac{2}{3\beta(-\rho_2)}}$.

3.2 Suppose that $\rho_2 < 0$, $(c,g) \in \Gamma_3^L$ i.e. $\rho_2 < 0$, $c < \frac{3\rho_2-2}{2\rho_2}$, $g = g_3(c)$. In this case, we get $\psi_2 = \psi_s$, $h_2 = h_{s0} = \frac{2\eta^2(3c\rho_2-2\eta)}{3\alpha^2\rho_2^3}$. When $h = h_{s0}$, system (2.1) has a periodic orbit to the point $S_0(\psi_s, 0)$ and around the center point $A_2(\psi_1, 0)$, see Fig. 2 (2-7), (2-8). Substituting $h = h_{s0}$ into (2.2) yields the following algebraic equations for this periodic orbit

$$y = \pm\sqrt{\frac{2}{3\beta(-\rho_2)}}\sqrt{(\psi_0-\psi)(\psi-\psi_s)}\,, \tag{3.6}$$

where ψ_s, ψ_0 are given above and $\psi_0 > \psi_s$.

Similarly, substituting (3.6) into the first equation of (2.1) to integrate along this orbit, we obtain a smooth periodic wave solution of (1.3):

$$v(x-ct) = \psi(x-ct) = \frac{1}{2}\left[(\psi_s+\psi_0)-(\psi_s-\psi_0)\cos\omega(x-ct)\right], \tag{3.7}$$

where $\omega = \sqrt{\frac{2}{3\beta(-\rho_2)}}$.

3.3 Suppose that (1) $0 < \rho_2 < \frac{2}{3}$, $(c,g) \in \Gamma_2^R$ i.e. $0 < \rho_2 < \frac{2}{3}$, $c > \frac{3\rho_2-2}{2\rho_2}$, $g = g_2(c)$; (2) $\rho_2 > \frac{2}{3}$, $(c,g) \in \Gamma_2^R$ i.e. $\rho_2 > \frac{2}{3}$, $c > \frac{3\rho_2-2}{2\rho_2}$, $g = g_2(c)$. In these two cases, we get $\psi_2 = \frac{3\rho_2-2}{2\alpha\rho_2} < \psi_s$ and $h_2 = h_s = \frac{2\eta(\eta^2-3c\rho_2\eta+3g\alpha\rho_2^2)}{3\alpha^2\rho_2^3}$. When $h = h_2 = h_s$, system (2.1) has two heterclinic orbits connect three saddle points $A_2(\psi_2, 0)$ and $S_\pm(\psi_s, Y_\pm)$, see Fig. 3 (3-3), (3-4) and Fig. 5 (5-2). Substituting $h = h_s$ into (2.2) yields the following algebraic equations for these two heterclinic orbits

$$y = \pm\frac{2\alpha\rho_2\psi - 3\rho_2 + 2}{\alpha\rho_2\sqrt{6\beta\rho_2}} = \pm\frac{2}{\sqrt{6\beta\rho_2}}(\psi-\psi_2)\,. \tag{3.8}$$

Similarly, substituting (3.8) into the first equation of (2.1) to integrate along these two orbits, we obtain a non-smooth solitary cusp wave solution of peak type of (1.3):

$$v(x-ct) = \psi(x-ct) = \psi_2 + (\psi_s-\psi_2)\exp\left(-\frac{2|x-ct|}{\sqrt{6\beta\rho_2}}\right). \tag{3.9}$$

3.4 Suppose that (1) $0<\rho_2<\frac{2}{3}$, $(c,g)\in\Gamma_2^L$ i.e. $0<\rho_2<\frac{2}{3}$, $c<\frac{3\rho_2-2}{2\rho_2}$, $g=g_2(c)$; (2) $\rho_2>\frac{2}{3}$, $(c,g)\in\Gamma_2^L$ i.e. $\rho_2>\frac{2}{3}$, $c<\frac{3\rho_2-2}{2\rho_2}$, $g=g_2(c)$. In these two cases, we have $\psi_1=\frac{1}{\alpha}(\frac{3}{2}-\frac{2}{\rho_2})>\psi_s$ and $h_1=h_s=\frac{2\eta(\eta^2-3c\rho_2\eta+3g\alpha\rho_2^2)}{3\alpha^2\rho_2^3}$. When $h=h_1=h_s$, system (2.1) has two heterclinic orbits connect three saddle points $A_1(\psi_2,0)$ and $S_\pm(\psi_s,Y_\pm)$, see Fig. 3 (3-9) and Fig. 5 (5-7), (5-8). Substituting $h=h_s$ into (2.2) yields the following algebraic equations for these two heterclinic orbits

$$y=\pm\frac{-2\alpha\rho_2\psi+3\rho_2-2}{\alpha\rho_2\sqrt{6\beta\rho_2}}=\pm\frac{2}{\sqrt{6\beta\rho_2}}(\psi_1-\psi)\,. \tag{3.10}$$

Similarly, substituting (3.10) into the first equation of (2.1) to integrate along these two orbits, we obtain a non-smooth solitary cusp wave solution of valley type of (1.3):

$$v(x-ct)=\psi(x-ct)=\psi_1-(\psi_1-\psi_s)\exp\left(-\frac{2|x-ct|}{\sqrt{6\beta\rho_2}}\right)\,. \tag{3.11}$$

4 Implicit expressions of periodic wave solutions which is defined by $H(\psi,y)=0$

By the phase portraits of (2-2)-(2-6), (2-8), (2-10), (3-2), (3-4), (3-6), (5-8) and (5-10) in Fig. 2-Fig. 5, it is easy to know that there is a periodic annuli through the point $O(0,0)$. This periodic annuli is defined by $H(\psi,y)=0$. By using the elliptic function integral method, see [9, 10] and their references, we derive the implicit expressions of periodic wave solutions of (1.3). See the below computational process and results. Here, we only consider the case of $\rho_2<0$, see Fig. 2. The other cases are similar to $\rho_2<0$, see Fig. 3-Fig. 5.

4.1.1 Suppose that $\rho_2<0$, $(c,g)\in\Gamma_3^{R+}$ i.e. $\rho_2<0$, $c>\frac{3\rho_2-2}{2\rho_2}$, $g=g_3(c)>0$. In this case, there is $\psi_1=\psi_s=\frac{3\rho_2(c-1)+2}{\alpha\rho_2}$. And, when $h=0$, system (2.1) has a periodic orbit to the point $O(0,0)$ and around the center point $A_2(\psi_2,0)$, see Fig. 2 (2-3). From $H(\psi,y)=0$, we obtain the following algebraic equations for this periodic orbit

$$y=\pm\sqrt{\frac{2}{3\beta(-\rho_2)}}\sqrt{\frac{(\psi_M-\psi)(\psi_m-\psi)(\psi-0)}{\psi_s-\psi}}\,, \tag{4.1}$$

where $\psi_m=\frac{3\rho_2c+\sqrt{3(5\rho_2c-6\rho_2+4)(3\rho_2c-6\rho_2+4)}}{2\alpha\rho_2}$, $\psi_M=\frac{3\rho_2c-\sqrt{3(5\rho_2c-6\rho_2+4)(3\rho_2c-6\rho_2+4)}}{2\alpha\rho_2}$ and $0<\psi<\psi_m<\psi_s<\psi_M$.

Substituting (4.1) into the first equation of (2.1) yields

$$\pm\sqrt{\frac{\psi_s-\psi}{(\psi_M-\psi)(\psi_m-\psi)(\psi-0)}}d\psi=\sqrt{\frac{2}{3\beta(-\rho_2)}}d\xi\,. \tag{4.2}$$

Integrating (4.2) along this periodic orbit, we get

$$\int_{\psi}^{\psi_m}\sqrt{\frac{\psi_s-\psi}{(\psi_M-\psi)(\psi_m-\psi)(\psi-0)}}d\psi=\sqrt{\frac{2}{3\beta(-\rho_2)}}\int_{\xi}^{0}d\xi,\qquad \xi>0 \tag{4.3}$$

and

$$-\int_{\psi_m}^{\psi}\sqrt{\frac{\psi_s-\psi}{(\psi_M-\psi)(\psi_m-\psi)(\psi-0)}}d\psi=\sqrt{\frac{2}{3\beta(-\rho_2)}}\int_{0}^{\xi}d\xi,\qquad \xi\leq 0 \tag{4.4}$$

By (4.2) and (4.3), we obtain

$$\int_{\psi}^{\psi_m}\sqrt{\frac{\psi_s-\psi}{(\psi_M-\psi)(\psi_m-\psi)(\psi-0)}}d\psi=\sqrt{\frac{2}{3\beta(-\rho_2)}}|\xi|\,. \tag{4.5}$$

By using the elliptic integral formulas citelon12, we obtain

$$\int_{\psi}^{\psi_m}\sqrt{\frac{\psi_s-\psi}{(\psi_M-\psi)(\psi_m-\psi)(\psi-0)}}d\psi=(\psi_s-\psi_m)e_0\int_{0}^{u_0}\frac{du}{1-\alpha_0^2sn^2u}\,, \tag{4.6}$$

where $e_0=\frac{2}{\sqrt{\psi_s(\psi_M-\psi_m)}}$, $u_0=sn^{-1}\left(\sqrt{\frac{\psi_s(\psi_m-\psi)}{\psi_m(\psi_s-\psi)}},k_0\right)$, $k_0=\sqrt{\frac{\psi_m(\psi_M-\psi_s)}{\psi_s(\psi_M-\psi_m)}}$, $k_0^2<\alpha_0^2=\frac{\psi_m}{\psi_s}<1$.
And

$$\int_{0}^{u_0}\frac{du}{1-\alpha_0^2sn^2u}=\Pi(u_0,\alpha_0^2)\,. \tag{4.7}$$

By (4.5), (4.6) and (4.7), we obtain a smooth periodic wave solution of (1.3):

$$\Pi\left((sn^{-1}\sqrt{\frac{\psi_s(\psi_m-\psi)}{\psi_m(\psi_s-\psi)}},k_0),\alpha_0^2\right)=\frac{1}{(\psi_s-\psi_m)}\sqrt{\frac{\psi_s(\psi_M-\psi_m)}{6\beta(-\rho_2)}}|\xi|\,, \tag{4.8}$$

where $sn^{-1}(*\ ,\ *)$ is the inverse function of $sn(*\ ,\ *)$ which is the Jacobian elliptic function, $\Pi(*\ ,\ *)$ is Legendre's incomplete elliptic integral of the third kind.

4.1.2 Suppose that $\rho_2<0$, $(c,g)\in\Gamma_3^{R-}$ i.e. $\rho_2<0$, $c>\frac{3\rho_2-2}{2\rho_2}$, $g=g_3(c)<0$. In this case, there is $\psi_1=\psi_s=\frac{3\rho_2(c-1)+2}{\alpha\rho_2}$. And, when $h=0$, system (2.1) has a periodic orbit to the point $O(0,0)$ and around the center point $A_2(\psi_2,0)$, see Fig. 2 (2-4). From $H(\psi,y)=0$, we get the following algebraic equations for this periodic orbit

$$y=\pm\sqrt{\frac{2}{3\beta(-\rho_2)}}\sqrt{\frac{(\psi_M-\psi)(0-\psi)(\psi-\psi_m)}{\psi_s-\psi}}\,, \tag{4.9}$$

where $\psi_m,\ \psi_M$ are given above and $\psi_m < \psi < 0 < \psi_s < \psi_M$.

Corresponding to (4.9), we obtain a smooth periodic wave solution of (1.3):

$$\Pi\left((sn^{-1}\sqrt{\frac{(\psi_s-\psi_m)\psi}{\psi_m(\psi_s-\psi)}},k_1),\alpha_1^2\right)=\frac{1}{\psi_s}\sqrt{\frac{\psi_M(\psi_s-\psi_m)}{6\beta(-\rho_2)}}|\xi|\,, \tag{4.10}$$

where the computational process is similar to (4.2)-(4.8) and $k_1^2=\frac{-\psi_m(\psi_M-\psi_s)}{\psi_M(\psi_s-\psi_m)}$, $\alpha_1^2=\frac{-\psi_m}{\psi_s-\psi_m}<1$.

4.1.3 Suppose that $\rho_2<0,\ (c,g)\in\Gamma_3^{L-}$ i.e. $\rho_2<0,\ c<\frac{3\rho_2-2}{2\rho_2},\ g=g_3(c)<0$. In this case, there is $\psi_2=\psi_s=\frac{3\rho_2(c-1)+2}{\alpha\rho_2}$. And, when $h=0$, system (2.1) has a periodic orbit to the point $O(0,0)$ and around the center point $A_1(\psi_1,0)$, see Fig. 2 (2-8). From $H(\psi,y)=0$, we get the following algebraic equations for this periodic orbit

$$y=\pm\sqrt{\frac{2}{3\beta(-\rho_2)}}\sqrt{\frac{(\psi_M-\psi)(\psi-0)(\psi-\psi_m)}{\psi-\psi_s}}\,, \tag{4.11}$$

where $\psi_m,\ \psi_M$ are given above and $\psi_m<\psi_s<0<\psi<\psi_M$.

Substituting (4.11) into the first equation of (2.1) yields

$$\pm\sqrt{\frac{\psi_s-\psi}{(\psi_M-\psi)(\psi_m-\psi)(\psi-0)}}d\psi=\sqrt{\frac{2}{3\beta(-\rho_2)}}d\xi\,. \tag{4.12}$$

Integrating (4.12) along this periodic orbit, we get

$$\int_{\psi}^{\psi_M}\sqrt{\frac{\psi-\psi_s}{(\psi_M-\psi)(\psi-0)(\psi-\psi_m)}}d\psi=\sqrt{\frac{2}{3\beta(-\rho_2)}}|\xi|\,, \tag{4.13}$$

By using the elliptic integral formulas [11], we obtain

$$\int_{\psi}^{\psi_M}\sqrt{\frac{\psi-\psi_s}{(\psi_M-\psi)(\psi-0)(\psi-\psi_m)}}d\psi=(\psi_M-\psi_s)e_2\int_0^{u_2}\frac{dn^2u du}{1-\alpha_2^2sn^2u}\,, \tag{4.14}$$

where $e_2=\frac{2}{\sqrt{-\psi_m(\psi_M-\psi_s)}}$, $u_2=sn^{-1}\left(\sqrt{\frac{-\psi_m(\psi_M-\psi)}{\psi_M(\psi-\psi_m)}},k_2\right)$, $k_2^2=\frac{\psi_M(\psi_s-\psi_m)}{-\psi_m(\psi_M-\psi_s)}$, $\alpha_2^2=\frac{\psi_M}{\psi_m}<0$.

And

$$\int_0^{u_2}\frac{dn^2u du}{1-\alpha_2^2sn^2u}=\frac{1}{\alpha_2^2}[k_2^2u_2+(\alpha_2^2-k_2^2)\Pi(u_2,\alpha_2^2)]\,. \tag{4.15}$$

By (4.13), (4.14) and (4.15), we obtain a smooth periodic wave solution of (1.3):

$$k_2^2sn^{-1}(\sqrt{\frac{-\psi_m(\psi_M-\psi)}{\psi_M(\psi-\psi_m)}},k_2)+(\alpha_2^2-k_2^2)\Pi\left((sn^{-1}\sqrt{\frac{-\psi_m(\psi_M-\psi)}{\psi_M(\psi-\psi_m)}},k_2),\alpha_2^2\right)=\Omega_1|\xi|\,, \tag{4.16}$$

where $\Omega_1 = \alpha_2^2\sqrt{\frac{-\psi_m}{6\beta(-\rho_2)(\psi_M-\psi_s)}}$.

4.2.1 Suppose that $\rho_2 < 0,\ (c,g) \in III^+$ i.e. $\rho_2 < 0,\ 0 < g < g_3(c)$. In this case, when $h = 0$, system (2.1) has a periodic orbit to the point $O(0,0)$ and around the center point $A_2(\psi_2, 0)$, see Fig. 2 (2-5). From $H(\psi, y) = 0$, we get the following algebraic equations for this periodic orbit

$$y = \pm\sqrt{\frac{2}{3\beta(-\rho_2)}}\sqrt{\frac{(\psi_G-\psi)(\psi_l-\psi)(\psi-0)}{\psi_s-\psi}}, \tag{4.17}$$

where $\psi_l = \frac{3c-\sqrt{9c^3-12\alpha g}}{2\alpha}$, $\psi_G = \frac{3c+\sqrt{9c^3-12\alpha g}}{2\alpha}$ and $0 < \psi < \psi_l < \psi_s < \psi_G$.

Corresponding to (4.17), we obtain a smooth periodic wave solution of (1.3):

$$\Pi\left((sn^{-1}\sqrt{\frac{\psi_s(\psi_l-\psi)}{\psi_l(\psi_s-\psi)}}, k_3), \alpha_3^2\right) = \frac{1}{(\psi_s-\psi_l)}\sqrt{\frac{\psi_s(\psi_G-\psi_l)}{6\beta(-\rho_2)}}|\xi|, \tag{4.18}$$

where $k_3^2 = \sqrt{\frac{\psi_l(\psi_G-\psi_s)}{\psi_s(\psi_G-\psi_l)}}$, $k_3^2 = \sqrt{\frac{\psi_l(\psi_G-\psi_s)}{\psi_s(\psi_G-\psi_l)}}$, $k_3^2 < \alpha_3^2 = \frac{\psi_l}{\psi_s} < 1$.

4.2.2 Suppose that (1) $\rho_2 < 0,\ (c,g) \in III^-$ i.e. $\rho_2 < 0,\ g < g_3(c) < 0$; (2) $\rho_2 < 0,\ (c,g) \in II^-$ i.e. $\rho_2 < 0,\ g_3(c) < g < 0$. In these two cases, when $h = 0$, system (2.1) has a periodic orbit to the point $O(0,0)$ and around the center point $A_2(\psi_2, 0)$, see Fig. 2 (2-6), (2-2). From $H(\psi, y) = 0$, we get the following algebraic equations for this periodic orbit

$$y = \pm\sqrt{\frac{2}{3\beta(-\rho_2)}}\sqrt{\frac{(\psi_G-\psi)(0-\psi)(\psi-\psi_l)}{\psi_s-\psi}}, \tag{4.19}$$

where $\psi_l,\ \psi_G$ are given above and $\psi_l < \psi < 0 < \psi_s < \psi_G$.

Corresponding to (4.21), we obtain a smooth periodic wave solution of (1.3):

$$\Pi\left((sn^{-1}\sqrt{\frac{(\psi_s-\psi_l)\psi}{\psi_l(\psi_s-\psi)}}, k_4), \alpha_4^2\right) = \frac{1}{\psi_s}\sqrt{\frac{\psi_G(\psi_s-\psi_l)}{6\beta(-\rho_2)}}|\xi|, \tag{4.20}$$

where $k_4^2 = \frac{-\psi_l(\psi_G-\psi_s)}{\psi_G(\psi_s-\psi_l)}$, $\alpha_4^2 = \frac{-\psi_l}{\psi_s-\psi_l} < 1$.

4.2.3 Suppose that $\rho_2 < 0,\ (c,g) \in IV^-, \Gamma_2^{L-}, V^-$ i.e. $\rho_2 < 0,\ c < \frac{3\rho_2-2}{2\rho_2},\ g_3(c) < g < 0$. In this case, when $h = 0$, system (2.1) has a periodic orbit to the point $O(0,0)$ and around the center point $A_1(\psi_1, 0)$, see Fig. 2 (2-10). From $H(\psi, y) = 0$, we get the following algebraic equations for this periodic orbit

$$y = \pm\sqrt{\frac{2}{3\beta(-\rho_2)}}\sqrt{\frac{(\psi_G-\psi)(\psi-0)(\psi-\psi_l)}{\psi-\psi_s}}, \tag{4.21}$$

where $\psi_l,\ \psi_G$ are given above and $\psi_l < \psi_s < 0 < \psi < \psi_G$.

Corresponding to (4.21), we obtain a smooth periodic wave solution of (1.3):

$$k_5^2 sn^{-1}(\sqrt{\frac{-\psi_l(\psi_G-\psi)}{\psi_G(\psi-\psi_l)}}, k_5) + (\alpha_5^2 - k^2)\Pi\left((sn^{-1}\sqrt{\frac{-\psi_l(\psi_G-\psi)}{\psi_G(\psi-\psi_l)}}, k_5), \alpha_5^2\right) = \Omega_2|\xi|\,, \tag{4.22}$$

where $\Omega_2 = \alpha_5^2\sqrt{\frac{-\psi_l}{6\beta(-\rho_2)(\psi_G-\psi_s)}}$, $k_5^2 = \frac{\psi_G(\psi_s-\psi_l)}{-\psi_l(\psi_G-\psi_s)}$, $\alpha_5^2 = \frac{\psi_G}{\psi_m} < 0$.

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