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Dynamics of a Nonlinear Difference Equation[2]

ABSTRACT. In this paper the dynamics for a third-order rational difference equation is considered. The rule for the trajectory structure of solutions of this equation is clearly described out. The successive lengths of positive and negative semicycles of nontrivial solutions of this equation are found to occur periodically with prime period 7. And the rule is $3^+, 2^-, 1^+, 1^-$ in a period. By utilizing the rule, the positive equilibrium point of the equation is verified to be globally asymptotically stable.

KEY WORDS. rational difference equation, semicycle, cycle length, global asymptotic stability.

1 Introduction and Preliminaries

The study of rational difference equations of order greater than one is quite challenging and rewarding because some prototypes for the development of the basic theory of the global behavior of nonlinear difference equations of order greater than one come from the results for rational difference equations. For this, see, for example, [1, 2] and the papers in the journal of " Advances in Difference Equations änd the references cited therein. Furthermore, there have not been any effective general methods to deal with the global behavior of rational difference equations of order greater than one so far. Therefore, the study of rational difference equations of order greater than one is worth further consideration.

Recently, M. R. S. Kulenović et al [3], Tim Nesemann [4] and Yang et.al [6, 7] investigated the global asymptotic stability of some second or higher order rational difference equations.

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From the known work, one can see that it is extremely difficult to understand thoroughly the trajectory structure of solutions of rational difference equations although they have simple forms (or expressions). One can refer to [1]-[10], especially [1, 2] for examples to illustrate this.

In this paper we consider the following third - order rational difference equation

$$x_{n+1} = \frac{x_{n-1} + x_{n-2} + a}{x_{n-1}x_{n-2} + 1 + a}, \quad n = 0, 1, 2, \cdots, \tag{1.1}$$

where $a \in [0, \infty)$ and the initial values $x_{-2}, x_{-1}, x_0 \in (0,\ \infty)$,

Mainly, by analyzing the rule for the length of semi-cycle to occur successively, we describe clearly out the rule for the trajectory structure of its solutions and further derive the global asymptotic stability of positive equilibrium of equation (1.1). Whereas, it is extremely difficult to use those methods in the known literature, such as [1]-[7], to obtain the rule of trajectory structure of solutions of equation (1.1).

It is easy to see that the positive equilibrium $\bar{x}$ of equation (1.1) satisfies

$$\bar{x} = \frac{2\bar{x} + a}{\bar{x}^2 + 1 + a},$$

from which one can see that equation (1.1) has a unique positive equilibrium $\bar{x} = 1$.

Here, for convenience of readers, we give some corresponding definitions, also review some results which will be useful in our investigation of the behavior of solutions of Eq. (1.1). Let I be some interval of real numbers and let $f : I \times I \to I$ be a continuously differentiable function. Then for every group of initial conditions $x_{-2}, x_{-1}, x_0 \in I$, the difference equation

$$x_{n+1} = f(x_{n-1},\ x_{n-2}), \quad n = 0, 1, 2, \cdots, \tag{E}$$

has a unique solution $\{x_n\}_{n=-2}^{\infty}$.

A point $\bar{x}$ is called an equilibrium point of Eq. (E) if $\bar{x} = f(\bar{x},\ \bar{x})$. That is, $x_n = \bar{x}$, for $n \geq 0$, is a solution of Eq. (E), or, equivalently, $\bar{x}$ is a fixed point of f.

Definition 1.1 *Let $\bar{x}$ be an equilibrium point of Eq. (E).*

(a) *The equilibrium $\bar{x}$ is called stable if, for every $\epsilon > 0$, there exists $\delta > 0$ such that if $x_{-2}, x_{-1}, x_0 \in I$ and $|x_{-2} - \bar{x}| + |x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \delta$, then $|x_n - \bar{x}| < \epsilon$ for all ≥ 1.*

(b) *The equilibrium $\bar{x}$ is called locally asymptotically stable if it is stable and if there exists $\gamma > 0$ such that if $x_{-2}, x_{-1}, x_0 \in I$ and $|x_{-2} - \bar{x}| + |x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \gamma$, then $\lim_{n\to\infty} x_n = \bar{x}$.*

(c) *The equilibrium* $\bar{x}$ *is called a global attractor if*

$$\lim_{n\to\infty} x_n = \bar{x} \quad \textit{for any} \quad x_{-2}, x_{-1},\ x_0 \in I.$$

(d) *The equilibrium* $\bar{x}$ *is called globally asymptotically stable if it is stable and is a global attractor.*

(e) *The equilibrium* $\bar{x}$ *is called unstable if it is not stable.*

(f) *The equilibrium* $\bar{x}$ *is called a repeller if there exists* $\gamma > 0$ *such that for* $x_{-2}, x_{-1}, x_0 \in I$ *and* $|x_{-2} - \bar{x}| + |x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \gamma$*, there exists* $N \geq -2$ *such that* $|x_N - \bar{x}| \geq \gamma$.

Clearly, a repeller is an unstable equilibrium.

Let

$$p = \frac{\partial f(\bar{x},\ \bar{x})}{\partial u} \qquad \text{and} \qquad q = \frac{\partial f(\bar{x},\ \bar{x})}{\partial v},$$

where $f(u,\ v)$ is the function in Eq. (E) and $\bar{x}$ is an equilibrium of the equation. Then the equation

$$y_{n+1} = py_{n-1} + qy_{n-2}, \qquad n = 0, 1, \cdots$$

is called the linearized equation associated with Eq. (E) about the equilibrium point $\bar{x}$.

Definition 1.2 *A positive semicycle of a solution* $\{x_n\}_{n=-2}^{\infty}$ *of equation* (1.1) *consists of a "string" of terms* $\{x_l, x_{l+1}, \cdots, x_m\}$*, all greater than or equal to the equilibrium* $\bar{x}$*, with* $l \geq -2$ *and* $m \leq \infty$ *such that*

$$\textit{either} \quad l = -2 \quad \textit{or} \quad l > -2 \quad \textit{and} \quad x_{l-1} < \bar{x}$$

and

$$\textit{either} \quad m = \infty \quad \textit{or} \quad m < \infty \quad \textit{and} \quad x_{m+1} < \bar{x}.$$

A negative semicycle of a solution $\{x_n\}_{n=-2}^{\infty}$ *of equation* (1.1) *consists of a "string" of terms* $\{x_l, x_{l+1}, \cdots, x_m\}$*, all less than* $\bar{x}$*, with* $l \geq -2$ *and* $m \leq \infty$ *such that*

$$\textit{either} \quad l = -2 \quad \textit{or} \quad l > -2 \quad \textit{and} \quad x_{l-1} \geq \bar{x}$$

and

$$\textit{either} \quad m = \infty \quad \textit{or} \quad m < \infty \quad \textit{and} \quad x_{m+1} \geq \bar{x}.$$

The length of a semicycle is the number of the total terms contained in it.

Definition 1.3 *A solution* $\{x_n\}_{n=-2}^{\infty}$ *of equation* (1.1) *is said to be eventually trivial if* x_n *is eventually equal to* $\bar{x} = 1$*; Otherwise, the solution is said to be nontrivial.*

A solution $\{x_n\}_{n=-2}^{\infty}$ *of equation* (1.1) *is said to be eventually positive(negative) if* x_n *is eventually great (less) than* $\bar{x} = 1$*;*

For the other concepts in this paper, see [1, 2].

2 Main Results and Their Proofs

In this section we will formulate our main results in this paper, that is, with respect to the nontrivial solutions, oscillation and non-oscillation and global asymptotic stability for equation (1.1).

2.1 Nontrivial solution

Theorem 2.1 *A positive solution* $\{x_n\}_{n=-3}^{\infty}$ *of equation* (1.1) *is eventually trivial if and only if*

$$(x_{-2}-1)(x_{-1}-1)(x_0-1)=0\,. \tag{2.1}$$

Proof: Sufficiency. Assume that (2.1) holds. Then it follows from equation (1.1) that the following conclusions hold.

i) If $x_{-2}=1$, then $x_n=1$ for $n\geq 3$;

ii) If $x_{-1}=1$, then $x_n=1$ for $n\geq 1$;

iii) If $x_0=1$, then $x_n=1$ for $n\geq 2$.

Necessity. Conversely, assume that

$$(x_{-2}-1)(x_{-1}-1)(x_0-1)\neq 0\,. \tag{2.2}$$

Then one can show that

$$x_n\neq 1 \quad \text{for any} \quad n\geq 1.$$

Assume the contrary that for some $N\geq 1$,

$$x_N=1 \quad \text{and that} \quad x_n\neq 1 \quad \text{for} \quad -2\leq n\leq N-1\,. \tag{2.3}$$

Clearly,

$$1=x_N=\frac{x_{N-2}+x_{N-3}+a}{x_{N-2}x_{N-3}+1+a}\,,$$

which implies $(x_{N-2}-1)(x_{N-3}-1)=0$, which contradicts (2.3).

Remark 2.2 Theorem 2.1 actually demonstrates that a positive solution $\{x_n\}_{n=-2}^{\infty}$ of equation (1.1) is eventually nontrivial if and only if $(x_{-2}-1)(x_{-1}-1)(x_0-1)\neq 0$. Therefore, if a solution $\{x_n\}_{n=-2}^{\infty}$ is nontrivial, then $x_n\neq 1$ for $n\geq -2$.

Next we consider some properties of nontrivial solutions of equation (1.1).

2.2 Oscillation and Non-oscillation

Before stating the oscillation and non-oscillation of solutions, we need the following key lemma.

Lemma 2.3 *For any nontrivial positive solution $\{x_n\}_{n=-2}^{\infty}$ of equation (1.1), the following conclusions are true:*

(a) $(x_{n+1}-1)(x_{n-1}-1)(x_{n-2}-1)<0$ *for* $n \geq 0$*;*

(b) $(x_{n+1}-x_{n-1})(x_{n-1}-1)<0$ *for* $n \geq 0$*;*

(c) $(x_{n+1}-x_{n-2})(x_{n-2}-1)<0$ *for* $n \geq 0$.

Proof: In view of equation (1.1), we can see that

$$x_{n+1}-1=-\frac{(x_{n-1}-1)(x_{n-2}-1)}{x_{n-1}x_{n-2}+1+a}, \quad n=0,1,2,\cdots$$

and

$$x_{n+1}-x_{n-1}=\frac{(1-x_{n-1})[a+x_{n-2}(1+x_{n-1})]}{x_{n-1}x_{n-2}+1+a}, \quad n=0,1,2,\cdots,$$

from which inequalities (a) and (b) follow. The proof for inequality (c) is similar to that of inequality (b). So the proof is complete.

Theorem 2.4 *There exist non-oscillatory solutions of equation (1.1), which must be eventually negative. There don't exist eventually positive non-oscillatory solutions of equation (1.1).*

Proof: Consider a solution of equation (1.1) with $x_{-2}<1$, $x_{-1}<1$ and $x_0<1$. We then know from Lemma 2.3 (a) that $x_n<1$ for $n \geq -2$. So, this solution is just a non-oscillatory solution and furthermore eventually negative.

Suppose that there exist eventually positive non-oscillatory solutions of equation (1.1). Then, there exists a positive integer N such that $x_n>1$ for $n \geq N$. Thereout, for $n \geq N+2$, $(x_{n+1}-1)(x_{n-1}-1)(x_{n-2}-1)>0$. This contradicts Lemma 2.3 (a). So, There don't exist eventually positive non-oscillatory solutions of equation (1.1), as desired.

We now analyze the rule for trajectory structure of strictly oscillatory solutions of equation (1.1).

Theorem 2.5 *Let $\{x_n\}_{n=-2}^{\infty}$ be a strictly oscillatory solution of equation (1.1). Then the rule for the lengths of positive and negative semi-cycles of this solution to successively occur is* $\cdots, 3^+, 2^-, 1^+, 1^-, 3^+, 2^-, 1^+, 1^-, 3^+, 2^-, 1^+, 1^-, 3^+, 2^-, 1^+, 1^-, 3^+, 2^-, 1^+, 1^-, 3^+, 2^-, 1^+, 1^-, \cdots$.

Proof: By Lemma 2.3 (a), one can see that the length of a negative semi-cycle is not larger than 2, whereas, the length of a positive semi-cycle is at most 3. Based on the strictly oscillatory character of the solution, we see, for some integer $p \geq 0$, one of the following two cases must occur:

Case 1: $x_{p-2} > 1, x_{p-1} < 1, x_p > 1$;

Case 2: $x_{p-2} > 1, x_{p-1} < 1, x_p < 1$.

If Case 1 occurs, it follows from Lemma 2.3 (a) that $x_{p+1} > 1$, $x_{p+2} > 1$, $x_{p+3} < 1$, $x_{p+4} < 1, x_{p+5} > 1$, $x_{p+6} < 1$, $x_{p+7} > 1$, $x_{p+8} > 1, x_{p+9} > 1$, $x_{p+10} < 1$, $x_{p+11} < 1$, $x_{p+12} > 1$, $x_{p+13} < 1$, $x_{p+14} > 1$, $x_{p+15} > 1$, $x_{p+16} > 1$, $x_{p+17} < 1$, $x_{p+18} < 1$, $x_{p+19} > 1$, $x_{p+20} < 1$, $x_{p+21} > 1$, $x_{p+22} > 1$, $x_{p+23} > 1$, $x_{p+24} < 1$, $x_{p+25} < 1$, $x_{p+26} > 1$, $x_{p+27} < 1$, $x_{p+28} > 1$, $x_{p+29} > 1$, $x_{p+30} > 1$, $x_{p+31} < 1$, $x_{p+32} < 1$, $x_{p+33} > 1$, $x_{p+34} < 1$, $x_{p+35} > 1$, $x_{p+36} > 1$, $x_{p+37} > 1$, $x_{p+38} < 1$, $x_{p+39} < 1$, $x_{p+40} > 1$, $x_{p+41} > 1$, $\cdots$, which means that the rule for the lengths of positive and negative semi-cycles of the solution of equation (1.1) to successively occur is $\cdots, 3^+, 2^-, 1^+, 1^-, 3^+, 2^-, 1^+, 1^-, 3^+, 2^-, 1^+, 1^-, 3^+, 2^-, 1^+, 1^-, 3^+, 2^-, 1^+, 1^-, 3^+, 2^-, 1^+, 1^-, \cdots$.

If Case 2 happens, then Lemma 2.3 (a) tells us that $x_{p+1} > 1$, $x_{p+2} < 1$, $x_{p+3} > 1$, $x_{p+4} > 1$, $x_{p+5} > 1$, $x_{p+6} < 1$, $x_{p+7} < 1$, $x_{p+8} > 1, x_{p+9} < 1$, $x_{p+10} > 1$, $x_{p+11} > 1$, $x_{p+12} > 1$, $x_{p+13} < 1$, $x_{p+14} < 1$, $x_{p+15} > 1$, $x_{p+16} < 1$, $x_{p+17} > 1$, $x_{p+18} > 1$, $x_{p+19} > 1$, $x_{p+20} < 1$, $x_{p+21} < 1$, $x_{p+22} > 1$, $x_{p+23} < 1$, $x_{p+24} > 1$, $x_{p+25} > 1$, $x_{p+26} > 1$, $x_{p+27} < 1$, $x_{p+28} < 1$, $x_{p+29} > 1$, $x_{p+30} < 1$, $x_{p+31} > 1$, $x_{p+32} > 1$, $x_{p+33} > 1$, $x_{p+34} < 1$, $x_{p+35} < 1$, $x_{p+36} > 1$, $x_{p+37} < 1$, $x_{p+38} > 1$, $x_{p+39} > 1$, $x_{p+40} > 1$, $x_{p+41} < 1$, $x_{p+42} < 1$, $x_{p+43} < 1$, $x_{p+44} > 1$, $\cdots$. This shows the rule for the numbers of terms of positive and negative semicycles of the solution of equation (1.1) to successively occur still is $\cdots 3^+, 2^-, 1^+, 1^-, 3^+, 2^-, 1^+, 1^-, 3^+, 2^-, 1^+, 1^-, 3^+, 2^-, 1^+, 1^-, 3^+, 2^-, 1^+, 1^-, 3^+, 2^-, 1^+, 1^-, \cdots$.

Hence, the proof is complete.

Remark 2.6 It is well known to all that the two cases in the proof of Theorem 2.5 are caused by the perturbation of the initial around the equilibrium point. So, the theorem 2.5 actually indicates that the perturbation of the initial values may lead to the variation of the trajectory structure rule for the solutions of equation (1.1).

2.3 Global Asymptotic Stability

First, we consider the local asymptotic stability for unique positive equilibrium point $\bar{x}$ of equation (1.1). We have the following results.

Theorem 2.7 *Then the positive equilibrium of equation (1.1) is locally asymptotically stable.*

Proof: The linearized equation of equation (1.1) about the positive equilibrium $\bar{x} = 1$ is

$$y_{n+1} = 0 \times y_n + 0 \times y_{n-1} + 0 \times y_{n-2}, \quad n = 0, 1, \cdots .$$

By virtue of [2, Remark 1.3.7], $\bar{x}$ is locally asymptotically stable. The proof is complete.

We now are in a position to study the global asymptotic stability of positive equilibrium point $\bar{x}$.

Theorem 2.8 *The positive equilibrium point of equation (1.1) is globally asymptotically stable.*

Proof: We must prove that the positive equilibrium point $\bar{x}$ of equation (1.1) is both locally asymptotically stable and globally attractive. Theorem 2.7 has shown the local asymptotic stability of $\bar{x}$. Hence, it remains to verify its global attractivity. That is, it suffices to prove that every solution $\{x_n\}_{n=-3}^{\infty}$ of equation (1.1) converges to $\bar{x}$ as $n \to \infty$, i.e., to prove

$$\lim_{n\to\infty} x_n = \bar{x} = 1 . \tag{2.4}$$

We can divide the solutions into two kinds of types.

i) Trivial solutions;

ii) Nontrivial solutions.

If the solution is a trivial solution, then it is obvious for (2.4) to hold because $x_n = 1$ eventually.

If the solution is a nontrivial solution, then we can further divide the solution into two cases.

a) Non-oscillatory solution;

b) Oscillatory solution.

If case a) happens, then it follows from Theorem 2.4 that the solution is actually an eventually negative one. Accordingly, there exists a positive integer N such that $x_n < 1$ for $n \geq N$. From Lemma 2.3 (b), we know that two subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ of the solution $\{x_n\}_{n=-2}^{\infty}$ are increasing and have upper bound 1. So, the limits $\lim_{n\to\infty} x_{2n}$ and $\lim_{n\to\infty} x_{2n+1}$ exist and are finite, denoted by L and M, respectively. It is clear from equation (1.1) that

$$x_{2n+1} = \frac{x_{2n-1} + x_{2n-2} + a}{x_{2n-1}x_{2n-2} + 1 + a} \qquad \text{and} \qquad x_{2n+2} = \frac{x_{2n} + x_{2n-1} + a}{x_{2n}x_{2n-1} + 1 + a}.$$

Taking limits on both sides of the above equalities produces

$$M = \frac{M + L + a}{LM + 1 + a} \qquad \text{and} \qquad L = \frac{L + M + a}{LM + 1 + a}.$$

Solving these two equations, we get $L = M = 1$. This manifests that (2.4) is valid for non-oscillatory solutions.

Thus, it suffices to prove that (2.4) holds for the solution to be oscillatory, i.e., case b) occurs.

Consider now $\{x_n\}$ to be strictly oscillatory about the positive equilibrium point $\bar{x}$ of equation (1.1). By virtue of Theorem 2.5, we know that the rule for the lengths of positive and negative semi-cycles which occur successively is $\cdots, 3^+, 2^-, 1^+, 1^-, 3^+, 2^-, 1^+, 1^-, 3^+, 2^-, 1^+, 1^-, \cdots$.

For simplicity, for some nonnegative integer p, we denote by $\{x_p, x_{p+1}, x_{p+2}\}^+$ the terms of a positive semi-cycle of length three, followed by $\{x_{p+3}, x_{p+4}\}^-$ a negative semi-cycle with length two, then a positive semi-cycle $\{x_{p+5}\}^+$ and a negative semi-cycle $\{x_{p+6}\}^-$, and so on. Namely, the rule for the positive and negative semi-cycles of the solution to occur successively can be periodically expressed as follows:
$\{x_{p+7n}, x_{p+7n+1}, x_{p+7n+2}\}^+$, $\{x_{p+7n+3}, x_{p+7n+4}\}^-$, $\{x_{p+7n+5}\}^+$, $\{x_{p+7n+6}\}^-$, $n = 0, 1, \cdots$.

From Lemma 2.3 (b) and (c) the following results can be derived straightforward:

(i) $x_{p+7n+7} < x_{p+7n+5} < x_{p+7n+2} < x_{p+7n}$;

(ii) $x_{p+7n+6} > \max\{x_{p+7n+4},\ x_{p+7n+3}\}$.

From i), one can see that $\{x_{p+7n}\}$ is monotonically decreasing and has lower bound 1. So, the limit $\lim_{n\to\infty} x_{p+7n}$ exists and is finite, denoted by L. Moreover, it follows from i) that

$$\lim_{n\to\infty} x_{p+7n+5} = \lim_{n\to\infty} x_{p+7n+2} = \lim_{n\to\infty} x_{p+7n} = L\,. \tag{2.5}$$

According to the taking values of variable in positive and negative semi-cycles and equation (1.1), we also have

$$x_{p+7n+3} = \frac{x_{p+7n+1} + x_{p+7n} + a}{x_{p+7n+1}x_{p+7n} + 1 + a} > \frac{1}{x_{p+7n+1}}\,,$$

and

$$x_{p+7n+4} = \frac{x_{p+7n+2} + x_{p+7n+1} + a}{x_{p+7n+2}x_{p+7n+1} + 1 + a} > \frac{1}{x_{p+7n+2}}\,.$$

So, we further obtain

$$x_{p+7n+7} = \frac{x_{p+7n+5} + x_{p+7n+4} + a}{x_{p+7n+5}x_{p+7n+4} + 1 + a} < \frac{1}{x_{p+7n+4}} < x_{p+7n+2} \tag{2.6}$$

and

$$x_{p+7n+8} = \frac{x_{p+7n+6} + x_{p+7n+5} + a}{x_{p+7n+6}x_{p+7n+5} + 1 + a} < \frac{1}{x_{p+7n+6}} < \frac{1}{x_{p+7n+3}} < x_{p+7n+1}\,. \tag{2.7}$$

we see by (2.5) and (2.6) that $\lim_{n\to\infty} x_{p+7n+4} = 1/L$.

(2.7) indicates that $\{x_{p+7n+1}\}$ is monotonically decreasing and has lower bound 1. So, the limit $\lim_{n\to\infty} x_{p+7n+1}$ exists and is finite, denoted by M. Furthermore, it is clear from (2.7) that

$$\lim_{n\to\infty} x_{p+7n+6} = \lim_{n\to\infty} x_{p+7n+3} = \frac{1}{M}. \tag{2.8}$$

Now it's sufficient for us to verify that $L = M = 1$. To this end, noting

$$x_{p+7n+6} = \frac{x_{p+7n+4} + x_{p+7n+3} + a}{x_{p+7n+4}x_{p+7n+3} + 1 + a},$$

taking the limits on both sides of the above equality, we obtain $\frac{1}{M} = \frac{1/L+1/M+a}{1/L\times 1/M+1+a}$. Solving this equation, we can derive $M = 1$.

By taking the limits on both sides of

$$x_{p+7n+5} = \frac{x_{p+7n+3} + x_{p+7n+2} + a}{x_{p+7n+3}x_{p+7n+2} + 1 + a},$$

we have $L = \lim_{n\to\infty} x_{p+7n+5} = 1$.

Up to this, we have shown $\lim_{n\to\infty} x_{p+7n+k} = 1, k = 0, 1, 2\cdots, 6$, which indicates $\lim_{n\to\infty} x_n = 1$. So, the proof for Theorem 2.8 is complete.

2.4 Rule of Trajectory Structure

Finally, we can sum the general rule for the trajectory structure of solutions of equation (1.1) as follows.

Theorem 2.9 *The rule for the trajectory structure of any solution of equation (1.1) is as follows.*

I). *The solution is either eventually trivial or;*

II). *The solution is eventually nontrivial and further either*

II-1). *The solution is eventually negative non-oscillatory or;*

II-2). *The solution is strictly oscillatory and moreover, the successive lengths for positive and negative semi-cycles occur periodically with prime period 7 and in a period the rule is* $3^+, 2^-, 1^+, 1^-$.

The positive equilibrium point of equation (1.1) is a global attractor of all its solutions.

It follows from the results stated previously that Theorem 2.9 is true.

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