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New identities for Bell's polynomials New approaches

ABSTRACT. In this work we suggest a new approach to the determination of new identities for Bell's polynomials, based on the Lagrange inversion formula, and the binomial sequences. This approach allows the easy recovery of known identities and deduction of some new identities including these polynomials.

KEY WORDS. Bell's polynomials, Bell's numbers, Lagrange inversion formula, binomial sequences.

1 Introduction

Using a proof by recurrence, Salim KHELIFA and Yves CHERRUAULT gave the following identity on Bell's polynomials [3].

Theorem 1 *For n and $k \in \mathbb{N}^*$,$k \leq n$, it holds*

$$B_{n,k}\left(1^0, 2^1, 3^2, \ldots\right) = \binom{n-1}{k-1} n^{n-k}. \tag{1}$$

This identity allowed the authors to demonstrate a new theorem of convergence for the Adomian decomposition method [4], but the proof is excessively long (7 pages) and consequently it requires another proof. Here we propose two new identities, with shorter proofs. The first uses the Lagrange inversion formula, having as an immediate consequence the identity (1), and the second uses binomial sequences, begetting new identities, and the known identities in the literature.

Definition 2 *The Bell polynomials are the polynomials $B_{n,k}(x_1, x_2, \ldots)$ in an infinite number of variables $x_1, x_2, \ldots$, defined by (see [2], p. 133)*

$$\frac{1}{k!}\left(\sum_{m \geq 1} x_m \frac{t^m}{m!}\right)^k = \sum_{n \geq k} B_{n,k} \frac{t^n}{n!}, \quad k = 0, 1, 2, \ldots \tag{2}$$

Their exact expression is (see [2], p. 134)

$$B_{n,k}\left(x_1,x_2,\ldots\right)=\sum_{\pi(n)}\frac{n!}{k_1!k_2!\cdots(1!)^{k_1}(2!)^{k_2}\cdots}x_1^{k_1}x_2^{k_2}\ldots,$$

where $\pi(n)$ denotes a partition of n, with $k_1+2k_2+\ldots=n$; k_i is of course, the number of parts of size i. Also $k_1+k_2+\ldots=k$ is the number of parts in the partition.

2 Main results

2.1 Method based on the Lagrange inversion formula

Let f be an analytic function about the origin such that $f(0)\neq 0$ and for n and $m\in\mathbb{N}$ let

$$f_n(m)=\begin{cases} D^n\left[f(w)\right]^m\big|_{w=0} & if\; n\geq 1\\ (f(0))^m & if\; n=0\end{cases}$$

where D is the differential operator $\dfrac{\mathrm{d}}{\mathrm{d}\,w}$.

Theorem 3 *For n and $k\in\mathbb{N}^*$, $k\leq n$, it holds*

$$B_{n,k}\left(f_0(1),f_1(2),f_2(3),\ldots\right)=\binom{n-1}{k-1}f_{n-k}(n)\;.$$

Proof: For $z\in\mathbb{C}$, let us consider the equation of the unknown $w\in\mathbb{C}$,

$$w-zf(w)=0\,.$$

This equation admits a unique solution $w=g(z)$ around the origin (see [1], p. 234) and for any analytic function F around the origin we have by Lagrange inversion formula

$$F(g(z))=F(0)+\sum_{n\geq 1}D^{n-1}\left\{F'(w)\left[f(w)\right]^n\right\}\Big|_{w=0}\frac{z^n}{n!}\,. \tag{3}$$

If we choose $F(w)=w$, we get from (3)

$$\begin{aligned} g(z)&=\sum_{n\geq 1}D^{n-1}\left[f(w)\right]^n\big|_{w=0}\frac{z^n}{n!}\\ &=\sum_{n\geq 1}f_{n-1}(n)\frac{z^n}{n!}\,. \end{aligned}$$

Thus from (2) we have

$$\begin{aligned}\frac{1}{k!}\left(g\left(z\right)\right)^{k} &= \frac{1}{k!}\left(\sum_{n\geq 1} f_{n-1}(n)\frac{z^{n}}{n!}\right)^{k}\\ &= \sum_{n\geq k} B_{n,k}\left(f_{0}(1), f_{1}(2), f_{2}(3),\ldots\right)\frac{z^{n}}{n!}\,.\end{aligned}$$

On the other hand, if we choose $F\left(w\right)=\frac{w^{k}}{k!}$, we get by (3)

$$\begin{aligned}\frac{1}{k!}\left(g\left(z\right)\right)^{k} &= \frac{1}{(k-1)!}\sum_{n\geq 1} D^{n-1}\left\{w^{k-1}\left[f\left(w\right)\right]^{n}\right\}\Big|_{w=0}\frac{z^{n}}{n!}\\ &= \frac{1}{(k-1)!}\sum_{n\geq 1} D^{n-1}\left\{w^{k-1}\sum_{j\geq 0} D^{j}\left[f\left(w\right)\right]^{n}\Big|_{w=0}\frac{w^{j}}{j!}\right\}\Bigg|_{w=0}\frac{z^{n}}{n!}\\ &= \frac{1}{(k-1)!}\sum_{n\geq 1} D^{n-1}\left\{\sum_{j\geq 0}\frac{f_{j}\left(n\right)}{j!}w^{j+k-1}\right\}\Bigg|_{w=0}\frac{z^{n}}{n!}\\ &= \frac{1}{(k-1)!}\sum_{n\geq 1} D^{n-1}\left\{\sum_{j\geq k-1}\frac{f_{j-k+1}\left(n\right)}{(j-k+1)!}w^{j}\right\}\Bigg|_{w=0}\frac{z^{n}}{n!}\\ &= \frac{1}{(k-1)!}\sum_{n\geq k}\frac{(n-1)!}{(n-k)!}f_{n-k}\left(n\right)\frac{z^{n}}{n!}\\ &= \sum_{n\geq k}\binom{n-1}{k-1}f_{n-k}\left(n\right)\frac{z^{n}}{n!}\,.\end{aligned}$$

□

Corollary 4 *Let $a\in\mathbb{R}$. We have for all n and $k\in\mathbb{N}^{*}$, $k\leq n$,*

$$B_{n,k}\left(\left(1a\right)^{0},\left(2a\right)^{1},\left(3a\right)^{2},\ldots\right)=\binom{n-1}{k-1}\left(an\right)^{n-k}\,.$$

Proof: We have just to apply Theorem 3 by putting $f\left(w\right)=e^{aw}$, that gives

$$f_{n}\left(m\right)=\begin{cases}\left(am\right)^{n} & \text{if } n\geq 1\\ 1 & \text{if } n=0\end{cases}$$

□

Remark 5

1) If we choose $a=1$ we find the identity (1).

2) It is obvious that the identity of Corollary 4 is not the only consequence of Theorem 3, because it depends on the choice of f. If, for instance, we choose the function $f(w) = 1 + aw$, we get

$$f_n(m) = \begin{cases} a^n [m]_n & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \end{cases}$$

where $[m]_n = m(m-1)\cdots(m-n+1)$. Thus we have

$$B_{n,k}\left(1!a^0, 2!a^1, 3!a^2, \ldots\right) = a^{n-k}\binom{n-1}{k-1}\frac{n!}{k!}.$$

If we choose :

△ $a = 1$, we recover the known identity

$$B_{n,k}(1!, 2!, 3!, \ldots) = \binom{n-1}{k-1}\frac{n!}{k!}.$$

△ $a = 0$, we get

$$B_{n,k}(1, 0, 0, \ldots) = 0, \quad \textit{except} \;\; B_{n,n} = 1.$$

2.2 Method based on binomial sequences

A sequence of definite functions $(\varphi_n(x))_n$ on a subset I of $\mathbb{R}$ is called binomial if,

$$\varphi_n(x+y) = \sum_{k=0}^{n}\binom{n}{k}\varphi_k(x)\,\varphi_{n-k}(y), \forall x, y \in I.$$

Theorem 6 *Let $(\varphi_n(x))_n$ be a binomial sequence defined on I, $\mathbb{N} \subseteq I \subseteq \mathbb{R}$, with $\varphi_0 \neq 0$. Then for all n and $k \in \mathbb{N}^*$, $k \leq n$, we have*

$$B_{n,k}(\varphi_0(1), 2\varphi_1(1), 3\varphi_2(1), \ldots) = \binom{n}{k}\varphi_{n-k}(k).$$

Proof: Let by Φ_x denote the exponential generating function associated to the sequence $(\varphi_n(x))_n$, i.e.

$$\Phi_x(t) = \sum_{n \geq 0}\varphi_n(x)\frac{t^n}{n!}.$$

(*We suppose, of course, that the radius of convergence satisfies $R > 0$.*)

The sequence $(\varphi_n(x))_n$ is binomial, then we have, from Cauchy product

$$\Phi_{x+y}(t) = \Phi_x(t) \cdot \Phi_y(t), \forall x, y \in I.$$

Hence

$$\Phi_k(t) = (\Phi_1(t))^k, \forall k \in \mathbb{N}^*. \tag{4}$$

It comes then, on the one hand

$$
\begin{aligned}
\frac{1}{k!}\left(t\Phi_1(t)\right)^k &= \frac{1}{k!}\left(\sum_{n\geq 0}\varphi_n(1)\frac{t^{n+1}}{n!}\right)^k \\
&= \frac{1}{k!}\left(\sum_{n\geq 1} n\varphi_{n-1}(1)\frac{t^n}{n!}\right)^k \\
&= \sum_{n\geq k} B_{n,k}\left(\varphi_0(1), 2\varphi_1(1), 3\varphi_2(1), \ldots\right)\frac{t^n}{n!}\,.
\end{aligned}
$$

On the other hand by (4), we have

$$
\begin{aligned}
\frac{1}{k!}\left(t\Phi_1(t)\right)^k &= \frac{1}{k!}t^k\Phi_k(t) \\
&= \frac{1}{k!}\sum_{n\geq 0}\varphi_n(k)\frac{t^{n+k}}{n!} \\
&= \sum_{n\geq k}\binom{n}{k}\varphi_{n-k}(k)\frac{t^n}{n!}\,.
\end{aligned}
$$

□

Application

Let $S(n,k)$ denote the Stirling number of the second kind, and put

$$B_n(x) = \sum_{k=0}^{n} S(n,k)\, x^k\,.$$

The sequence $\left(B_n(x)\right)_n$ is defined in $\mathbb{R}$, where $B_0(x) \equiv 1$ and $B_n(1) = B_n$, the Bell numbers.

Corollary 7 *We have*

$$B_{n,k}\left(B_0, 2B_1, 3B_2, \ldots\right) = \binom{n}{k}\sum_{j=0}^{n-k} S(n-k, j)\, k^j\,.$$

Proof: It is well known and easily verified that

$$\sum_{n=0}^{+\infty} B_n(x)\frac{z^n}{n!} = \exp\left(x\left(e^z - 1\right)\right)\,. \tag{5}$$

In fact, from (5) it follows that

$$\sum_{n=0}^{\infty} B_n(x+y)\frac{z^n}{n!} = \exp\left((x+y)\left(e^z-1\right)\right) = \sum_{n=0}^{+\infty} B_n(x)\frac{z^n}{n!}\sum_{n=0}^{+\infty} B_n(y)\frac{z^n}{n!}\,.$$

Therefore

$$B_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} B_k(x) B_{n-k}(y) \cdot$$

Thus the sequence $(B_n(x))_n$ is binomial and the result is proved by means of Theorem 6. $\square$

Remark 8 Corollary 7 is not the only consequence of Theorem 6. It all depends on the choice of binomial sequence. If we choose for example the binomial sequence defined on $\mathbb{R}$ by $\varphi_n(x) = x^n$, we recover the known identity

$$B_{n,k}(1,2,3,\ldots) = \binom{n}{k} k^{n-k}.$$

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