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On the Difference Equation $x_{n+1} = (\beta x_n + \gamma x_{n-1})/(\gamma x_n + \beta x_{n-1})$

ABSTRACT. The difference equation in the title is solved by means of functions, which can be represented as composed functions of the exponential function and a function being odd with respect to one or two arbitrary parameters. In the case $\beta = 1/4$, $\gamma = 3/4$ there is given a conjecture concerning a solution of a new type. A second conjecture concerns the existence of asymptotically 3-periodic solutions. Though the difference equation is of second order, we point out singular cases where three initial values can be prescribed.

KEY WORDS. Nonlinear difference equations, odd functions, asymptotic behaviour, 3-periodic solutions, three initial values, conjectures.

Rational difference equations of second order are systematically investigated in the book M. R. S. Kulenović and G. Ladas [6], where also various applications of these equations are pointed out.

Here, we consider the special case

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-1}}{\gamma x_n + \beta x_{n-1}} \tag{1}$$

with $\beta^2 \neq \gamma^2$ and integer n. According to the classification in [6] it belongs to the type $(2, 2)$ or to the type $(1, 1)$. In the case of positive coefficients a detailed stability and semicycle analysis of the positive solutions of (1) is carried out in [6, Chapter 6.9].

In this paper we study arbitrary solutions of (1) with respect to its structure and its asymptotic behaviour. These solutions can have zeros, but not at consecutive points.

It can easily be seen that (1) has the following property:

Proposition 1 *If $x_n = w_n$ is a special solution of (1), then $x_n = \frac{1}{w_n}$ is also a solution of (1).*

Here we allow that a solution equals infinity, but not at consecutive points. This means that the forbidden set of the initial values (x_{-1}, x_0) is only $\{(0,0), (\infty,\infty)\}$, cf. [6, pp. 2.17].

Without loss of generality we use for the coefficients the normalization

$$\beta + \gamma = 1\,, \tag{2}$$

so that we can eliminate γ, and it follows $\beta \neq \frac{1}{2}$. Equation (1) has the equilibrium $\overline{x} = 1$, and about this equilibrium it has the linearized equation

$$y_{n+1} = (2\beta - 1)(y_n - y_{n-1})$$

with the characteristic equation

$$\lambda^2 + (1 - 2\beta)(\lambda - 1) = 0\,. \tag{3}$$

At first we study solutions with one arbitrary parameter, and afterwards with two parameters. Moreover, we consider the exceptional case $\beta = \frac{1}{3}$, we ask for asymptotically 3-periodic solutions, and finally we point out cases with three given initial values.

One parameter. Let $\lambda = z$ be a solution of (3) with $|z| \neq 1$. Then (1) possesses a solution of the form

$$x_n = 1 + \sum_{j=1}^{\infty} c_j a^j z^{nj} \tag{4}$$

with $c_1 = 1$, arbitrary a and $|c_j| \leq M^{j-1}$ for a certain constant M, cf. [2]. Hence, (4) converges for

$$|az^n| < \frac{1}{M}\,, \tag{5}$$

i. e. for sufficiently large n in case of $|z| < 1$, and for sufficiently large $-n$ in case of $|z| > 1$. The series in (4) is simultaneously an asymptotic expansion as $n \to +\infty$ resp. $n \to -\infty$.

Proposition 2 *If x_n is the solution (4) of (1), then $c_2 = \frac{1}{2}$ and, under the condition (5), the solution $\frac{1}{x_n}$ has the expansion (4) with $-a$ instead of a. Moreover, under the sharpening of (5)*

$$|az^n| < \frac{1}{M+1} \tag{6}$$

there exists a function $f_n(a)$ being odd in a such that

$$x_n = \exp(f_n(a))\,. \tag{7}$$

Proof: Under the condition (5) it follows from (4) that

$$\frac{1}{x_n} = 1 - az^n + (1-c_2)a^2z^{2n} + \dots\,. \tag{8}$$

According to Proposition 1 the left-hand side is also a solution of (1), and the right-hand side must have the form (4), i. e. the expansion in (8) is the expansion (4) with $-a$ instead of a.

Let $x_n = F_n(a)$ and therefore $\frac{1}{x_n} = F_n(-a)$. Under the condition (6) it is $|F_n(a)-1| < 1$, so that $f_n(a) = \ln(F_n(a))$ exists, and this function satisfies $f_n(a) = -f_n(-a)$. From this and

$$f_n(a) = \ln(1 + az^n + c_2a^2z^{2n} + \dots) = az^n + \left(c_2 - \frac{1}{2}\right)a^2z^{2n} + \dots$$

it follows that $c_2 = \frac{1}{2}$ ■

The functions $F_n(a)$ and $f_n(a)$ are holomorphic under the condition (6). Hence, the analytic continuation of (7) remains a solution of (1).

Example 1 In the case $\beta = \frac{1}{4}$ we can choose the solution $\lambda = \frac{1}{2}$ of (3) and obtain by means of the DERIVE system

$$x_n = \exp\left\{\frac{a}{2^n} + \frac{1}{108}\frac{a^3}{2^{3n}} + \frac{19}{71280}\frac{a^5}{2^{5n}} + \frac{68437}{6951510720}\frac{a^7}{2^{7n}} + \dots\right\}$$

and for the coefficient in (4)

$$c_3 = \frac{19}{108}\,,\ c_4 = \frac{11}{216}\,,\ c_5 = \frac{943}{71280}\,,\ c_6 = \frac{4159}{1283040}\,,\ c_7 = \frac{764869}{993072960}\,.$$

Note that (1) in this example corresponds to [6, (6.66)] with $p = 1/3$ and $q = 3$, however, neither [6, (6,67)] nor [6, (6,68)] are satisfied.

Two parameters. Let $\lambda = z$ and $\lambda = s$ be two different solutions of (3), and assume that $\lambda = z^js^\ell$ is no solution of (3) for all non-negative integers j, ℓ with $j+\ell \geq 2$, then (1) has also a solution of the form

$$x_n = 1 + \sum_{1\leq j+\ell} c_{j\ell}\, a^jz^{nj}b^\ell s^{n\ell} \tag{9}$$

with $c_{10} = c_{01} = 1$, arbitrary a, b, and $|c_{j\ell}| \leq M^{j+\ell-1}$ for a certain constant M, which is convergent for $|z| < 1$, $|s| < 1$ and n large, $|z| > 1$, $|s| > 1$ and $-n$ large, cf. [2] in the real, and [3] in the complex case, and also for $|z| < 1 < |s|$, if $|a|$ and $|b|$ are sufficiently small and

$$\frac{\ln(|a|M)}{-\ln|z|} < n < \frac{-\ln(|b|M)}{\ln s}\,.$$

Instead of a detailed analysis we only mention that with some more effort Proposition 2 can be generalized to these cases, and that also the analytic continuation can be applied.

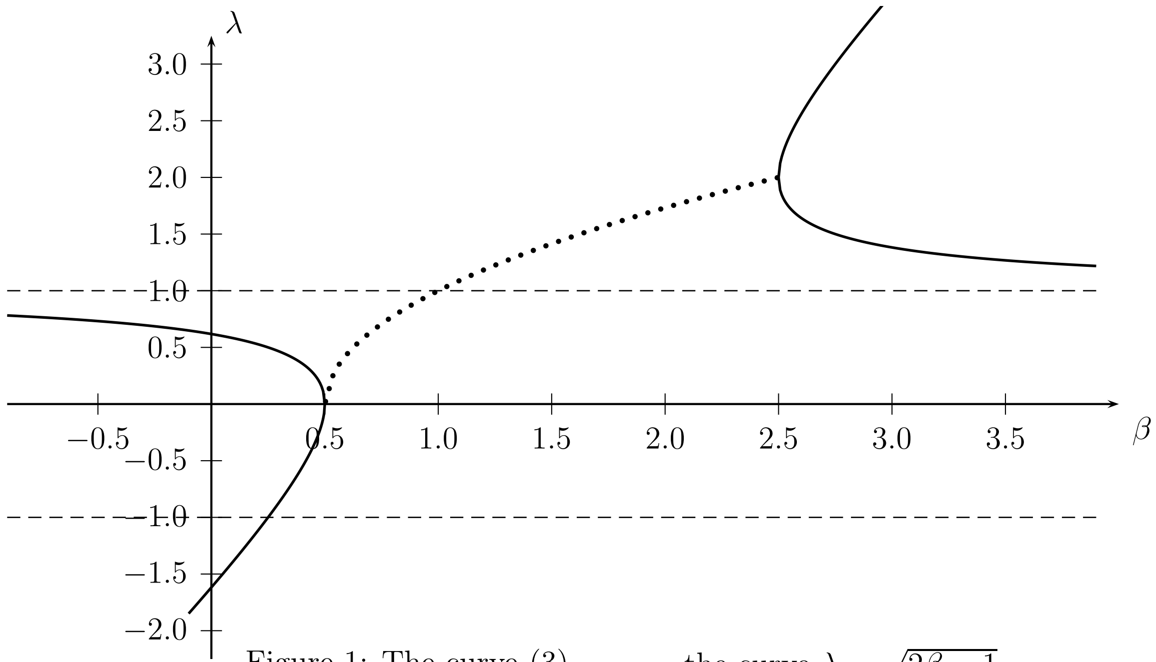

Figure 1: The curve (3) ——— , the curve $\lambda = \sqrt{2\beta - 1}$ ······ and the straight lines $\lambda = \pm 1$ - - - -

Figure 2 shows the real branches of the curve (3), and for $\frac{1}{2} < \beta < \frac{5}{2}$ the curve $\lambda = \sqrt{2\beta - 1}$ $(= |z| = |s|)$ concerning the complex branches. For $\frac{1}{4} < \beta < \frac{1}{2}$ there are two real, and for $\frac{1}{2} < \beta < 1$ two complex solutions with $|\lambda| < 1$. For $1 < \beta < \frac{5}{2}$ there are two complex, and for $\frac{5}{2} < \beta$ two real solutions with $|\lambda| > 1$. For $\beta < \frac{1}{4}$ there is one solution with $\frac{1}{2} < \lambda < 1$ and one solution with $\lambda < -1$.

In the two cases, where (1) belongs to the type $(1, 1)$, we have elementary solutions of the form

$$x_n = \exp\{az^n + bs^n\}\,, \tag{10}$$

which visualize Proposition 2, and which can be expanded in the form (9) for all a, b and all n. The first case is $\beta = 0$ with $z = -\frac{1}{2} + \frac{\sqrt{5}}{2}$, $s = -\frac{1}{2} - \frac{\sqrt{5}}{2}$ and therefore $|z| < 1 < |s|$, cf. [6, Chapter 3.3]. The second case is $\beta = 1$ with $z = e^{i\pi/3}$, $s = e^{-i\pi/3}$ and $|z| = |s| = 1$, cf. [6, Chapter 3.2] and [3, Example 4]. Of course, this solution (10) can also be written in the real form

$$x_n = \exp\left\{c\,\cos\frac{n\pi}{3} + d\,\sin\frac{n\pi}{3}\right\}$$

with arbitrary real parameters c, d.

In some of the excluded cases with $z^k s^\ell = 1$ for some integers k, ℓ, there also exist solutions of the form (9), however, with polynomial coefficients, cf. [2].

Example 2 Let us come back to Example 1 with $\beta = \frac{1}{4}$, but concerning the second solution $\lambda = -1$ of (5). In this case equation (1) should be expected to have a 2-periodic solution, however, it turns out that such a solution must be the constant $x_n = 1$.

Hence, we expect that (1) has a solution of the form

$$x_n = 1 + u_n + (-1)^n v_n \tag{11}$$

with functions u_n, v_n tending to zero as $n \to \infty$. In order to find the asymptotic behaviour of these functions, we use the heuristic method from [1], assuming $n = t$ as a continuous variable, replacing u_{n+k} according to Taylor approximately by $u + ku'$, assuming that $u' = o(u)$, and proceeding analogously with v_n. Then (1) can be replaced approximately by

$$(1 + u + u' - (-1)^n(v + v'))(4 + 4u - u' + (-1)^n(2v + v')) = 4 + 4u - 3u' + (-1)^n(3v' - 2v)\,.$$

Comparing coefficients of $(-1)^n$ we obtain

$$\begin{aligned} 6u' + 4u + 4u^2 + 3uu' - u'^2 - 2v^2 - 3vv' - v'^2 &= 0\,, \\ -6v' - 2uv - 3uv' + 3vu' + 2u'v' &= 0\,. \end{aligned}$$

Cancelling all terms of smaller order and dividing by 2, these equations reduce to

$$\begin{aligned} 2u &= v^2\,, \\ -3v' &= uv\,. \end{aligned}$$

Integration yields

$$u = \frac{3}{2t}\,, \quad v = \pm\sqrt{\frac{3}{t}}\,, \tag{12}$$

disregarding the constant of integration. A further analysis shows that we can expect an improvement of (11) with (12) (taking the sign +) in the form

$$x_n = 1 + \frac{3}{2n} + (a \ln n + b)\frac{1}{n^2} + (-1)^n \left(\sqrt{\frac{3}{n}} + (c \ln n + d)\frac{1}{\sqrt{n^3}} \right) \tag{13}$$

up to smaller terms as $n \to \infty$. By means of the DERIVE system we find

$$a = -\frac{3}{8}\,, \quad b = \sqrt{3}d - \frac{9}{8}\,, \quad c = -\frac{\sqrt{3}}{8}\,, \tag{14}$$

where d is an arbitrary constant, cf. [7, Remark 1], and

$$x_{n+1} - \frac{x_n + 3x_{n-1}}{3x_n + x_{n-1}} \sim -\frac{3}{32}\,\frac{\ln^2 n}{n^3}\,.$$

Conjecture 1 *There exists a solution x_n of (1) such that the expansion (13) with (14) is valid up to $o\left(\frac{1}{n^2}\right)$.*

However, similarly as in [8, Conjecture 1] we cannot prove it. Obviously, a solution x_n having the finite asymptotic expansion (13) as $n \to \infty$ is oscillating about the equilibrium 1. From (13) it follows that $\frac{1}{x_n}$ has the asymptotic expansion (13) with $-\sqrt{3}$, $-d$ instead of $\sqrt{3}$, d and that

$$x_n = \exp\left\{(-1)^n \left(\sqrt{\frac{3}{n}} - \frac{1}{8}\left(\sqrt{3}\ln n - 8d + 4\sqrt{3}\right)\frac{1}{\sqrt{n^3}} + \frac{3}{80}\left(5\sqrt{3}\ln n - 40d + 18\sqrt{3}\right)\frac{1}{\sqrt{n^5}}\right)\right\}$$

both up to smaller terms as $n \to \infty$, where the argument of the exponential function is an odd function of $\sqrt{n}$.

Asymptotically 3-periodic solutions. Since 2-periodic solutions were already investigated in [6, Section 6.9.1], we ask for 3-periodic solutions. It can easily be seen that

$$\ldots, -1, -1, 1, -1, -1, 1, \ldots \tag{15}$$

is such a solution of (1) for all considered coefficients.

In connection with (15) it makes sense to ask for asymptotically 3-periodic solutions of the form

$$\left.\begin{array}{lcrcl} x_{3n-1} & = & -1 & + & a\lambda^n \\ x_{3n} & = & -1 & + & b\lambda^n \\ x_{3n+1} & = & 1 & + & c\lambda^n \end{array}\right\} \tag{16}$$

up to $O(\lambda^{2n})$, cf. [1, Section 5]. Substituting (16) into (1) we find

$$\begin{aligned} (1 + c\lambda^n)(\gamma(-1 + b\lambda^n) + \beta(-1 + a\lambda^n)) &= \beta(-1 + b\lambda^n) + \gamma(-1 + a\lambda^n) \\ (-1 + a\lambda^{n+1})(\gamma(1 + c\lambda^n) + \beta(-1 + b\lambda^n)) &= \beta(1 + c\lambda^n) + \gamma(-1 + b\lambda^n) \\ (-1 + b\lambda^{n+1})(\gamma(-1 + a\lambda^{n+1}) + \beta(1 + c\lambda^n)) &= \beta(-1 + c\lambda^{n+1}) + \gamma(1 + c\lambda^n) \end{aligned}$$

again up to $O(\lambda^{2n})$, and, comparing the coefficients of λ^n, it follows

$$\left.\begin{array}{rcrcrcl} \delta a & - & \delta b & - & c & = & 0 \\ \delta\lambda a & + & b & + & c & = & 0 \\ \lambda a & - & \delta\lambda b & + & c & = & 0 \end{array}\right\} \tag{17}$$

using the notation $\delta = \beta - \gamma = 2\beta - 1$ according to (2). The homogeneous system (17) has a non-trivial solution, if its determinant

$$\begin{vmatrix} \delta & -\delta & -1 \\ \delta\lambda & 1 & 1 \\ \lambda & -\delta\lambda & 1 \end{vmatrix} = \delta^2\lambda^2 + (2\delta^2 - \delta + 1)\lambda + \delta$$

vanishes, i.e. with the notation $\eta = \frac{1}{\delta}$, if

$$\lambda^2 + (2 - \eta + \eta^2)\lambda + \eta = 0\,. \tag{18}$$

If the condition (18) is satisfied, then (17) has the solution

$$b = \frac{1+\lambda}{1-\eta}\,a\,, \qquad c = \frac{\eta+\lambda}{\eta(\eta-1)}\,a \tag{19}$$

with arbitrary a.

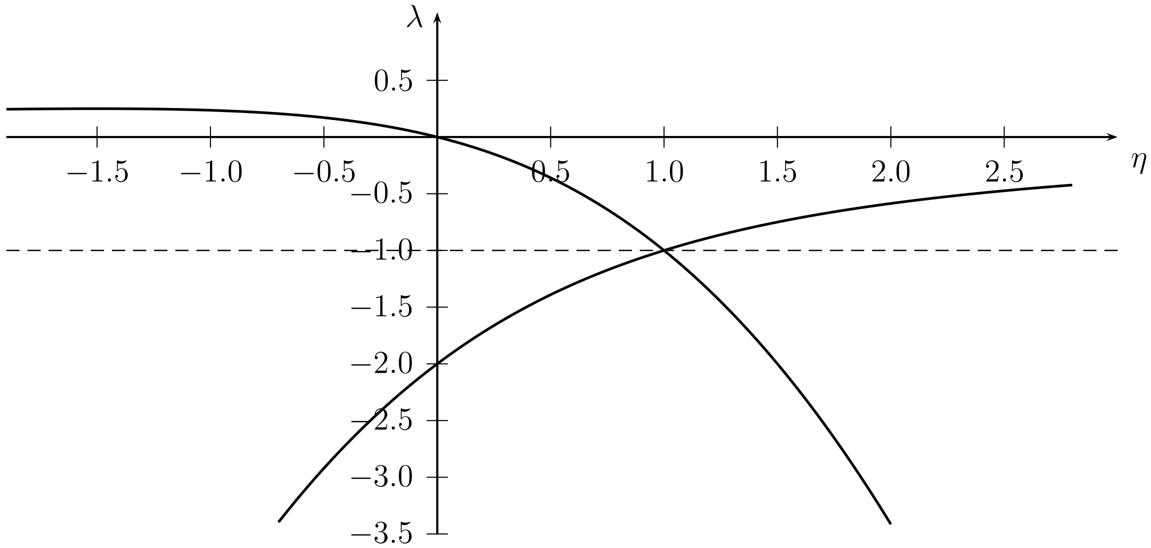

Figure 2: The curve (18) ——, and the straight line $\lambda = -1$ - - - -

Figure 2 of the curve (18) shows that for all $\eta \neq 1$, i.e. $\beta \neq 1$, there exists one solution of (18) with $|\lambda| < 1$, and another one with $\lambda < -1$. Since for all finite β it is $\eta \neq 0$, it follows that always $\lambda \neq 0$. We expect that for $\beta \neq 1$ there are two solutions of (1) with the asymptotic behaviour (16) for $n \to \infty$ in case of $|\lambda| < 1$ resp. for $n \to -\infty$ in case of $|\lambda| > 1$, and with regard to (4) we make the

Conjecture 2 *For $\beta \neq 1$ there exist two solutions of (1) such that x_{3n-1}, x_{3n}, x_{3n+1} can be expanded into power series in λ^n with the first terms (16) and an arbitrary a. The other parameters are determined by (18), (19) and $\eta = \frac{1}{2\beta-1}$.*

Conjecture 2 comes true in the case $\beta = 0$, where $\eta = -1$. Namely, denoting the 3-periodic solution (15) by ε_n, then (1) has besides of (10) also the solution $\varepsilon_n x_n$, cf. [4, Setion 4.2] as well as [5, p. 175], and the two solutions of (18) are z^3 and s^3 with z and s from (10).

Let us remark that the curve (18) attains its maximum $\lambda = \frac{1}{4}$ at $\eta = -\frac{3}{2}$, where $b = \frac{a}{2}$, $c = -\frac{a}{3}$ and $\beta = \frac{1}{6}$.

Three initial values. Let x_n be a solution of Equation (1) for non-negative integers n. In order to continue this solution to negative n, it is appropriate to write (1) in the form

$$x_{n-1} = x_n \frac{\gamma x_{n+1} - \beta}{\gamma - \beta x_{n+1}}\,, \tag{20}$$

which is singular for $x_{n+1} = \frac{\gamma}{\beta}$. In the case that the initial values (x_{-1}, x_0) are neither $\left(0, \frac{\gamma}{\beta}\right)$ nor $\left(\infty, \frac{\beta}{\gamma}\right)$, the value x_{-2} is uniquely determined by means of (20). Otherwise, this value x_{-2} remains indetermined and can be prescribed arbitrarily, disregarding the countably many cases where the continuation by means of (20) satisfies $x_{n-1} \in \{0, \infty\}$ for some negative n. In particular, we have to avoid the case $x_{-2} = x_{-1}$ in order to avoid the (shifted) forbidden set.

Analogously, if the initial values (x_{-1}, x_0) are given in such a way that for a negative integer m the pair (x_{m-1}, x_m) is either $\left(0, \frac{\gamma}{\beta}\right)$ or $\left(\infty, \frac{\beta}{\gamma}\right)$, then we can choose x_{m-2} as a third arbitrary initial value subject to an analogous restriction as before.

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