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# Iterative Processes with Random Errors for Fixed Point of Φ-Pseudocontractive Operator*

ABSTRACT. The purpose of this paper is to introduce Φ-pseudo-contractive operators—a class of operators which is much more general than the important class of strongly pseudo-contractive operators and $\phi$-strongly pseudocontractive operators, and to study problems of approximating fixed points by Ishikawa and Mann iterative processes with random errors for Φ-pseudocontractive operators. As applications, the iterative approximative methods for the solution of equation with Φ-accretive operator are obtained. The results presented in this paper improve, generalize and unify the corresponding results of Chang [3]-[4], Chidume [5]-[10], Deng [12], Ding [13]-[14], Liu [16], Osilike [18], Xu [19], Zhou [20].

KEY WORDS AND PHRASES. Duality mapping, Mann iteration sequence, Ishikawa iteration sequence, Φ -pseudocontractive operator.

## 1 Introduction and Preliminaries

Throughout this paper, we assume that $X$ is a real Banach space with dual $X^*$, $(\cdot,\ \cdot)$ denotes the generalized duality pairing. The mapping $J : X \to 2^{X^*}$ defined by

$$Jx = \{j \in X^* : (x, j) = \|x\|\|j\|, \|j\| = \|x\|\}^{[1]} \quad \forall x \in X \tag{1.1}$$

is called the normalized duality mapping.

We recall the following two iterative processes due to Ishikawa [15] and Mann [17], respectively.

(a) Let $K$ be a nonempty convex subset of $X$, and $T : K \longrightarrow K$ be a mapping. For any given $x_0 \in K$ the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n, \quad y_n = (1 - \beta_n)x_n + \beta_n T x_n, \quad (n \geq 0)$$

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is called Ishikawa iteration sequence, where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0,1]$ satisfying some conditions.

(b) In particular, if $\beta_n = 0$ for all $n \geq 0$ in (a), then $\{x_n\}$ defined by

$$x_0 \in K, \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \qquad (n \geq 0)$$

is called the Mann iteration sequence.

The consideration of error terms is an important part of any theory of iteration methods. For this reason, Xu [19] introduced the following definitions.

(A) Let $K$ be a nonempty convex subset of $X$ and $T : K \longrightarrow K$ a mapping. For any given $x_0 \in K$ the sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + \beta_n T y_n + \gamma_n u_n, \quad y_n = \hat{\alpha}_n x_n + \hat{\beta}_n T x_n + \hat{\gamma}_n v_n \quad (n \geq 0) \tag{1.2}$$

is called Ishikawa iteration sequence with random errors. Here $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in $K$; $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\hat{\alpha}_n\}, \{\hat{\beta}_n\}$ and $\{\hat{\gamma}_n\}$ are six sequences in $[0,1]$ satisfying

$$\alpha_n + \beta_n + \gamma_n = \hat{\alpha}_n + \hat{\beta}_n + \hat{\gamma}_n = 1, \quad for \;\; all \;\; n \geq 0.$$

(B) In particular, if $\hat{\beta}_n = \hat{\gamma}_n = 0$ for all $n \geq 0$ in (A), the $\{x_n\}$ defined by

$$x_0 \in K, \quad x_{n+1} = \alpha_n x_n + \beta_n T x_n + \gamma_n u_n, \quad (n \geq 0) \tag{1.3}$$

is called Mann iteration sequence with random errors .

Note that the Ishikawa and Mann iterative processes are all special cases of the Ishikawa and Mann iterative processes with random errors.

Now, we introduce $\Phi$-pseudocontractive operators as follows.

**Definition 1.1** *Let $K$ be nonempty subset of $X$. An operator $T : K \to X$ is said to be $\Phi$-pseudocontractive, if there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ and $j(x-y) \in J(x-y)$ such that*

$$(Tx - Ty,\ j(x-y)) \leq \|x-y\|^2 - \Phi(\|x-y\|) \quad \forall x, y \in K. \tag{1.4}$$

*An operator $A : K \to X$ is said to be $\Phi$-accretive, if*

$$(Ax - Ay,\ j(x-y)) \geq \Phi(\|x-y\|) \ \ \forall x, y \in K. \tag{1.5}$$

**Remark 1.1** Obvious, if a $\Phi$-pseudocontractive operator has a fixed point then it is unique.

The pseudocontractive operator is intimately connected with accretive operator [11]. It is easy to verify that the operator $T$ is $\Phi$-pseudoaccretive if and only if $I - T$ is $\Phi$-accretive where $I$ is a identity mapping on $X$. Hence, the mapping theory for accretive operators is intimately connected with the fixed point theory for pseudocontraction operators.

We like to point out: every $\phi$-strongly pseudocontractive operator must be the $\Phi$-pseudocontractive operator with $\Phi$:$[0, \infty) \to [0, \infty)$ defined by $\Phi(s) = \phi(s)s$, and every strongly pseudocontractive operator is $\phi$-strongly pseudocontractive with $\phi : [0, \infty) \to [0, \infty)$ defined by $\phi(s) = ks$ where $k \in (0,\ 1)$.

In 1994, Chidume proved a related result that deals with the Ishikawa iterative approximation of the fixed point for the class of Lipschitz strictly pseudocontractive mappings in uniformly smooth Banach space. At the same time, he put forth an open problem: It is not known whether or not the Ishikawa iteration method converges for a continuous strongly pseudocontractive mapping. Recently, this open problem has been studied extensively by researchers (see, for example[3-4, 6-10, 12-14, 18-20]) in the case of $T$ is strongly pseudocontractive or $\phi$-strongly pseudocontractive operators respectively.

The objective of this paper is to introduce the $\Phi$-pseudocontractive operators — a class of operators which is much more general than the important class of strongly pseudocontractive operators and $\phi$-strongly pseudocontractive operators, and to study problems of approximating fixed point by Ishikawa and Mann iterative processes with random errors for $\Phi$-pseudocontractive operators. We will prove that the answer of Chidume's open problem is affirmative if $X$ is an arbitrary Banach space and $T : K \to K \subset X$ is uniformly continuous $\Phi$-quasicontractive. furthermore, if $X$ is an uniformly smooth Banach space and $T$ may be not continuous, the answer of Chidume's open problem also is affirmative. As applications, the iterative approximation methods for the solution of equation with $\Phi$-accretive operator are obtained. The results presented in this paper improve, generalize and unify results of Chang [3]-[4], Chidume [5]-[10], Deng [12], Ding [13]-[14], Liu [16], Osilike [18], Xu [19], Zhou [20].

The following two Lemmas play crucial roles in the proofs of our main results.

**Lemma 1.1 ([4])** *If $X$ be a real Banach space then there exists $j(x + y) \in J(x + y)$ such that*

$$\|x + y\|^2 \leq \|x\|^2 + 2(y,\ j(x + y)) \quad \forall x,\ y \in X. \tag{1.6}$$

**Lemma 1.2 ([2](Browder))** *$X$ is uniformly smooth(equivalently $X^*$ is uniformly convex) Banach space if and only if $J$ is single-valued and uniformly continuous on any bounded subset of $X$.*

## 2 The Convergence Theorems in Arbitrary Banach Space

If $X$ is an arbitrary real Banach space with dual $X^*$, we can prove following theorems.

**Theorem 2.1** *Let $X$ be an arbitrary real Banach space with dual $X^*$ and $K \subset X$ a nonempty bounded convex subset. Let $T : K \to K$ be an uniformly continuous $\Phi$-pseudocontractive mapping. Suppose the Ishikawa iteration sequence $\{x_n\}$ with random errors be defined by (1.2) with parameters*

(i) $\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \hat{\beta}_n = \lim_{n\to\infty} \hat{\gamma}_n = 0$ *and* $\sum_{n=0}^{+\infty} \beta_n = +\infty$;

(ii) $\gamma_n = o(\beta_n)$.

*If $F(T) \neq \emptyset$ then for arbitrary $x_0 \in K$, $\{x_n\}$ converges strongly to unique fixed point of $T$.*

**Proof:** From Remark 1.1, we have that $F(T) = \{q\}$. Putting $M = \sup\{\|x\| : x \in K\} + \|q\|$. Since $\|y_n - x_{n+1}\| = \|(\hat{\alpha}_n - \alpha_n)x_n + \hat{\beta}_n T x_n + \hat{\gamma}_n v_n - \beta_n T y_n - \gamma_n u_n)\| \to 0(as\ n \to \infty)$, therefore,

$$e_n := \|Ty_n - Tx_{n+1}\| \to 0(as\ n \to \infty)$$

by the uniformly continuity of $T$.

Let $2\sigma = \inf\{\|x_{n+1} - q\| : n \geq 0\}$. If $\sigma > 0$, then $\Phi(\|x_{n+1} - q\|) > \Phi(\sigma)$ for all $n \geq 0$. From the conditions (i) and (ii) there exists an integer $N_0 > 0$ such that

$$0 \leq \gamma_n, \beta_n \leq \frac{1}{6} \quad and \quad o(\beta_n) \leq \beta_n \Phi(\sigma) \quad \forall n \geq N_0. \tag{2.7}$$

By (1.4), (1.6) and (2.7) we have

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|\alpha_n(x_n - q) + \beta_n(Ty_n - q) + \gamma_n(u_n - q)\|^2 \\ &\leq \alpha_n^2\|x_n - q\|^2 + 2\beta_n(Ty_n - q,\ j(x_{n+1} - q)) \\ &\quad +2\gamma_n(u_n - q,\ j(x_{n+1} - q)) \\ &\leq \alpha_n^2\|x_n - q\|^2 + 2\beta_n(Ty_n - Tx_{n+1},\ j(x_{n+1} - q)) \\ &\quad +2\beta_n(Tx_{n+1} - q,\ j(x_{n+1} - q)) + 2M^2\gamma_n \\ &\leq (1 - \beta_n - \gamma_n)^2\|x_n - q\|^2 + 2\beta_n\|x_{n+1} - q\|^2 \\ &\quad -2\beta_n\Phi(\|x_{n+1} - q\|) + 2M\beta_n e_n + 2M^2\gamma_n \\ &\leq \|x_n - q\|^2 + \tfrac{3}{2}M^2\beta_n^2 + 3M\beta_n e_n + 3M^2\gamma_n \\ &\quad -2\Phi(\|x_{n+1} - q\|)\beta_n \\ &= \|x_n - q\|^2 + o(\beta_n) - 2\Phi(\|x_{n+1} - q\|)\beta_n \end{aligned} \tag{2.8}$$

for all $n \geq N_0$. It follows from (2.8) that

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 + o(\beta_n) - 2\Phi(\sigma)\beta_n \leq \|x_n - q\|^2 + -\Phi(\sigma)\beta_n$$

for all $n \geq N_0$. By induction, we obtain

$$\Phi(\sigma)\sum_{j=N}^{+\infty}\beta_j \leq \|x_N - q\|^2 \leq M^2. \tag{2.9}$$

(2.9) is in contradiction with $\sum_{j=0}^{+\infty}\beta_j = +\infty$. From this contradiction, we get $\sigma = 0$. Therefore, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \to q$ as $j \to \infty$. For any given $\varepsilon > 0$ there exists an integer $j_0 \geq N_0$ such that $\|x_{n_j} - q\| < \varepsilon$ for all $j \geq j_0$. If $j_0$ is fixed, we will prove that $\|x_{n_{j_0}+k} - q\| < \varepsilon$ for all integers $k \geq 1$.

The proof is by induction. For $k = 1$, suppose $\|x_{n_{j_0}+1} - q\| \geq \varepsilon$. It follows from (2.8) and $\Phi(\|x_{n_{j_0}+1} - q\|) \geq \Phi(\varepsilon)$ that

$$\varepsilon^2 \leq \|x_{n_{j_0}+1} - q\|^2 \leq \|x_{n_{j_0}} - q\|^2 + o(\beta_{n_{j_0}}) - 2\beta_{n_{j_0}}\Phi(\varepsilon) \leq \|x_{n_{j_0}} - q\|^2 < \varepsilon^2.$$

It is a contradiction. Hence, $\|x_{n_{j_0}+1} - q\| < \varepsilon$ holds for $k = 1$. Assume now that $\|x_{n_{j_0}+p} - q\| < \varepsilon$ for some integer $p > 1$. We prove $\|x_{n_{j_0}+p+1} - q\| < \varepsilon$. Again, assuming the contrary, Using (2.8), $\Phi(\|x_{n_{j_0}+p+1} - q\|) > \Phi(\varepsilon)$ and (2.7), as above, it leads to a contradiction as follows

$$\varepsilon^2 \leq \|x_{n_{j_0}+p+1} - q\|^2 \leq \|x_{n_{j_0}+p} - q\|^2 + o(\beta_{n_{j_0}+p}) - 2\beta_{n_{j_0}+p}\Phi(\varepsilon) \leq \|x_{n_{j_0}+p} - q\|^2 < \varepsilon^2$$

Where $n_{j_0} + p \geq n_{j_0} \geq j_0 \geq N_0$. Therefore, $\|x_{n_{j_0}+k} - q\| < \varepsilon$ holds for all integers $k \geq 1$, so that $x_{n_{j_0}+k} \to q$ as $k \to \infty$.
The Proof is completed.□

**Remark 2.1** Theorem 2.1 improves a number of results (for example, Theorem 3.4 of [7] and Theorem 4 of [11]). A prototype for $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, $\{\hat{\alpha}_n\}, \{\hat{\beta}_n\}$ and $\{\hat{\gamma}_n\}$ in Theorem 2.1 is

$$\alpha_n = \frac{n^2 + 3n + 1}{(n+2)^2}, \quad \beta_n = \frac{1}{n+2}, \quad \gamma_n = \frac{1}{(n+2)^2}, \quad \hat{\alpha}_n = \frac{n+1}{n+3}$$

and

$$\hat{\beta}_n = \hat{\gamma}_n = \frac{1}{n+3} \quad \forall n \geq 0.$$

**Theorem 2.2** *Let $X$, $K$ and $T$ be as in Theorem 2.1. If $q$ is a fixed point of $T$ in $K$ and the Mann iteration sequence $\{x_n\}$ is defined by (1.3) with parameters*

(i) $\lim_{n\to\infty}\beta_n = 0$ *and* $\sum_{n=0}^{+\infty}\beta_n = +\infty$;

(ii) $\gamma_n = o(\beta_n)$ *then $\{x_n\}$ converges strongly to unique fixed point of $T$.*

**Theorem 2.3** *Suppose that $K \subset X$ is a nonempty bounded convex subset with $K + K \subseteq K$ and $A : K \to K$ is an uniformly continuous $\Phi$-accretive operator. For any given $f \in K$ the equation $Ax = f$ has unique solution in $K$.*

**Proof:** We define $S : K \to K$ by $Sx = f + x - Ax$ for all $x \in K$. It is easy to see that $S$ is uniformly continuous $\Phi$-pseudocontractive. Clearly, $q$ is a fixed point of $S$ in $K$ if and only if that $q$ is a solution of the equation $Ax = f$. It follows from Theorem 2.1 or Theorem 2.2 above that the equation $Ax = f$ has unique solution in $K$.
The proof is completed. □

## 3 The Convergence Theorems in Uniformly Smooth Banach Space

Let $X$ be a real uniformly smooth Banach space. Now we prove the following theorems.

**Theorem 3.1** *Suppose that $K \subset X$ is a nonempty bounded convex subset and $T : K \to K$ is a $\Phi$-pseudocontractive operator. If $T$ has a fixed point and the Ishikawa iteration sequence $\{x_n\}$ is defined by (1.2) with parameters*

(i) $\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \hat{\beta}_n = 0$ *and* $\sum_{n=0}^{+\infty} \beta_n = +\infty$;

(ii) $\hat{\gamma}_n = o(\hat{\beta}_n)$ *and* $\gamma_n = o(\beta_n)$,

*then iteration sequence $\{x_n\}$ converges strongly to unique fixed point of $T$.*

**Proof:** From Definition 1.1, we know that $F(T)$ is singleton. Setting $F(T) = \{q\}$ and $M = \sup\{\|x\| : x \in K\} + \|q\|$. Since $\|(y_n - q) - (x_{n+1} - q)\| = \|(\hat{\alpha}_n - \alpha_n)x_n + \hat{\beta}_n T x_n + \hat{\gamma}_n v_n - \beta_n T y_n - \gamma_n u_n\| \to 0$ $(as\ n \to \infty)$ and $\|(y_n - q) - (x_n - q)\| = \|(\hat{\alpha}_n - 1)x_n + \hat{\beta}_n T x_n + \hat{\gamma}_n v_n\| \to 0$ $(as\ n \to \infty)$, thus the uniformly continuity of $j$ ensures that

$$e_n := \|j(y_n - q) - j(x_{n+1} - q)\| \to 0\ (as\ n \to \infty)$$

and

$$s_n := \|j(y_n - q) - j(x_n - q)\| \to 0\ (as\ n \to \infty).$$

Using (1.4) and (1.6), we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|\alpha_n(x_n - q) + \beta_n(Ty_n - q) + \gamma_n(u_n - q)\|^2 \\
&\le \|\alpha_n(x_n - q)\|^2 + 2\beta_n(Ty_n - q,\ j(x_{n+1} - q)) \\
&\quad +2\gamma_n(u_n - q,\ j(x_{n+1} - q)) \\
&\le \|\alpha_n(x_n - q)\|^2 + 2\beta_n(Ty_n - q,\ j(y_n - q)) \\
&\quad +2\beta_n(Ty_n - q,\ j(x_{n+1} - q) - j(y_n - q)) \\
&\quad +2M^2\gamma_n \\
&\le \alpha_n^2\|x_n - q\|^2 + 2\beta_n\|y_n - q\|^2 - 2\beta_n\Phi(\|y_n - q\|) \\
&\quad +2M\beta_n e_n + 2M^2\gamma_n \\
&\le (1 - \beta_n)^2\|x_n - q\|^2 + 2\beta_n\|y_n - q\|^2 \\
&\quad -2\beta_n\Phi(\|y_n - q\|) + o(\beta_n)
\end{aligned} \tag{3.10}$$

for all $n \geq 0$. Similarly,

$$\begin{array}{rcl} \|y_n - q\|^2 & = & \|\hat{\alpha}_n(x_n - q) + \hat{\beta}_n(Tx_n - q) + \hat{\gamma}_n(v_n - q)\|^2 \\ & \leq & \|\hat{\alpha}_n(x_n - q)\|^2 + 2\hat{\beta}_n(Tx_n - q,\ j(y_n - q)) \\ & & +2\hat{\gamma}_n(u_n - q,\ j(y_n - q)) \\ & \leq & \hat{\alpha}_n^2\|x_n - q\|^2 + 2\hat{\beta}_n(Tx_n - q,\ j(x_n - q)) \\ & & +2\hat{\beta}_n(Tx_n - q,\ j(y_n - q) - j(x_n - q)) + 2M^2\hat{\gamma}_n \\ & \leq & \hat{\alpha}_n^2\|x_n - q\|^2 + 2\hat{\beta}_n\|x_n - q\|^2 - 2\hat{\beta}_n\Phi(\|x_n - q\|) \\ & & +2M\hat{\beta}_n s_n + 2M^2\hat{\gamma}_n \\ & \leq & \|x_n - q\|^2 + M^2\hat{\beta}_n^2 + 2M\hat{\beta}_n s_n + 2M^2\hat{\gamma}_n \\ & \leq & \|x_n - q\|^2 + o(\hat{\beta}_n) \end{array} \tag{3.11}$$

for all $n \geq 0$. Substituting (3.11) into (3.10) and simplifying, we obtain

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 + o(\beta_n) - 2\beta_n\Phi(\|y_n - q\|) \quad \forall n \geq 0 \tag{3.12}$$

where $o(\beta_n) \geq 0$. Let $2\sigma = \inf\{\|y_n - q\| :\ n \geq 0\}$. If $\sigma > 0$, then $\Phi(\|y_n - q\|) > \Phi(\sigma) > 0$ for all $n \geq 0$, and so, there exists an integer $N > 0$ such that $o(\beta_n) < \beta_n\Phi(\sigma)$ for all $n \geq N$. It follows from (3.12) that

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - \beta_n\Phi(\sigma) \quad \forall n \geq N.$$

By induction, we obtain

$$\Phi(\sigma)\sum_{j=N}^{+\infty}\beta_j \leq \|x_N - q\|^2 \leq M^2. \tag{3.13}$$

(3.13) is in contradiction with $\sum_{j=0}^{+\infty}\beta_j = +\infty$. It follows from the contradiction that $\sigma = 0$. Therefore, there exists a subsequence $\{y_{n_j}\} \subset \{y_n\}$ such that $y_{n_j} \to q$ as $j \to \infty$. Since $\lim_{j\to\infty}\|x_{n_j} - q\| = \lim_{j\to\infty}\hat{\alpha}_{n_j}\|x_{n_j} - q\| \leq \lim_{j\to\infty}\|y_{n_j} - q\| + M\lim_{j\to\infty}(\hat{\beta}_{n_j} + \hat{\gamma}_{n_j}) = 0$, the subsequence $\{x_{n_j}\}$ converges strongly to $q$. So, we know that maybe $\{x_n\}$ converges to $q$ and we cannot assure $\{x_n\}$ is not convergent. But, there are other conditions of $\{x_n\}$, such that $\{x_n\}$ converges to $q$. Since $x_{n_j} \to q$ as $j \to \infty$, for any given $\varepsilon > 0$ there exists an integer $j_0 > 0$ such that $\|x_{n_j} - q\| < \varepsilon$ for all $j \geq j_0$, and $2M(|\alpha_n - \hat{\alpha_n}| + \beta_n + \hat{\beta_n} + \gamma_n + \hat{\gamma_n}) < \varepsilon$ and $o(\beta_n) \leq \beta_n\Phi(\varepsilon/2)$ for all $n \geq n_{j_0}$. If $j_0$ is fixed, we will prove that $\|x_{n_{j_0}+k} - q\| < \varepsilon$ for all integers $k \geq 1$.

The proof is by induction. For $k = 1$, suppose $\|x_{n_{j_0}+1} - q\| \geq \varepsilon$. Then, (1.2) implies that $\|y_{n_{j_0}} - q\| > \varepsilon/2$. In fact, we have

$$\varepsilon \leq \|x_{n_{j_0}+1} - q\| \leq \|y_{n_{j_0}} - q\| + M(|\alpha_{n_{j_0}} - \hat{\alpha_{n_{j_0}}}| + \beta_{n_{j_0}} + \hat{\beta_{n_{j_0}}} + \gamma_{n_{j_0}} + \hat{\gamma_{n_{j_0}}}) < \|y_{n_{j_0}} - q\| + \varepsilon/2.$$

From $\Phi(\|y_{n_{j_0}} - q\|) > \Phi(\varepsilon/2)$ and using (3.12), we obtain

$$\varepsilon^2 \leq \|x_{n_{j_0}+1} - q\|^2 \leq \|x_{n_{j_0}} - q\|^2 + o(\beta_{n_{j_0}}) - 2\beta_{n_{j_0}}\Phi(\varepsilon/2) \leq \|x_{n_{j_0}} - q\|^2 < \varepsilon^2.$$

It is a contradiction. So, $\|x_{n_{j_0}+1}-q\|<\varepsilon$ holds for $k=1$. Assume now that $\|x_{n_{j_0}+p}-q\|<\varepsilon$ for some integer $p>1$. We prove $\|x_{n_{j_0}+p+1}-q\|<\varepsilon$. Again, assuming the contrary, as above, it leads to a contradiction. Hence, $\|x_{n_{j_0}+k}-q\|<\varepsilon$ holds for all integers $k\geq 1$, so that $x_n\to q$ as $n\to\infty$, i.e., $\lim_{k\to\infty}x_{n_{j_0}+k}=q$.
The Proof is completed. □

**Remark 3.1** Theorem 3.1 gives an affirmative answer to Chidume's open problem when $T$ is $\Phi$-quasicontractive. The corresponding results (see, for example, Theorem 3.3 of [4], Theorem 2 of [5], Theorem 3.1 of [13], Theorem 2 of [16] and Theorem 3.3 of [19]) are all special cases of Theorem 3.1 in the following senses:

1) $T$ may not be continuous, therefore, $T$ may not be Lipschitz, also;

2) $T$ may not be strongly pseudocontractive or $\phi$-strongly pseudocontractive;

3) the random errors of iterative processes have been considered appropriately;

4) the condition (iii) of Chidume's Theorem in [5] is dropped.

We like to point out: the iteration parameters $\{\alpha_n\}$, $\{\beta_n\},\{\gamma_n\},\{\hat{\alpha}_n\},\{\hat{\beta}_n\}$ and $\{\hat{\gamma}_n\}$ in Theorem 3.1 do not depend on any geometric structure of the Banach space $X$ and on any property of the operator $T$, but, the selection of the parameters is deal with the convergence rate of the iteration. A prototype for $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\hat{\alpha}_n\},\{\hat{\beta}_n\}$ and $\{\hat{\gamma}_n\}$ in our theorem is

$$\alpha_n=\hat{\alpha}_n=\frac{n^2+3n+1}{(n+2)^2},\quad \beta_n=\hat{\beta}_n=\frac{1}{n+2}$$

and

$$\gamma_n=\hat{\gamma}_n=\frac{1}{(n+2)^2}\quad \forall n\geq 0.$$

In the Theorem 3.1, if $\hat{\beta}_n=\hat{\gamma}_n=0$ for all $n\geq 0$, then we obtain a result that deals with the Mann iterative process with random errors as follows.

**Theorem 3.2** *Let $K$ be a nonempty bounded convex subset of $X$ and $T:K\to K\subset X$ a $\Phi$-pseudocontractive operator. If $T$ has a fixed point and the Mann iteration sequence $\{x_n\}$ is defined by (1.3) with parameters*

(i) $\lim_{n\to\infty}\beta_n=0$ *and* $\sum_{n=0}^{+\infty}\beta_n=+\infty$;

(ii) $\gamma_n=o(\beta_n)$.

*Then $\{x_n\}$ converges strongly to unique fixed point of $T$.*

**Theorem 3.3** *Suppose that $K \subset X$ is a nonempty bounded convex subset with $K+K \subseteq K$ and $A : K \to K$ is a $\Phi$-accretive operator. For any given $f \in K$ the equation $Ax = f$ has unique solution in $K$.*

**Proof:** We define $S : K \to K$ by $Sx = f + x - Ax$ for all $x \in K$. It is easy to see that $S$ is $\Phi$-pseudocontractive. Clearly, $q$ is a fixed point of $S$ if and only if that $q$ is a solution of the equation $Ax = f$. It follows from Theorem 2.1 or Theorem 2.2 above that the equation $Ax = f$ has an unique solution in $K$.
The proof is completed. □

## References


[1] **Asplund, E. :** *Positivity of duality mappings.* Bull. Amer. Math. Soc. **73**, 200-203 (1967)

[2] **Browder, F. E. :** *Nonlinear operations and nonlinear equations of evolution in Banach spaces.* Proc. Sympos. Pure Math. 18, No. 2 (1976)

[3] **Chang, S. S. :** *On chidume's open questions and approximation solutions of multivalued strongly accretive mapping equations in Banach spaces.* J. Math. Anal. Appl. **216**, 94-111 (1997)

[4] **Chang, S. S., Cho, Y. J., Lee, B. S.**, and **Kang, S. M. :** *Iterative approximations of fixed points and solutions for strongly accretive and strongly pseudocontractive mappings in Banach spaces.* J. Math. Anal. Appl. **224**, 149-165 (1998)

[5] **Chidume, C. E. :** *Approximation of fixed points of strongly pseudocontractive mappings.* Proc. Amer. Math. Soc. **120**, 545-551 (1994)

[6] **Chidume, C. E. :** *Iterative solutions of nonlinear equations in smooth Banach spaces.* Nonlinear Anal. **26**, N0:11, 1823-1834 (1996)

[7] **Chidume, C. E.**, and **Osilike, M. O. :** *Fixed point iterations for strictly hemicontractive maps in uniformly smooth Banach spaces.* Numer. Funct. Optim. **15**, No. 7-8, 779-790 (1994)

[8] **Chidume, C. E.**, and **Moore :** *The solution by iteration of nonlinear equations in uniformly smooth Banach spaces.* J. Math. Anal. Appl. **215**, 132-146 (1997)

[9] **Chidume, C. E. :** *Global iteration schemes for strongly pseudocontractive maps.* Proc. Amer. Math. Soc. **126**, 2641-2649 (1998)

[10] **Chidume, C. E. :** *Convergence theorems for strongly pseudocontractive and strongly accretive maps.* J. Math. Anal. Appl. **228**, 254-264 (1998)

[11] **Chidume, C. E. :** *A generalized steepest descent approximation for the zeros of m-accretive operators.* J. Math. Anal. Appl. **236**, 48-73 (1999)

[12] **Deng, L.**, and **Ding, X. P. :** *Iterative approximation of Lipschitz strictly pseudo-contractive mappings in uniformly smooth Banach spaces.* Nonlinear Anal. TMA. **24**, 981-987 (1995)

[13] **Ding, X. P. :** *Iterative process with random errors to locally strictly pseudo-contractive maps in Banach spaces.* Computers Math. Applic. Vol. **32** , No. 10, 91-97(1996)

[14] **Ding, X. P. :** *Iterative process with random errors to nonlinear $\Phi$- strongly accretive operator equations in arbitrary Banach spaces.* Computers Math. Applic. Vol. **33**, No. 8, 75-82 (1997)

[15] **Ishikawa, S. :** *Fixed point and iteration of a nonexpansive mapping in a Banach spaces.* Proc. Amer. Math. Soc. **73**, 65-71 (1976)

[16] **Liu, L. S. :** *Ishikawa and Mann iterative process with random errors for nonlinear strongly accretive mappings in Banach spaces.* J. Math. Anal. Appl. **194**, 114-125 (1995)

[17] **Mann, W. R. :** *Mean value methods in iteration.* Proc. Amer. Math. Soc. **4**, 506-510 (1953)

[18] **Osilike, M. O. :** *Iterative solutions of nonlinear $\phi$ -strongly accretive operator equations in arbitrary Banach spaces.* Nonlinear Analysis **36**, 1-9 (1999)

[19] **Xu, Y. G. :** *Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive operator equations.* J. Math. Anal. Appl. **224**, 91-101 (1998)

[20] **Zhou, H. :** *Some convergence theorems for the Ishikawa iterative sequences of certain nonlinear operators in uniformly smooth Banach spaces.* Acta. Math. Sinica. **40**, 751-758 (1997)

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