Rostock. Math. Kolloq. **59**, 29–40 (2005)

Subject Classification (AMS)
47H05, 47H06, 47H10, 47H14

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Strong Convergence and pseudo Stability for Operators of the ϕ-accretive type in uniformly smooth Banach Spaces

ABSTRACT. Let X be a uniformly Banach space and let $T : X \to X$ be a ϕ-strongly quasi-accretive operator. It is proved that, under suitable conditions, the Ishikawa iterative process with errors both converges strongly to the unique zero of T and is pseudo stable. A few related results deal with the convergence and stability of the Ishikawa iterative process with errors to the solutions of the equations $Tx = f$ and $x + Tx = f$, respectively, when $T : X \to X$ is ϕ-strongly accretive. Our results extend, improve, and unify the results due to Chidume [2], [3] and Zhou [18].

KEY WORDS AND PHRASES. Ishikawa iterative process with errors, ϕ-strongly quasi-accretive operator, ϕ-strongly accretive operator, stability, uniformly smooth Banach space.

1 Introduction

Let X be a Banach space with norm $\|\cdot\|$ and the dual space X^*. The normalized duality mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) = \{f \in X^* : \mathrm{Re}\langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad x \in X,$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. It is known that if X is uniformly smooth, then J is single valued and is uniformly continuous on any bounded subset of X.

The symbols $D(T), R(T), F(T), N(T)$ stand for the domain, the range, the fixed point set and the kernel of T, respectively, where $N(T) = \{x \in D(T); Tx = 0\}$.

Let $T : D(T) \subseteq X \to X$ be an operator and I denote the identity mapping on X.

[1]This research was financially supported by Changwon National University in 2004

Definition 1.1 (i) *T is called to be strongly accretive if there exists a constant $k \in (0,1)$ such that for each $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ satisfying*

$$Re\langle Tx - Ty, j(x-y)\rangle \geq k\|x-y\|^2;$$

(ii) *T is said to be ϕ-strongly accretive if there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for each $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ satisfying*

$$Re\langle Tx - Ty, j(x-y)\rangle \geq \phi(\|x-y\|)\|x-y\|;$$

(iii) *T is said to be ϕ-strongly quasi-accretive if $N(T) \neq \emptyset$ and if there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for each $x \in D(T)$ and $y \in N(T)$, there exists $j(x-y) \in J(x-y)$ satisfying*

$$Re\langle Tx, j(x-y)\rangle \geq \phi(\|x-y\|)\|x-y\|.$$

The classes of operators appearing Definition 1.1 have been used and studied by several authors (see, e.g., [1]-[4], [8], [10], [12]-[16], [18]). It is known that the classes of strongly accretive operators and ϕ-strongly accretive operators with a nonempty kernel are proper subclasses of the classes of ϕ-strongly accretive operators and ϕ-strongly quasi-accretive operators, respectively.

Let us recall the following iterative schemes due to Mann [11], Ishikawa [9] and Liu [10], respectively.

Definition 1.2 (i) *Let $D(T)$ be a convex subset of X with $D(T) = R(T)$. For any given $x_0 \in D(T)$, the sequence $\{x_n\}_{n=0}^{\infty}$ in $D(T)$ defined by*

$$y_n = (1-\beta_n)x_n + \beta_n Tx_n, \ \ x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n, n \geq 0$$

is called the Ishikawa iteration sequence, where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$ satisfying certain conditions;

(ii) *If $\beta_n = 0$ for all $n \geq 0$ in (i), then the sequence $\{x_n\}_{n=0}^{\infty}$ in $D(T)$ defined by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, \ \ n \geq 0,$$

is called the Mann iterative sequence;

(iii) *For any given $x_0 \in D(T)$, the sequence $\{x_n\}_{n=0}^{\infty}$ in $D(T)$ defined by*

$$y_n = (1-\beta_n)x_n + \beta_n Tx_n + v_n, \ \ x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n + u_n, \ \ n \geq 0,$$

is called the Ishikawa iteration sequence with errors, where $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are sequences in X and $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$ satisfying suitable conditions;

(iv) *If, $\beta_n = \|v_n\| = 0$ for all $n \geq 0$ in (iii), then the sequence $\{x_n\}_{n=0}^{\infty}$ in $D(T)$ now defined by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n + u_n, \ n \geq 0,$$

is called the Mann iteration sequence with errors.

It is clear that the Ishikawa and Mann iterative sequences are all special cases of the Ishikawa iterative sequences with errors.

Let $T : X \to X$ be an operator and $\{\alpha_n\}_{n=0}^{\infty}$ be sequences in $[0, 1]$. Assume that $x_0 \in X$ and $x_{n+1} = f(T, \alpha_n, x_n)$ defines an iteration scheme which produces a sequence $\{x_n\}_{n=0}^{\infty} \subset X$. Suppose that, furthermore, that $F(T) \neq \emptyset$ and that $\{x_n\}_{n=0}^{\infty}$ converges strongly to $q \in F(T)$. Let $\{y_n\}_{n=0}^{\infty}$ be any sequence in X and define $\{\varepsilon_n\}_{n=0}^{\infty} \subset [0, \infty)$ by $\varepsilon_n = \|y_{n+1} - f(T, \alpha_n, y_n)\|$.

Definition 1.3 (i) *The iterative scheme $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = f(T, \alpha_n, x_n)$ is called T-stable if $\lim_{n\to\infty} \varepsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = q$;*

(ii) *The iterative scheme $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = f(T, \alpha_n, x_n)$ is called almost T-stable if $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ implies that $\lim_{n\to\infty} y_n = q$;*

(iii) *The iterative scheme $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = f(T, \alpha_n, x_n)$ is called pseudo T-stable if $\lim_{n\to\infty} \alpha_n = 0$ and $\varepsilon_n = o(\alpha_n)$ implies that $\lim_{n\to\infty} y_n = q$.*

Osilike [16] pointed out that T-stability implies almost T-stability, and the converse does not hold in general. Clearly, an iteration scheme $\{x_n\}_{n=0}^{\infty}$ which is T-stable is pseudo T-stable. In section 2, we shall show that an iteration which is pseudo T-stable may fail to be T-stable.

Several researchers proved that the Mann iterative scheme, Ishikawa iterative scheme, the Mann iterative scheme with errors and Ishikawa iterative scheme with errors can be used to approximate solutions of the equations $Tx = f$ and $x + Tx = f$, where T is continuous strongly accretive or continuous ϕ-strongly accretive operators (see, e.g. [2]-[4], [12], [15], [18]).

Rhoades [17] obtained that the Mann and Ishikawa iterative schemes may exhibit different behaviors for different classes of nonlinear mappings. Several stability results for certain classes of nonlinear mappings have been established by a few researchers (see, e.g. [5]-[7], [13], [14], [16]). Harder and Hicks [7] revealed the importance of investigating the stability of various iteration schemes for various classes of nonlinear mappings. In [13], [14] and [16], Osilike established the stability and almost stability of certain Mann and Ishikawa iteration procedures for the classes of Lipschitz strongly accretive operators and Lipschitz ϕ-strongly accretive operators in real q-uniformly smooth Banach spaces and real Banach spaces, respectively.

For ϕ-strongly quasi-accretive operators without Lipschitz assumption, the possibility of establishing corresponding stability results has not been explored yet within our knowledge.

The aim of this paper is to establish the strong convergence and pseudo stability of the Ishikawa iterative scheme with errors to zeros of ϕ-strongly quasi-accretive operators in uniformly smooth Banach spaces. A few related results deal with the strong convergence and pseudo stability of the Ishikawa iterative scheme with errors to solutions of the equation $Tx = f$ and $x + Tx = f$, respectively, where $T : X \to X$ is ϕ-strongly accretive. The convergence results in this paper are generalizations and improvements of the corresponding results due to Chidume [2], [3] and Zhou [18].

We shall make use of the following result.

Lemma 1.1 ([1]) *Let X be a Banach space. Then for all $x, y \in X$, $j(x+y) \in J(x+y)$*

$$\|x + y\|^2 \leq \|x\|^2 + 2\mathrm{Re}\langle y,\ j(x + y)\rangle.$$

2 Main results

Theorem 2.1 *Let X be a uniformly Banach space and let $T : X \to X$ be a ϕ-strongly quasi-accretive operator. Suppose that the range $(I - T)$ of either or T is bounded and that $S = I - T$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0, 1]$ and $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are sequences in X satisfying*

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \|v_n\| = 0; \tag{2.1}$$

$$\sum_{n=0}^{\infty} \alpha_n = \infty; \tag{2.2}$$

$$\|u_n\| = o(\alpha_n). \tag{2.3}$$

Suppose that $\{x_n\}_{n=0}^{\infty}$ is the sequence generated from arbitrary $x_0 \in X$ by

$$z_n = (1 - \beta_n)x_n + \beta_n Sx_n + v_n, \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Sz_n + u_n, \quad n \geq 0. \tag{2.4}$$

Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique zero q of T and it is pseudo $(I - T)$-stable.

Proof: Since T is ϕ-strongly quasi-accretive, it follows that $N(T)$ is a singleton, say, $\{q\}$. It is easy to see that S has a unique fixed point q, and that

$$\mathrm{Re}\langle Sx - q, j(x - q)\rangle \leq \|x - q\|^2 - \phi(\|x - q\|)\|x - q\|, \quad x \in X. \tag{2.5}$$

Now we show that $R(S)$ is bounded. In fact, if $R(I-T)$ is bounded, so is $R(S)$; if $R(T)$ is bounded, then

$$\|Sx\| \leq \|x-q\| + \|q\| + \|Tx\| \leq \phi^{-1}(\|Tx\|) + \|q\| + \|Tx\|$$

for all $x \in X$. That is, $R(S)$ is bounded. Using (2.1) and (2.3), we conclude that there exists a nonnegative sequence $\{r_n\}_{n=0}^{\infty}$ such that

$$\|u_n\| = r_n \alpha_n, \quad n \geq 0; \tag{2.6}$$

$$\lim_{n\to\infty} r_n = 0. \tag{2.7}$$

Let $A = \mathrm{diam}R(S) + \|x_0 - q\|$ and $B = A + \sup\{\|v_n\| : n \geq 0\} + \sup\{r_n : n \geq 0\}$. Next we show by induction that

$$\|x_n - q\| \leq A + \sup\{r_n : n \geq 0\} \leq B, \quad n \geq 0. \tag{2.8}$$

Obviously, (2.8) is true for $n = 0$. Suppose that (2.8) is true for some $n \geq 0$. It follows from (2.4) and (2.6) that

$$\begin{aligned}
\|x_{n+1} - q\| \leq& (1-\alpha_n)\|x_n - q\| + \alpha_n\|Sz_n - q\| + \|u_n\| \\
\leq& (1-\alpha_n)[A + \sup\{r_n : n \geq 0\}] + \alpha_n A + \alpha_n r_n \\
\leq& A + \sup\{r_n : n \geq 0\}.
\end{aligned}$$

Hence (2.8) is true for all $n \geq 0$.

In view of (2.4) and (2.8), we infer that

$$\begin{aligned}
\|z_n - q\| \leq& (1-\beta_n)\|x_n - q\| + \beta_n\|Sx_n - q\| + \|v_n\| \\
\leq& (1-\beta_n)[A + \sup\{r_n : n \geq 0\}] + \beta_n A + \|v_n\| \\
\leq& B
\end{aligned} \tag{2.9}$$

for all $n \geq 0$. It follows from Lemma 1.1, (2.4), (2.8) and (2.9) that

$$\begin{aligned}
\|z_n - q\|^2 =& \|(1-\beta_n)(x_n - q) + \beta_n(Sx_n - q) + v_n\|^2 \\
\leq& (1-\beta_n)^2\|x_n - q\|^2 + 2\beta_n \mathrm{Re}\langle Sx_n - q, j(z_n - q)\rangle \\
&+ 2\mathrm{Re}\langle v_n, j(z_n - q)\rangle \\
\leq& (1-\beta_n)^2\|x_n - q\|^2 + 2B^2 B\beta_n + 2B\|v_n\|
\end{aligned} \tag{2.10}$$

for all $n \geq 0$. Using Lemma 1.1, (2.4)-(2.6) and (2.8)-(2.10), we get that

$$\begin{aligned}
\|x_{n+1} - q\|^2 =& \|(1-\alpha_n)(x_n - q) + \alpha_n(Sz_n - q) + u_n\|^2 \\
\leq& (1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n \mathrm{Re}\langle Sz_n - q, j(x_{n+1} - q)\rangle \\
&+ 2\mathrm{Re}\langle u_n, j(x_{n+1} - q)\rangle \\
\leq& (1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n \mathrm{Re}\langle Sz_n - q, j(z_n - q)\rangle \\
&+ 2\alpha_n \mathrm{Re}\langle Sz_n - q, j(x_{n+1} - q) - j(z_n - q)\rangle + 2B\|u_n\| \\
\leq& (1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n[\|z_n - q\|^2 - \phi(\|z_n - q\|)\|z_n - q\|] \\
&+ 2\alpha_n B\|j(x_{n+1} - q) - j(z_n - q)\| + 2B\|u_n\| \\
\leq& [(1-\alpha_n)^2 + 2\alpha_n(1-\beta_n)^2]\|x_n - q\|^2 + 4B\alpha_n\beta_n + 4B^2B\alpha_n\|v_n\| \\
&- 2\alpha_n\phi(\|z_n - q\|)\|z_n - q\| \\
&+ 2\alpha_n B\|j(x_{n+1} - q) - j(z_n - q)\| + 2B\|u_n\| \\
\leq& \|x_n - q\|^2 - 2\alpha_n\phi(\|z_n - q\|)\|z_n - q\| + \alpha_n t_n
\end{aligned} \tag{2.11}$$

for all $n \geq 0$, where

$$t_n = B^2\beta_n + 4B\beta_n + 4B\|v_n\| + 2B\|j(x_{n+1} - q) - j(z_n - q)\| + 2Br_n, \quad n \geq 0.$$

Since j is uniformly continuous on each bounded subset of X and

$$\begin{aligned}
\|x_{n+1} - q - (z_n - q)\| \leq& \alpha_n\|x_n - Sz_n\| + \beta_n\|x_n - Sx_n\| + \|u_n\| + \|v_n\| \\
\leq& 2B(\alpha_n + \beta_n) + \|u_n\| + \|v_n\| \to 0
\end{aligned}$$

as $n \to \infty$, it follows that $\lim_{n\to\infty} \|j(x_{n+1} - q) - j(z_n - q)\| = 0$. Thus, by (2.1), (2.6) and (2.7) we have

$$\lim_{n\to\infty} t_n = 0. \tag{2.12}$$

Put $\inf\{\|z_n - q\| : n \geq 0\} = r$. We claim that $r = 0$. Otherwise $r > 0$. Thus (2.12) ensures that there exists a positive integer N such that $t_n \leq \phi(r)r$ for all $n \geq N$. From (2.11) we obtain that for all $n \geq N$,

$$\begin{aligned}
\|x_{n+1} - q\|^2 \leq& \|x_n - q\|^2 - 2\alpha_n\phi(r)r + \alpha_n\phi(r)r \\
\leq& \|x_n - q\|^2 - \alpha_n\phi(r)r,
\end{aligned}$$

which implies that

$$\phi(r)r\sum_{n=N}^{\infty}\alpha_n \leq \sum_{n=N}^{\infty}(\|x_n - q\|^2 - \|x_{n+1} - q\|^2) = \|x_N - q\|^2.$$

That is, $\sum_{n=0}^{\infty} \alpha_n < \infty$ contradicting (2.2). Therefore $r = 0$. Thus there exists a subsequence $\{\|z_{n_k} - q\|\}_{k=0}^{\infty}$ of $\{\|z_n - q\|\}_{n=0}^{\infty}$ such that $\lim_{k\to\infty} \|z_{n_k} - q\| = 0$. It follows from (2.1), (2.4), (2.6) and (2.7) that

$$\begin{aligned}\|x_{n_k} - q\| \le &\|z_{n_k} - q\| + \beta_{n_k}\|x_{n_k} - Sx_{n_k}\| + \|v_{n_k}\| \\ \le &\|z_{n_k} - q\| + 2B\beta_{n_k} + \|v_{n_k}\| \to 0\end{aligned}$$

as $k \to \infty$. That is,

$$\lim_{k\to\infty} \|x_{n_k} - q\| = 0. \tag{2.13}$$

By virtue of (2.1)-(2.3), (2.12) and (2.13), we conclude that for given $\varepsilon > 0$, there exists positive numbers k_0 and $p = n_{k_0}$ such that

$$\begin{aligned}&\|x_p - q\| \le \varepsilon, \quad \max\{\alpha_n, \beta_n\} \le \frac{\varepsilon}{16B}, \\ &\max\{\|u_n\|, \|v_n\|\} \le \frac{\varepsilon}{8}, \quad t_n \le \phi(\frac{1}{2}\varepsilon)\varepsilon, \quad n \ge p.\end{aligned} \tag{2.14}$$

By induction we show that

$$\|x_{p+m} - q\| \le \varepsilon, \quad m \ge 0. \tag{2.15}$$

Note that (2.14) ensures that (2.15) holds for $m = 0$. Suppose that (2.15) holds for some $m \ge 0$. If $\|x_{p+m+1} - q\| > \varepsilon$, then (2.14), (2.8) and (2.4) yield that

$$\begin{aligned}\|x_{p+m} - q\| \ge &\|x_{p+m+1} - q\| - \alpha_{p+m}\|Sz_{p+m} - x_{p+m}\| - \|u_{p+m}\| \\ > &\varepsilon - \frac{\varepsilon}{16B} \cdot 2B - \frac{\varepsilon}{8} = \frac{3}{4}\varepsilon\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}\|z_{p+m} - q\| \ge &\|x_{p+m} - q\| - \beta_{p+m}\|Sx_{p+m} - x_{p+m}\| - \|v_{p+m}\| \\ > &\frac{3}{4}\varepsilon - \frac{\varepsilon}{16B} \cdot 2B - \frac{\varepsilon}{8} = \frac{1}{2}\varepsilon.\end{aligned} \tag{2.17}$$

It follows from (2.11), (2.14), (2.16) and (2.17) that

$$\begin{aligned}\varepsilon^2 < &\|x_{p+m+1} - q\|^2 \\ \le &\|x_{p+m} - q\|^2 - 2\alpha_{p+m}\phi(\|z_{p+m} - q\|)\|z_{p+m} - q\| + \alpha_{p+m}t_{p+m} \\ \le &\varepsilon^2 - \alpha_{p+m}\phi(\frac{1}{2}\varepsilon)\varepsilon + \alpha_{p+m}\phi(\frac{1}{2}\varepsilon)\varepsilon = \varepsilon^2,\end{aligned}$$

which is impossible. Hence $\|x_{p+m+1} - q\| \le \varepsilon$. That is, (2.15) holds for all $m \ge 0$. Thus (2.15) yields that $\lim_{n\to\infty} x_n = q$.

Let $\{y_n\}_{n=0}^{\infty}$ be any given sequence in X and define $\{\varepsilon_n\}_{n=0}^{\infty}$ by

$$w_n = (1-\beta_n)y_n + \beta_n S y_n + v_n, \quad n \geq 0; \tag{2.18}$$

$$\varepsilon_n = \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n T w_n - u_n\|, \quad n \geq 0.$$

Put $p_n = y_{n+1} - (1-\alpha_n)y_n - \alpha_n T w_n - u_n$. Then

$$y_{n+1} = (1-\alpha_n)y_n + \alpha_n T w_n + u_n + p_n, \quad n \geq 0. \tag{2.19}$$

Suppose that $\varepsilon_n = o(\alpha_n)$. By (2.3), we get that

$$\|u_n + p_n\| \leq \|u_n\| + \varepsilon_n = o(\alpha_n),$$

which implies that $\|u_n + p_n\| = o(\alpha_n)$. It follows from the above conclusion that the sequence $\{y_n\}_{n=0}^{\infty}$ defined by (2.18) and (2.19) converges strongly to q. That is, $\{x_n\}_{n=0}^{\infty}$ is pseudo T-stable. This completes the proof.

Theorem 2.2 *Let X, $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ be as in Theorem 2.1. Let $T : X \to X$ be a ϕ-strongly accretive operator and the range of either $(I - T)$ or T be bounded. Suppose that the equation $Tx = f$ has a solution for a given $f \in X$ and that $Sx = f + x - Tx$ for all $x \in X$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ generated from an arbitrary $x_0 \in X$ by (2.4) converges strongly to the unique solution of the equation $Tx = f$ and it is pseudo S-stable.*

Proof: Since T is ϕ-strongly accretive and the equation $Tx = f$ has a solution, it follows that the equation $Tx = f$ has a unique solution. The rest of the proof is identical the proof of Theorem 2.1 and is therefore omitted. This completes the proof.

Theorem 2.3 *Let X, $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ be as in Theorem 2.1. Let $T : X \to X$ be a ϕ-strongly accretive operator and the range of either $(I + T)$ or T be bounded. Suppose that the equation $x + Tx = f$ has a solution for a given $f \in X$ and that $Sx = f - Tx$ for all $x \in X$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ generated from an arbitrary $x_0 \in X$ by (2.4) converges strongly to the unique solution of the equation $x + Tx = f$ and it is pseudo S-stable.*

Proof: Let $A = I + T$. Then A is ϕ-strongly accretive and the range either A or $(I - A)$ is bounded. Clearly, $x + Tx = f$ becomes $Ax = f$ and $Sx = f - Tx = f + x - Ax$ for all $x \in X$. Hence Theorem 2.3 follows from Theorem 2.2. This completes the proof.

Remark 2.1 The boundedness of $R(T)$ or $R(I-T)$ in Theorems 2.1 and 2.2 can be replaced by the boundedness of $\{Tx_n\}_{n=0}^{\infty}$ and $\{Tz_n\}_{n=0}^{\infty}$ or $\{x_n - Tx_n\}_{n=0}^{\infty}$ and $\{z_n - Tz_n\}_{n=0}^{\infty}$.

Remark 2.2 The convergence result in Theorem 2.2 extends, improves and unifies Theorems 1 and 2 of [2], Theorems 7 and 8 of [3] and Theorem 1 of [18] in the following ways:

(a) The Mann iterative schemes in [2, 3] and the Ishikawa iterative schemes in [2, 3, 18] are replaced by the more general Ishikawa iterative scheme with errors.

(b) The strongly accretive operators in [2], [3] and [18] are replaced by the more general ϕ-strongly accretive operators;

(c) That T is Lipschitz in [2] is omitted;

(d) The assumptions of $\alpha_n \leq \beta_n$ in [2], [3], [18], $\sum_{n=0}^{\infty} c_n b(c_n) < \infty$ in [2], [3], $\sum_{n=0}^{\infty} \alpha_n b(\alpha_n) < \infty$ in [2], [3] are superfluous;

(e) The boundedness hypotheses of $R(I-T)$ in [2], [18] and $R(T)$ in [3] are replaced by the boundedness of either $R(I-T)$ or $R(T)$;

The following example reveals that the convergence result in Theorem 2.2 extends properly the corresponding results in [2], [3] and [18].

Example 2.1 Let $X = (-\infty, \infty)$ with the usual norm. Then for any $q > 1$, X is real q-uniformly smooth Banach space. Define $T : X \to X$ by

$$Tx = \begin{cases} x - 1, & \text{if } x < -1 \\ x - \sqrt{-x}, & \text{if } x \in [-1, 0) \\ x, & \text{if } x \in [0, \infty). \end{cases}$$

Clearly $R(T) = X$, $R(I-T)$ is bounded and T is continuous. Note that

$$\lim_{x \to 0^-} \frac{Tx - T0}{x - 0} = \lim_{x \to 0^-} \left(1 + \frac{1}{\sqrt{-x}}\right) = \infty.$$

Hence T is not Lipschitz. Take $\phi(t) = \frac{1}{2}t$ for all $t \geq 0$. In order to prove that T is ϕ-strongly accretive, that is,

$$\langle Tx - Ty, j(x-y) \rangle \geq \phi(\|x - y|)\|x - y\|, \quad x, y \in X. \tag{2.20}$$

We have to consider the following cases.

Case 1. Let $x, y \in (-\infty, -1)$ or $x, y \in [0, \infty)$. Then

$$\langle Tx - Ty, j(x-y) \rangle = (x-y)^2;$$

Case 2. Let $x, y \in [-1, 0)$. Then

$$\langle Tx - Ty, j(x-y)\rangle = [x - y - (\sqrt{-x} - \sqrt{-y})](x-y) = \left(1 + \frac{1}{\sqrt{-x}+\sqrt{-y}}\right)(x-y)^2;$$

Case 3. Let $x \in (-\infty, -1)$, $y \in [-1, 0)$. Then

$$\langle Tx - Ty, j(x-y)\rangle = [x - 1 - (y - \sqrt{-y})(x-y) = (x-y)^2 + (1 - \sqrt{-y})(y-x);$$

Case 4. Let $x \in (-\infty, -1)$, $y \in [0, \infty)$. Then

$$\langle Tx - Ty, j(x-y)\rangle = (x - 1 - y)(x-y) = (x-y)^2 + (y-x);$$

Case 5. Let $x \in [-1, 0)$, $y \in [0, \infty)$. Then

$$\langle Tx - Ty, j(x-y)\rangle = (x - \sqrt{-x} - y)(x-y) = (x-y)^2 + \sqrt{-x}(y-x).$$

Therefore (2.20) holds. Since $R(T) = X$, it follows that the equation $Tx = f$ has a solution for any $f \in X$. Set

$$\alpha_n = (1+n)^{-\frac{1}{2}}, \quad \beta_n = (2+2n)^{-\frac{1}{2}}, \quad u_n = (1+n)^{-1}, \quad v_n = (1+n)^{-\frac{1}{3}}, \quad n \geq 0.$$

Then all the assumptions of Theorem 2.2 are fulfilled. But Theorems 1 and 2 in [2], Theorems 7 and 8 in [3], and Theorem 1 in [18] are not applicable since $R(T)$ is unbounded, T is not Lischitz, and $\alpha_n > \beta_n$ for each $n > 0$.

Remark 2.3 Theorems 11 and 12 in [3] are special cases of our Theorem 2.3.

Remark 2.4 For $T : X \to X$ a ϕ-strongly quasi-accretive operator, Theorem 2.1 proves that the Ishikawa iterative scheme with errors considered in Theorem 2.1 is pseudo $(I-T)$-stable. The following example reveals that the iterative scheme is not $(I-T)$-stable.

Example 2.2 Let $X = (-\infty, \infty)$ with the usual norm, $T = I$ and $u_n = v_n = 0$ for all $n \geq 0$. Clearly,

$$\mathrm{Re}\langle Tx - Ty, j(x-y)\rangle = \|x-y\|^2 \geq \phi(\|x-y\|)\|x-y\|, \quad x \in X, \ y \in N(T),$$

where $\phi(t) = \frac{1}{2}t$ for all $t \geq 0$. It follows from Theorem 2.1 that the sequence $\{x_n\}_{n=0}^{\infty}$ generated from an arbitrary $x_0 \in X$ by

$$z_n = (1-\beta_n)x_n + \beta_n S x_n + v_n, \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n S z_n + u_n, \quad n \geq 0,$$

converges strongly to the unique zero 0 of T and is pseudo $(I-T)$-stable. Next we prove that it is not pseudo $(I-T)$-stable. Let $y_n = \frac{n}{1+n}$ for all $n \geq 0$. Then

$$\varepsilon_n = \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n S w_n - u_n\| \leq \|y_{n+1} - y_n\| + \alpha_n\|y_n\| \to 0$$

as $n \to \infty$. That is, $\lim_{n\to\infty} \varepsilon_n = 0$. But, $\lim_{n\to\infty} y_n = 1 \notin N(T) = \{0\}$.

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received: January 7, 2005

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