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Finite and conditional completeness properties of generalized ordered sets

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ABSTRACT. In particular, we show that if X is a set equipped with a transitive relation $\leq$, then the following completeness properties are equivalent:

(1) $\operatorname{lb}(\{x, y\}) \neq \emptyset$ for all $x, y \in X$, and $\inf(A) \neq \emptyset$ for all $A \subset X$ with $A \neq \emptyset$ and $\operatorname{lb}(A) \neq \emptyset$;

(2) $\inf(\{x, y\}) \neq \emptyset$ for all $x, y \in X$, and $\inf(A) \neq \emptyset$ for all $A \subset X$ with $A \neq \emptyset$, $\operatorname{lb}(A) \neq \emptyset$ and $\operatorname{ub}(A) \neq \emptyset$.

Thus, we obtain a substantial generalization of a basic theorem of Garrett Birkhoff which says only that in a conditionally complete lattice every nonempty subset which has a lower bound has a greatest lower bound.

KEY WORDS AND PHRASES. Generalized ordered sets, lower bound and infimum completenesses.

Introduction

Throughout this paper, X will denote an arbitrary set equipped with an arbitrary binary relation $\leq$. Thus, X may be considered as a generalized ordered set or an ordered set without axioms.

The set X will be called reflexive, transitive, antisymmetric and total if the relation $\leq$ has the corresponding property. If X is total, then for any $x, y \in X$ we have either $x \leq y$ or $y \leq x$. Thus, in particular, X is reflexive.

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For any $A \subset X$, the members of the families

$$\operatorname{lb}(A) = \big\{\, x \in X : \quad \forall\ a \in A : \quad x \leq a \,\big\}$$

and

$$\operatorname{ub}(A) = \big\{\, x \in X : \quad \forall\ a \in A : \quad a \leq x \,\big\}$$

are called the lower and upper bounds of A in X, respectively. And the members of the families

$$\min(A) = A \cap \operatorname{lb}(A), \qquad \max(A) = A \cap \operatorname{ub}(A),$$
$$\inf(A) = \max\big(\operatorname{lb}(A)\big), \qquad \sup(A) = \min\big(\operatorname{ub}(A)\big)$$

are called the minima, maxima, infima and suprema of A in X, respectively.

First, we show that the following extension of [2, Lemma 2.23, p. 46] is true.

Lemma *If X is transitive, and moreover $A_i \subset X$ and $\inf(A_i) \neq \emptyset$ for all $i \in I$, then*

$$\operatorname{lb}\Big(\bigcup_{i \in I} A_i\Big) = \operatorname{lb}\Big(\bigcup_{i \in I} \inf(A_i)\Big) \quad \text{and} \quad \inf\Big(\bigcup_{i \in I} A_i\Big) = \inf\Big(\bigcup_{i \in I} \inf(A_i)\Big).$$

Then, by using this lemma, we show that the following generalization of [1, Theorem 9, p. 115] is also true.

Theorem *If X is transitive, then the following completeness properties are equivalent:*

(1) $\operatorname{lb}\big(\{x, y\}\big) \neq \emptyset$ *for all* $x, y \in X$, *and* $\inf(A) \neq \emptyset$ *for all* $A \subset X$ *with* $A \neq \emptyset$ *and* $\operatorname{lb}(A) \neq \emptyset$;

(2) $\inf\big(\{x, y\}\big) \neq \emptyset$ *for all* $x, y \in X$, *and* $\inf(A) \neq \emptyset$ *for all* $A \subset X$ *with* $A \neq \emptyset$, $\operatorname{lb}(A) \neq \emptyset$ *and* $\operatorname{ub}(A) \neq \emptyset$.

Remark If in particular X is partially ordered, then by using the above lemma we also show that the following completeness properties are equivalent:

(1) $\inf(\{x, y\}) \neq \emptyset$ for all $x, y \in X$;

(2) $\inf(A) \neq \emptyset$ for every finite, nonvoid subset A of X.

In this respect, it is noteworthy that to prove a counterpart of the above equivalence for lb instead of inf, the transitivity of the relation $\leq$ is again sufficient.

1 Lower and upper bounds

Concerning lower and upper bounds, we shall only quote here the following simple theorems of [5] .

Theorem 1.1 *If* $A_i \subset X$ *for all* $i \in I$, *then*

$$\operatorname{lb}\Big(\bigcup_{i \in I} A_i\Big) = \bigcap_{i \in I} \operatorname{lb}(A_i) .$$

Corollary 1.2 *If* $A \subset B \subset X$, *then* $\operatorname{lb}(B) \subset \operatorname{lb}(A)$.

Proof: Note that $\operatorname{lb}(B) = \operatorname{lb}(A \cup B) = \operatorname{lb}(A) \cap \operatorname{lb}(B) \subset \operatorname{lb}(A)$.

Corollary 1.3 *If* $A \subset X$, *then* $\operatorname{lb}(A) = \bigcap_{a \in A} \operatorname{lb}(a)$, *where* $\operatorname{lb}(a) = \operatorname{lb}\big(\{a\}\big)$.

Theorem 1.4 *If* $A, B \subset X$, *then*

$$A \subset \operatorname{lb}(B) \quad \Longleftrightarrow \quad B \subset \operatorname{ub}(A) .$$

Corollary 1.5 *If* $A \subset X$, *then* $A \subset \operatorname{ub}\big(\operatorname{lb}(A)\big)$.

Proof: Clearly, $\operatorname{lb}(A) \subset \operatorname{lb}(A)$. Hence, by Theorem 1.4, the required inclusion already follows.

Theorem 1.6 *If* $A \subset X$, *then*

$$\min(A) = A \cap \inf(A) \qquad \textit{and} \qquad \inf(A) = \operatorname{lb}(A) \cap \operatorname{ub}\big(\operatorname{lb}(A)\big) .$$

Corollary 1.7 *If* $A \subset X$, *then* $\min(A) \subset \inf(A) \subset \operatorname{lb}(A) \subset \operatorname{lb}\big(\inf(A)\big)$.

Proof: By Theorem 1.6, we have not only $\min(A) \subset \inf(A) \subset \operatorname{lb}(A)$, but also $\inf(A) \subset \operatorname{ub}\big(\operatorname{lb}(A)$. Hence, by Theorem 1.4, the required inclusion already follows.

The importance of reflexivity, totality and antisymmetry will only be illuminated here by the following basic theorems of [6].

Theorem 1.8 *If* $\Phi = \operatorname{lb}$, $\min$ *or* $\inf$, *then the following assertions are equivalent :*

(1) X *is reflexive ;*

(2) $x \in \Phi(x)$ *for all* $x \in X$.

Theorem 1.9 *The following assertions are equivalent :*

(1) X *is reflexive;*

(2) $\min(x) \neq \emptyset$ *for all* $x \in X$*;*

(3) $\min(x) = \{x\}$ *for all* $x \in X$*.*

Theorem 1.10 *The following assertions are equivalent:*

(1) X *is total;*

(2) $\min\big(\{x, y\}\big) \neq \emptyset$ *for all* $x, y \in X$*.*

Theorem 1.11 *If X is reflexive and $\Phi = \min$ or $\inf$, then the following assertions are equivalent:*

(1) X *is antisymmetric;*

(2) $\operatorname{card}\big(\Phi(A)\big) \leq 1$ *for all* $A \subset X$*.*

Corollary 1.12 *If X is reflexive and antisymmetric, then $\inf(x) = \{x\}$ for all $x \in X$.*

2 The importance of transitivity

Concerning the importance of transitivity, we shall only quote here the following basic theorems of [6]. Hints for the proofs are included only for the reader's convenience.

Theorem 2.1 *The following assertions are equivalent:*

(1) X *is transitive;*

(2) $y \in \operatorname{lb}(x)$ *and* $z \in \operatorname{lb}(y)$ *imply* $z \in \operatorname{lb}(x)$ *for all* $x, y, z \in X$*;*

(3) $B \subset \operatorname{lb}(A)$ *and* $C \subset \bigcup_{b \in B} \operatorname{lb}(b)$ *imply* $C \subset \operatorname{lb}(A)$ *for all* $A, B \subset X$*.*

Proof: To prove the less obvious implication $(2) \Longrightarrow (3)$, suppose that (2) and the conditions of (3) hold. If $c \in C$, then since $C \subset \bigcup_{b \in B} \operatorname{ub}(b)$ there exists $b \in B$ such that $c \in \operatorname{lb}(b)$. Moreover, if $a \in A$, then since $b \in B \subset \operatorname{lb}(A)$ we have $b \in \operatorname{lb}(a)$. Hence, by using (2), we can already infer that $c \in \operatorname{lb}(a)$. Now, since $a \in A$ and $c \in C$ were arbitrary, it is clear that $c \in \operatorname{lb}(A)$, and thus $C \subset \operatorname{lb}(A)$. Therefore, (3) also holds.

From the above theorem, it is clear that in particular we also have

Corollary 2.2 *If* X *is transitive, then* $x \in \operatorname{lb}(A)$ *and* $y \in \operatorname{lb}(x)$ *imply* $y \in \operatorname{lb}(A)$ *for all* $A \subset X$ *and* $x, y \in X$.

Theorem 2.3 *If* X *is transitive, then*

$$\operatorname{lb}(x) = \operatorname{lb}(A) \qquad \text{for all} \qquad A \subset X \quad \text{and} \quad x \in \inf(A).$$

Proof: If $x \in \inf(A)$, then by Corollaries 1.7 and 1.2 we have $\operatorname{lb}(A) \subset \operatorname{lb}\big(\inf(A)\big) \subset \operatorname{lb}(x)$ even if X is not transitive.

Moreover, if $x \in \inf(A)$, then Corollary 1.7 we also have $x \in \operatorname{lb}(A)$. Hence, by Corollary 2.2, it is clear $y \in \operatorname{lb}(x)$ implies $y \in \operatorname{lb}(A)$. Therefore, $\operatorname{lb}(x) \subset \operatorname{lb}(A)$ is also true.

Corollary 2.4 *If* X *is transitive, then*

$$\operatorname{lb}(A) = \operatorname{lb}\big(\inf(A)\big) \qquad \text{for all} \qquad A \subset X \quad \text{with} \quad \inf(A) \neq \emptyset.$$

Proof: By Theorems 1.1 and 2.3, it is clear that

$$\operatorname{lb}\big(\inf(A)\big) = \bigcap_{x \in \inf(A)} \operatorname{lb}(x) = \bigcap_{x \in \inf(A)} \operatorname{lb}(A) = \operatorname{lb}(A).$$

Now, in addition to the results of [6], we can also easily prove the following

Theorem 2.5 *If* X *is transitive and* $A_i \subset X$ *for all* $i \in I$, *then*

$$\text{(1)}\ \operatorname{lb}\Big(\bigcup_{i \in I} A_i\Big) = \operatorname{lb}\bigg(\Big(\bigcup_{i \in J} A_i\Big) \cup \Big(\bigcup_{i \in I \setminus J} \inf(A_i)\Big)\bigg),$$

$$\text{(2)}\ \inf\Big(\bigcup_{i \in I} A_i\Big) = \inf\bigg(\Big(\bigcup_{i \in J} A_i\Big) \cup \Big(\bigcup_{i \in I \setminus J} \inf(A_i)\Big)\bigg),$$

where $J = \big\{ i \in I : \ \inf(A_i) = \emptyset \big\}$.

Proof: By Theorem 1.1 and Corollary 2.4, we have

$$\operatorname{lb}\Big(\bigcup_{i \in I} A_i\Big) = \bigcap_{i \in I} \operatorname{lb}(A_i) =$$
$$\Big(\bigcap_{i \in J} \operatorname{lb}(A_i)\Big) \cap \Big(\bigcap_{i \in I \setminus J} \operatorname{lb}(A_i)\Big) = \Big(\bigcap_{i \in J} \operatorname{lb}(A_i)\Big) \cap \Big(\bigcap_{i \in I \setminus J} \operatorname{lb}\big(\inf(A_i)\big)\Big) =$$
$$\operatorname{lb}\Big(\bigcup_{i \in J} A_i\Big) \cap \operatorname{lb}\Big(\bigcup_{i \in J} \inf(A_i)\Big) = \operatorname{lb}\bigg(\Big(\bigcup_{i \in J} A_i\Big) \cup \Big(\bigcup_{i \in I \setminus J} \inf(A_i)\Big)\bigg).$$

Hence, by the definition of inf, it is clear that (2) is also true.

From Theorem 2.5, we can at once get the following generalization of the second part of [2, Lemma 2.23, p. 46].

Corollary 2.6 *If* X *is transitive, and moreover* $A_i \subset X$ *and* $\inf(A_i) \neq \emptyset$ *for all* $i \in I$, *then*

(1) $\operatorname{lb}\Big(\bigcup_{i\in I} A_i\Big) = \operatorname{lb}\Big(\bigcup_{i\in I} \inf(A_i)\Big)$;

(2) $\inf\Big(\bigcup_{i\in I} A_i\Big) = \inf\Big(\bigcup_{i\in I} \inf(A_i)\Big)$.

3 Finite lower bound completenesses

Definition 3.1 *We say that*

(1) X *is two-lb-complete if* $\operatorname{lb}(\{x, y\}) \neq \emptyset$ *for all* $x, y \in X$;

(2) X *is two-inf-complete if* $\inf(\{x, y\}) \neq \emptyset$ *for all* $x, y \in X$;

(3) X *is finitely quasi-lb-complete if* $\operatorname{lb}(A) \neq \emptyset$ *for all finite, nonvoid subset* A *of* X;

(4) X *is finitely quasi-inf-complete if* $\inf(A) \neq \emptyset$ *for all finite, nonvoid subset* A *of* X.

Remark 3.2 By Corollary 1.7, it is clear that 'two-inf-completeness' implies 'two-lb-completeness', and 'finite quasi-inf-completeness' implies 'finite quasi-lb-completeness'.

Moreover, by using the well-orderedness of the set $\mathbb{N}$ of all natural numbers, we can prove the following

Theorem 3.3 *If* X *is transitive, then the following assertions are equivalent:*

(1) X *is two-lb-complete;*

(2) X *is finitely quasi-lb-complete.*

Proof: By Definition 3.1, it is clear that $(2) \Longrightarrow (1)$ even if X is not partially ordered.

To prove the converse implication, suppose on the contrary that (1) holds, but (2) does not hold. That is, $\operatorname{lb}(\{x, y\}) \neq \emptyset$ for all $x, y \in X$, and $\operatorname{lb}(A) = \emptyset$ for some finite, nonvoid subset A of X.

Denote by $\mathcal{A}$ the family of all finite, nonvoid subsets A of X such that $\operatorname{lb}(A) = \emptyset$. Then, by the above assumptions, it is clear that $\mathcal{A} \neq \emptyset$ and $\operatorname{card}(A) > 2$ for all $A \in \mathcal{A}$. Define

$$M = \{\operatorname{card}(A) : \quad A \in \mathcal{A}\}.$$

Then, we evidently have $\emptyset \neq M \subset \mathbb{N}$ such that $1 \notin M$ and $2 \notin M$.

Hence, since $\mathbb{N}$ is well-ordered, we can infer that $\min(M) \neq \emptyset$. Therefore, there exists $n \in \min(M)$. This implies that $n \in M$ and $n \in \text{lb}(M)$. Hence, it is clear that $2 < n \in \mathbb{N}$ such that $n \leq m$ for all $m \in M$. Moreover, we can also state that there exists $A \in \mathcal{A}$ such that $n = \text{card}(A)$.

Thus, we can choose $a \in A$, and define $B = A \setminus \{a\}$. Then, it is clear that B is a finite nonvoid subset of X such that $k = \text{card}(B) < \text{card}(A) = n$. Therefore, $\text{lb}(B) \neq \emptyset$ also holds. Namely, $\text{lb}(B) = \emptyset$ would imply that $B \in \mathcal{A}$. Hence, we could infer that $k = \text{card}(B) \in M$, and thus $n \leq k$, which would be a contradiction.

Now, we can choose $\beta \in \text{lb}(B)$ and $\gamma \in \text{lb}\big(\{a, \beta\}\big)$. Then, by Theorem 1.1, it is clear that $\gamma \in \text{lb}(a)$ and $\gamma \in \text{lb}(\beta)$. Hence, by using Corollary 2.2, we can infer that $\gamma \in \text{lb}(B)$. Therefore, by Theorem 1.1, we also have $\gamma \in \text{lb}(a) \cap \text{lb}(B) = \text{lb}\big(\{a\} \cup B\big) = \text{lb}(A)$. This contradiction proves that $(1) \Longrightarrow (2)$.

A particular case of the following theorem is usually considered to be quite obvious in the advanced theory of lattices. The proofs given here and in [4, p. 40] show that this attitude cannot be completely justified.

Theorem 3.4 *If X is partially ordered, then the following assertions are equivalent:*

(1) *X is two-inf-complete;*

(2) *X is finitely quasi-inf-complete.*

Proof: By Definition 3.1, it is clear that $(2) \Longrightarrow (1)$ even if X is not partially ordered.

To prove the converse implication, suppose on the contrary that (1) holds, but (2) does not hold. Denote by $\mathcal{A}$ the family of all finite, nonvoid subsets A of X such that $\inf(A) = \emptyset$. Then, by using a similar argument as in the proof of Theorem 3.3, we can see that there exists $A \in \mathcal{A}$ such that by choosing $a \in A$ and defining $B = A \setminus \{a\}$, we already have $\inf(B) \neq \emptyset$.

Now, by Theorem 1.11, it is clear that there exists $x \in X$ such that $\inf(B) = \{x\}$. Moreover, by Corollary 1.12, we also have $\inf\big(\{a\}\big) = \{a\}$. Hence, by using Corollary 2.6, we can infer that

$$\inf(A) = \inf\big(\{a\} \cup B\big) = \inf\big(\inf\big(\{a\}\big) \cup \inf(B)\big) = \inf\big(\{a\} \cup \{x\}\big) = \inf\big(\{a, x\}\big).$$

However, this is already a contradiction. Namely, by $A \in \mathcal{A}$, we have $\inf(A) = \emptyset$. While, by (1), we have $\inf\big(\{a, x\}\big) \neq \emptyset$. Therefore, the implication $(1) \Longrightarrow (2)$ is also true.

4 Conditional infimum completenesses

Definition 4.1 *We say that*

(1) X *is pseudo-inf-complete if* $\inf(A) \neq \emptyset$ *for all* $A \subset X$ *with* $\mathrm{lb}(A) \neq \emptyset$ *;*

(2) X *is semi-inf-complete if* $\inf(A) \neq \emptyset$ *for all* $A \subset X$ *with* $A \neq \emptyset$ *and* $\mathrm{lb}(A) \neq \emptyset$ *;*

(3) X *is almost pseudo-inf-complete if* $\inf(A) \neq \emptyset$ *for all* $A \subset X$ *with* $\mathrm{lb}(A) \neq \emptyset$ *and* $\mathrm{ub}(A) \neq \emptyset$ *;*

(4) X *is almost semi-inf-complete if* $\inf(A) \neq \emptyset$ *for all* $A \subset X$ *with* $A \neq \emptyset$, $\mathrm{lb}(A) \neq \emptyset$ *and* $\mathrm{ub}(A) \neq \emptyset$.

Remark 4.2 Thus, 'pseudo-inf-complete' implies both 'semi-inf-complete' and 'almost pseudo-inf-complete', and 'almost pseudo-inf-complete' implies 'almost-semi-inf-complete'.

Moreover, by using Corollary 2.6, we can also prove the following

Theorem 4.3 *If* X *is transitive and* $\mathrm{ub}(X) \neq \emptyset$, *then the following assertions are equivalent:*

(1) X *is two-lb-complete and pseudo-inf-complete;*

(2) X *is two-inf-complete and almost pseudo-inf-complete.*

Proof: By the corresponding definitions, it is clear that (1) $\Longrightarrow$ (2) even if X is not transitive or $\mathrm{ub}(X) = \emptyset$. Moreover, from Remark 3.2 we know that the first part (2) always implies that of (1). Therefore, to prove the converse implication (2) $\Longrightarrow$ (1), we need only show that (2) implies the second part of (1).

For this, assume that (2) holds, and moreover $A \subset X$ such that $\mathrm{lb}(A) \neq \emptyset$. If $A = \emptyset$, then by the corresponding definitions it is clear that

$$\inf(A) = \inf(\emptyset) = \max\big(\mathrm{lb}(\emptyset)\big) = \max(X) = \mathrm{ub}(X),$$

and thus $\inf(A) \neq \emptyset$. Therefore, we may assume that $A \neq \emptyset$, i.e., there exists $a \in A$. Define

$$B = \bigcup_{x \in A} \inf\big(\{a, x\}\big).$$

Then, by Corollary 2.6, it is clear that

$$\mathrm{lb}(B) = \mathrm{lb}\Big(\bigcup_{x \in A} \inf(\{a, x\}\Big) = \mathrm{lb}\Big(\bigcup_{x \in A} \{a, x\}\Big) = \mathrm{lb}(A).$$

Moreover, by using the duals of Theorems 1.1 and Corollary 1.2, and Corollaries 1.7 and 1.5, we can see that

$$\operatorname{ub}(B) = \operatorname{ub}\Big(\bigcup_{x\in A}\inf(\{a, x\})\Big) = \bigcap_{x\in A}\operatorname{ub}\big(\inf(\{a, x\})\big) \supset \bigcap_{x\in A}\operatorname{ub}\big(\operatorname{lb}(\{a, x\})\big) \supset \bigcap_{x\in A}\{a, x\} \supset \{a\}.$$

Therefore, $\operatorname{lb}(B) \neq \emptyset$ and $\operatorname{ub}(B) \neq \emptyset$ also hold. Thus, by the almost pseudo-inf-completeness of X, we also have $\inf(B) \neq \emptyset$.

Now, it remains to note that by Corollary 2.6 we also have

$$\inf(A) = \inf\Big(\bigcup_{x\in A}\{a, x\}\Big) = \inf\Big(\bigcup_{x\in A}\inf(\{a, x\}\Big) = \inf(B).$$

Therefore, $\inf(A) \neq \emptyset$ also holds, and thus X is pseudo-inf-complete.

The following theorem is a generalization of the first part of [1, Theorem 9, p. 115]. Our subsequent sketch of the proof shows that the two and a half line proof given there may only be considered as a hint.

Theorem 4.4 *If X is transitive, then the following assertions are equivalent:*

(1) *X is two-lb-complete and semi-inf-complete;*

(2) *X is two-inf-complete and almost semi-inf-complete.*

Proof: Again, it is clear that (1) $\Longrightarrow$ (2) even if X is not transitive. Moreover, the first part (2) always implies that of (1). Therefore, to prove the converse implication (2) $\Longrightarrow$ (1), we need only show that (2) implies the second part of (1).

For this, assume that (2) holds, and moreover $A \subset X$ such that $A \neq \emptyset$ and $\operatorname{lb}(A) \neq \emptyset$. Choose $a \in A$, and define

$$B = \bigcup_{x\in A}\inf\big(\{a, x\}\big).$$

Then, it is clear that $\emptyset \neq B \subset X$. Namely, by the two-inf-completeness of X and the definition of B, we evidently have $\emptyset \neq \inf\big(\{a, a\}\big) \subset B$.

Moreover, from the proof of Theorem 4.3, we can see that $\operatorname{lb}(B) \neq \emptyset$ and $\operatorname{ub}(B) \neq \emptyset$ also hold. Thus, by the almost semi-inf-completeness of X, we also have $\inf(B) \neq \emptyset$. Now, it remains to note that by the proof of Theorem 4.3, we also have $\inf(A) = \inf(B)$. Therefore, $\inf(A) \neq \emptyset$ also holds, and thus X is semi-inf-complete.

5 Two illustrating examples

Example 5.1 If $X = \{a, b, c\}$ such that we only have

$$a \le b, \qquad b \le c, \qquad c \le a \qquad \text{and} \qquad x \le x \qquad \text{for all} \qquad x \in X,$$

then X is total and antisymmetric. Moreover, X is two-inf-complete, but not finitely quasi-lb-complete. Thus, by Remark 3.2, X is also two-lb-complete, but not finitely quasi-inf-complete.

To check that X is not finitely quasi-lb-complete, note that

$$\operatorname{lb}(a) = \{a, c\}, \qquad \operatorname{lb}(b) = \{a, b\}, \qquad \operatorname{lb}(b) = \{b, c\}.$$

Therefore, by Corollary 1.3, we have

$$\operatorname{lb}(X) = \operatorname{lb}(a) \cap \operatorname{lb}(b) \cap \operatorname{lb}(c) = \emptyset,$$

and thus X is not finitely quasi-lb-complete.

Moreover, we can quite similarly see that

$$\operatorname{lb}\big(\{a, b\}\big) = \{a\}, \qquad \operatorname{lb}\big(\{a, c\}\big) = \{c\}, \qquad \operatorname{lb}\big(\{b, c\}\big) = \{b\}.$$

Hence, since by the dual of Theorem 1.8 we have $x \in \max(x)$ for all $x \in X$, it is already clear that

$$\inf\big(\{x, y\}\big) = \max\big(\operatorname{lb}\big(\{x, y\}\big)\big) \ne \emptyset$$

for all $x, y \in X$ with $x \ne y$. Moreover, by Theorem 1.8, we also have $x \in \inf(x)$, and hence $\inf(x) \ne \emptyset$ for all $x \in X$. Therefore, X is two-inf-complete.

Remark 5.2 In addition to Example 5.1 and Corollary 2.4, it is worth noticing that if X is reflexive, antisymmetric and

$$\operatorname{lb}(A) = \operatorname{lb}\big(\inf(A)\big)$$

for all $A \subset X$ with $\operatorname{card}(A) = 2$ and $\inf(A) \ne \emptyset$, then X is necessary transitive. Thus, by Theorem 3.4, X is finitely quasi-inf-complete if and only if it is two-inf-complete.

To check the transitivity of X, by Theorem 2.1 it is enough to show only that if $x \in X$,

$$y \in \operatorname{lb}(x), \qquad z \in \operatorname{lb}(y) \qquad \text{and} \qquad x \ne y,$$

then $z \in \operatorname{lb}(x)$. For this, note if $A = \{x, y\}$, then by Theorem 1.8 and Corollaries 1.3 and 1.5 we have

$$y \in \operatorname{lb}(x) \cap \operatorname{lb}(y) = \operatorname{lb}(A) \qquad \text{and} \qquad y \in A \subset \operatorname{ub}\big(\operatorname{lb}(A)\big).$$

Hence, by Theorems 1.6 and 1.11, it is clear that

$$y \in \mathrm{lb}\,(A) \cap \mathrm{ub}\,\big(\mathrm{lb}\,(A)\big) = \inf\,(A)\,, \qquad \text{and thus} \qquad \{y\} = \inf\,(A)\,.$$

Now, by using our former assumptions and observations, we can already easily see that

$$z \in \mathrm{lb}\,(y) = \mathrm{lb}\,\big(\inf\,(A)\big) = \mathrm{lb}\,(A) = \mathrm{lb}\,(x) \cap \mathrm{lb}\,(y) \subset \mathrm{lb}\,(x)\,.$$

Example 5.3 If $X = \{\,a,\, b,\, c,\, d\,\}$ such that we only have

$$a \leq a\,, \qquad a \leq c\,, \qquad a \leq d\,, \qquad b \leq d\,, \qquad c \leq d\,,$$

then X is transitive and antisymmetric. Moreover, X is almost semi-inf-complete, but not semi-inf-complete.

To check this, note that

$$\begin{array}{llll} \mathrm{lb}\,(a) = \{a\}\,, & \mathrm{lb}\,(b) = \emptyset\,, & \mathrm{lb}\,(c) = \{a\}\,, & \mathrm{lb}\,(d) = \{\,a,\, b,\, c\,\}\,; \\ \mathrm{ub}\,(a) = \{\,a,\, c,\, d\,\}\,, & \mathrm{ub}\,(b) = \{d\}\,, & \mathrm{ub}\,(c) = \{d\}\,, & \mathrm{ub}\,(d) = \emptyset\,. \end{array}$$

Hence, by Theorem 1.6 and the dual of Corollary 1.3, it is clear that

$$\inf\,(d) = \mathrm{lb}\,(d) \cap \mathrm{ub}\,\big(\mathrm{lb}\,(d)\big) = \mathrm{lb}\,(d) \cap \mathrm{ub}\,(a) \cap \mathrm{ub}\,(b) \cap \mathrm{ub}\,(c) = \emptyset\,.$$

Therefore, X is not semi-inf-complete.

Moreover, by Corollary 1.3, it is clear that, for any $A \subset X$,

$$\mathrm{lb}\,(A) \neq \emptyset \implies A \subset \{\,a,\, c,\, d\,\} \qquad \text{and} \qquad \mathrm{ub}\,(A) \neq \emptyset \implies A \subset \{\,a,\, b,\, c\,\}\,.$$

Therefore, if $A \neq \emptyset$, $\mathrm{lb}\,(A) \neq \emptyset$ and $\mathrm{ub}\,(A) \neq \emptyset$, then we necessarily have

$$A = \{a\} \qquad \text{or} \qquad A = \{c\} \qquad \text{or} \qquad A = \{\,a,\, c\,\}\,.$$

Hence, by Corollary 1.3, it is clear that $\mathrm{lb}\,(A) = \{a\}$. Moreover, by Theorem 1.6, it is clear that

$$\inf\,(A) = \mathrm{lb}\,(A) \cap \mathrm{ub}\,\big(\,\mathrm{lb}\,(A)\big) = \mathrm{lb}\,(a) \cap \mathrm{ub}\,(a) = \{a\}\,.$$

Therefore, X is almost semi-inf-complete.

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