Rostock. Math. Kolloq. **59**, 41–49 (2005)

Subject Classification (AMS)
39A10, 39A11, 41A60

Cengiz Çinar, Stevo Stević and Ibrahim Yalçinkaya

A Note on Global Asymptotic Stability of a Family of Rational Equations

ABSTRACT. In this note we prove that all positive solutions of the difference equations

$$x_{n+1} = \frac{1 + x_n \sum_{i=1}^{k} x_{n-i}}{x_n + x_{n-1} + x_n \sum_{i=2}^{k} x_{n-i}}, \quad n = 0, 1, ...,$$

where $k \in \mathbf{N}$, converge to the positive equilibrium $\bar{x} = 1$. The result generalizes the main theorem in the paper: Li Xianyi and Zhu Deming, Global asymptotic stability in a rational equation, *J. Differ. Equations Appl.* **9** (9), (2003), 833-839. We present a very short proof of the theorem. Also, we find the asymptotics of some of the positive solutions.

KEY WORDS AND PHRASES. rational difference equation, global asymptotic stability, equilibrium point, positive solution, asymptotics

1 Introduction

In [11], Xianyi and Deming prove that the positive equilibrium of the difference equation

$$x_{n+1} = \frac{x_n x_{n-1} + 1}{x_n + x_{n-1}}, \quad n = 0, 1, 2, ... \tag{1}$$

with positive initial values x_{-1}, x_0, is globally asymptotically stable.

In [1], Kruse and Nesemann, among other things, proved the following theorem:

Theorem A *Consider the difference equation*

$$x_{n+r} = f(x_{n+r-1}, ..., x_n), \quad n = 0, 1, ... \tag{2}$$

where $r \in \mathbf{N}$, $f : (0, \infty)^r \to (0, \infty)$ *is a continuous function with some unique positive equilibrium* $\bar{x}$. *Suppose that there is an* $m \in \mathbf{N}$ *such that for all solutions* (x_n) *of Eq.* (2)

$$(x_n - x_{n+m}) \left(\frac{\bar{x}^2}{x_n} - x_{n+m} \right) \leq 0$$

with equality if and only if $x_n = \bar{x}$. *Then* $\bar{x}$ *is globally asymptotically stable.*

In this note we consider a family of difference equations of the form

$$x_{n+1} = \frac{1 + x_n \sum_{i=1}^{k} x_{n-i}}{x_n + x_{n-1} + x_n \sum_{i=2}^{k} x_{n-i}}, \quad n = 0, 1, ..., \tag{3}$$

where $k \in \mathbf{N}$ and the initial conditions $x_{-k}, x_{-k+1}, ..., x_0$ are positive numbers. From the equation

$$\bar{x} = \frac{k\bar{x}^2 + 1}{(k-1)\bar{x}^2 + 2\bar{x}} \tag{4}$$

we see that $\bar{x} = 1$ is a unique positive equilibrium of Eq. (3).

We show that the positive solutions of Eq. (3) have some similar properties with the positive solutions of Eq. (1) and give a very short proof of the following result:

Theorem 1 *The positive equilibrium point $\bar{x}$ of Eq. (3) is globally asymptotically stable.*

This theorem generalizes the main result in [11], since for $k = 1$ Eq. (3) becomes Eq. (1).

For some other globally convergence results and their applications, see, for example, [5, 6, 7, 8, 9, 10].

In the last section we find the asymptotics of some solutions of Eq. (1).

2 Some properties of the positive solutions of Eq. (3)

In this section we prove several results concerning the positive solutions of Eq. (3).

Lemma 1 *A positive solution $(x_n)_{n=-k}^{\infty}$ of Eq. (3) is eventually equal to 1 if and only if*

$$(x_{-1} - 1)(x_0 - 1) = 0. \tag{5}$$

Proof: Assume that Eq. (5) holds. Then by Eq. (3), it is easy to see that the following conclusion is true: if $x_{-1} = 1$ or $x_0 = 1$, then $x_n = 1$ for $n \geq 1$.

Conversely, assume that $(x_{-1} - 1)(x_0 - 1) \neq 0$. We show

$$x_n \neq 1 \text{ for any } n \geq 1 \tag{6}$$

Let $x_N = 1$ with minimally chosen $N \geq 1$.

Clearly

$$1 = x_N = \frac{1 + x_{N-1} \sum_{i=1}^{k} x_{N-1-i}}{x_{N-1} + x_{N-2} + x_{N-1} \sum_{i=2}^{k} x_{N-1-i}}$$

which implies $(1 - x_{N-1})(1 - x_{N-2}) = 0$ and consequently $x_{N-1} = 1$ or $x_{N-2} = 1$, a contradiction with the choice of N and the condition $(x_{-1} - 1)(x_0 - 1) \neq 0$.

Lemma 2 *Let $(x_n)_{n=-k}^{\infty}$ be a positive solution of Eq. (3) which is not eventually equal to* 1. *Then the following statements are true:*

(i) $(x_{n+1} - x_n)(x_n - 1) < 0$ *for* $n \geq 0$,

(ii) $(x_{n+1} - 1)(x_n - 1)(x_{n-1} - 1) > 0$ *for* $n \geq 0$.

Proof: From Eq. (3), we obtain

$$x_{n+1} - x_n = \frac{(1 - x_n)(1 + x_n + x_n \sum_{i=2}^{k} x_{n-i})}{x_n + x_{n-1} + x_n \sum_{i=2}^{k} x_{n-i}}, \quad n = 0, 1, 2, \ldots \tag{7}$$

and

$$x_{n+1} - 1 = \frac{(x_n - 1)(x_{n-1} - 1)}{x_n + x_{n-1} + x_n \sum_{i=2}^{k} x_{n-i}}, \quad n = 0, 1, 2, \ldots \tag{8}$$

From (7) and (8), inequalities (i) and (ii) follow according to Lemma 1.

Remark 1 From Lemma 2 we see that the signs of $x_n - 1$, $n \geq 1$ of a positive solution (x_n) of Eq. (3) are determined by x_{-1} and x_0. Hence in the investigation of the semicycle analysis of positive solutions of Eq. (3) we will consider only the terms with the indices greater than or equal to -1.

A *positive semicycle* of a solution (x_n) of Eq.(3) consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$, all greater than or equal to $\bar{x}$, with $l \geq -1$ and $m \leq \infty$ and such that

$$\text{either} \quad l = -1, \quad \text{or} \quad l > -1 \quad \text{and} \quad x_{l-1} < \bar{x}$$

and

$$\text{either} \quad m = \infty, \quad \text{or} \quad m < \infty \quad \text{and} \quad x_{m+1} < \bar{x}.$$

A *negative semicycle* of a solution (x_n) of Eq. (3) consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$, all less than to $\bar{x}$, with $l \geq -1$ and $m \leq \infty$ and such that

$$\text{either} \quad l = -1, \quad \text{or} \quad l > -1 \quad \text{and} \quad x_{l-1} \geq \bar{x}$$

and

$$\text{either} \quad m = \infty, \quad \text{or} \quad m < \infty \quad \text{and} \quad x_{m+1} \geq \bar{x}.$$

The first semicycle of a solution starts with the term x_{-1} and is positive if $x_{-1} \geq \bar{x}$ and negative if $x_{-1} < \bar{x}$.

Lemma 3 *For Eq. (3), the following statements are true:*

(i) *There exists a positive solution with a semicycle of Eq. (3) which has an infinite number of terms and monotonically tends to the positive equilibrium point $\bar{x}$;*

(ii) *Every negative semicycle of a solution of Eq. (3), except perhaps for the first, has exactly two terms.*

(iii) *Every positive semicycle of an oscillatory solution of Eq. (3) has exactly one term.*

Proof:

(i) If $x_{-1} > 1$ and $x_0 > 1$, then by Lemma 2 (ii), it follows that $x_n > 1$, $n \geq -1$, i.e. this positive semicycle has infinite number of terms. By Lemma 2 (i), we see that x_n is strictly decreasing for $n \geq 0$. Hence, there is finite $\lim_{n\to\infty} x_n = l > 0$. From this and (4) it follows that $l = \bar{x} = 1$.

(ii) If x_s $(s \geq 0)$ is the first term of a negative semicycle, then from Lemma 2 (ii) we have

$$(x_{s+1} - 1)(x_s - 1)(x_{s-1} - 1) > 0$$

and consequently $x_{s+1} < 1$.

From this and since

$$(x_{s+2} - 1)(x_{s+1} - 1)(x_s - 1) > 0$$

it follows that $x_{s+2} > 1$, from which the result follows.

(iii) If x_p $(p \geq 0)$ is the first term of a positive semicycle of an oscillatory solution of Eq. (3), then from the inequality in Lemma 2 (ii) we have

$$(x_{p+1} - 1)(x_p - 1)(x_{p-1} - 1) > 0.$$

Since $x_{p-1} < 1$ it follows that $x_{p+1} < 1$, as desired.

From Lemmas 1, 2 and 3 it follows the following corollary.

Corollary 1 *Consider Eq. (3). Then a positive solution of Eq. (3) is either eventually equal to 1, or greater than 1 and monotonically tends to 1, or an oscillatory solution of Eq. (3), such that the positive semicycles of the solution have always one term, and the negative semicycles, disregarding the first one, two terms.*

3 Proof of Theorem 1

In this section we prove Theorem 1.

Proof: From (3) we have

$$\begin{aligned} \frac{1}{x_n} - x_{n+1} &= \frac{1}{x_n} - \frac{1 + x_n \sum_{i=1}^{k} x_{n-i}}{x_n + x_{n-1} + x_n \sum_{i=2}^{k} x_{n-i}} \\ &= \frac{(1 - x_n)(x_{n-1}(1 + x_n) + x_n \sum_{i=2}^{k} x_{n-i})}{x_n(x_n + x_{n-1} + x_n \sum_{i=2}^{k} x_{n-i})}. \end{aligned} \tag{9}$$

From (7) and (9) we have

$$(x_n - x_{n+1})\left(\frac{1}{x_n} - x_{n+1}\right) \le 0, \quad n = 0, 1, \dots .$$

with equality if and only if $x_n = 1$. From this and by Theorem A, we obtain that the positive equilibrium $\bar{x} = 1$ is globally asymptotically stable, as desired.

4 Asymptotics of solutions of Eq. (3)

In this section we find the asymptotics of some solutions of Eq. (3). We use the method described in [3], see also, [2] and [4].

4.1 Asymptotics of nonoscillatory solutions of Eq. (3)

According to Lemma 3 these solutions monotonically tend to 1 as $n \to \infty$. In order to find the asymptotics we make the ansatz $x_n = 1 + y_n$ with $y_n = o(1)$. Equation (3) implies

$$y_{n+1} = \frac{y_n y_{n-1}}{k + 1 + k y_n + \sum_{i=1}^{k} y_{n-i} + y_n \sum_{i=2}^{k} y_{n-i}}. \tag{10}$$

Note that Eq. (10) can be approximated by the equation

$$y_{n+1} = \frac{y_n y_{n-1}}{k+1}, \tag{11}$$

where first we look for positive solutions y_n which correspond to the condition $x_n > 1$ for $n \ge 0$. Taking the logarithm of (11) and making the change $z_n = \ln y_n$, we obtain

$$z_{n+1} - z_n - z_{n-1} = -\ln(k+1). \tag{12}$$

By standard methods it can be shown that the general solution of Eq. (12) has the form.

$$z_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n + \ln(k+1).$$

Hence the general solution of Eq. (11) reads

$$y_n = (k+1)e^{c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n}. \tag{13}$$

For real constants c_j this solution is positive, and it satisfies $y_n = o(1)$ if $c_1 < 0$. Without loss of generality we may assume that $c_1 = -1$, which is shown by a suitable shift of n.

This motivated us to make the ansatz

$$y_n = (k+1)\left(e^{-l^n} + b\psi_n\right), \tag{14}$$

with $\psi_n = \exp(-al^n)$, $a > 1$, where $l = (1+\sqrt{5})/2$.

Setting (14) into (10) and comparing the coeficients we obtain that $a = 1 + l^{1-k}$ and $b = 1$. Now after a shift of n to $n+k$ in (10) we apply Theorem 2.1 in [3]. Let

$$\varphi_n = (k+1)\left(e^{-l^n} + e^{-al^n}\right) \quad \text{and} \quad \psi_n = e^{-al^n}, \tag{15}$$

where a and l are as above and let

$$F(w_0, w_1, ..., w_{k+1}) = (k + 1 + kw_k + w_{k-1} + (w_k + 1)\sum_{i=0}^{k-2} w_i)w_{k+1} - w_k w_{k-1}.$$

The partial derivatives of the function F are the following

$$F_{w_0} = F_{w_1} = \cdots = F_{w_{k-2}} = w_{k+1}(w_k + 1),$$

$$F_{w_{k-1}} = w_{k+1} - w_k, \quad F_{w_k} = w_{k+1}(k + \sum_{i=0}^{k-2} w_i) - w_{k-1},$$

$$F_{w_{k+1}} = k + 1 + kw_k + w_{k-1} + (w_k + 1)\sum_{i=0}^{k-2} w_i.$$

Hence

$$\psi_{n+i}F_{w_i}(\varphi_n, ..., \varphi_{n+k+1}) \sim \psi_{n+i}\varphi_{n+k+1} \sim (k+1)e^{-l^n(al^i + l^{k+1})}$$

for $i = 0, 1, ..., k-2$,

$$\psi_{n+k-1}F_{w_{k-1}}(\varphi_n, ..., \varphi_{n+k+1}) \sim -\psi_{n+k-1}\varphi_{n+k} \sim -(k+1)e^{-l^n(al^{k-1} + l^k)},$$

$$\psi_{n+k}F_{w_k}(\varphi_n, ..., \varphi_{n+k+1}) \sim -\psi_{n+k}\varphi_{n+k-1} \sim -(k+1)e^{-l^n(al^k + l^{k-1})}$$

and

$$\psi_{n+k+1}F_{w_{k+1}}(\varphi_n, ..., \varphi_{n+k+1}) \sim (k+1)\psi_{n+k+1} = (k+1)e^{-l^n(al^{k+1})}.$$

Since $a = 1 + l^{1-k}$ it is easy to see that

$$l^{k+1} + 1 = al^{k-1} + l^k = \min\{al^i + l^{k+1}, (i = 0, 1, ..., k-2), al^{k-1} + l^k, al^k + l^{k-1}, al^{k+1}\},$$

where the minimum is attained at the last but two position.

Thus, for $f_n = e^{-l^n(l^{k+1}+1)}$ we obtain

$$\psi_{n+i}F_{w_i}(\varphi_n, ..., \varphi_{n+k+1}) \sim A_i f_n$$

where $A_i = 0$, $i = 0, 1, 2, ..., k-2, k, k+1$, and $A_{k-1} = -(k+1)$.

Now we prove that

$$F(\varphi_n, ..., \varphi_{n+k+1}) \sim (k+1)^2 e^{-(l^{k+1}+1+l^{1-k})l^n} = o(f_n). \tag{16}$$

For $w_i = \varphi_{n+i} = (k+1)s_i$, $i = 0, 1, ..., k+1$, with $s_i = e^{-l^{n+i}} + e^{-al^{n+i}}$ let $F = (k+1)^2 G$ with

$$G(s_0, s_1, ..., s_{k+1}) = s_{k+1}(1 + ks_k + s_{k-1} + (1 + (k+1)s_k) \sum_{i=0}^{k-2} s_i) - s_k s_{k-1}.$$

It follows

$$G(s_0, s_1, ..., s_{k+1}) = s_{k+1}(1 + s_0 + s_1) - s_k s_{k-1} + o\left(e^{(L+a)l^n}\right)$$

with $L = l^{k+1}$, since the terms $s_{k+1}s_k s_i$ with $i \geq 0$, the terms $s_{k+1}s_i$ for $i \geq 2$, and the terms $s_{k+1}s_k$ for $k \geq 2$ are contained in the remainder term. In the exponents of the terms of the product $s_k s_{k-1}$ there appear the factors of $-l^n$

$$l^k + l^{k-1} = L, \tag{17}$$

$$l^k + al^{k-1} = L + 1, \tag{18}$$

$$al^k + l^{k-1} = L + l \tag{19}$$

and

$$al^k + al^{k-1} = aL. \tag{20}$$

The corresponding factors concerning the product $s_{k+1}(1 + s_0 + s_1)$ are

$$(17),\ (20),\ (18),\ L + a,\ aL + 1,\ a(L+1),\ (19),\ L + al,\ aL + l,\ a(L + l).$$

The terms with a number cancel. The smallest term of the remaining ones is $L + a$. Hence (16) is proved.

From all above mentioned the conditions of Theorem 2.1 in [3] are satisfied for $m = k + 1$, hence for every $\varepsilon > 0$, Eq. (3) has a solution y_n in the stripe $\varphi_n - \varepsilon\psi_n \leq y_n \leq \varphi_n + \varepsilon\psi_n$ for sufficiently large $n_0 = n_0(\varepsilon)$, with φ_n and ψ_n defined in (15).

4.2 Asymptotics of oscillatory solutions of Eq. (3)

The signs of the terms of a solution of Eq. (11) depend on the initial conditions y_0 and y_1. It can easily be seen that the general nontrivial solution of Eq. (11) can be written as $v_n y_n$ where y_n is the positive solution (13) and v_n for $n \geq 0$ one of the four 3-periodic sequences in Table 1.

$v_n^{(i)}$	$v_0^{(i)}$	$v_1^{(i)}$	$v_2^{(i)}$	$v_3^{(i)}$	$v_4^{(i)}$	$v_5^{(i)}$	$v_6^{(i)}$	$v_7^{(i)}$	$v_8^{(i)}$	.	.	.
$v_n^{(1)}$	1	1	1	1	1	1	1	1	1	.	.	.
$v_n^{(2)}$	1	-1	-1	1	-1	-1	1	-1	-1	.	.	.
$v_n^{(3)}$	-1	1	-1	-1	1	-1	-1	1	-1	.	.	.
$v_n^{(4)}$	-1	-1	1	-1	-1	1	-1	-1	1	.	.	.

TABLE 1. Values of the sequences $v_n^{(i)}$, $i = 1, 2, 3, 4$.

These periodic sequences can be represented as $v_n = e^{i\pi t_n}$ where t_n is one of the solutions with integer values mod 2 of Fibonacci's equation in Table 2.

$t_n^{(i)}$	$t_0^{(i)}$	$t_1^{(i)}$	$t_2^{(i)}$	$t_3^{(i)}$	$t_4^{(i)}$	$t_5^{(i)}$	$t_6^{(i)}$	$t_7^{(i)}$	$t_8^{(i)}$	.	.	.
$t_n^{(1)}$	0	0	0	0	0	0	0	0	0	.	.	.
$t_n^{(2)}$	0	1	1	0	1	1	0	1	1	.	.	.
$t_n^{(3)}$	1	0	1	1	0	1	1	0	1	.	.	.
$t_n^{(4)}$	1	1	0	1	1	0	1	1	0	.	.	.

TABLE 2. $t_{n+1}^{(i)} = t_n^{(i)} + t_{n-1}^{(i)} \pmod 2,\ i = 1, 2, 3, 4.$

With some more effort it can be shown analogously as before that Eq. (10) has also solutions which behave asymptotically like the solutions $v_n y_n$ of (11). This result matches with Lemma 2 (ii), which is equivalent to $v_{n+1}v_nv_{n-1} > 0$ for $n \geq 0$, and it also matches with Lemma 3.

Acknowledgement. We would like to express our sincere thanks to Professor Lothar Berg for giving valuable suggestions during the preparation of this paper.

References

[1] **Kruse, N.**, and **Nesemann, T.** : *Global asymptotic stability in some discrete dynamical systems.* J. Math. Anal. Appl. **235**, 151-158 (1999)

[2] **Berg, L.** : *On the asymptotics of nonlinear difference equations.* Z. Anal. Anwendungen **21**, No.4, 1061-1074 (2002)

[3] **Berg, L.** : *Inclusion theorems for non-linear difference equations with applications.* J. Differ. Equations Appl. **10** (4), 399-408 (2004)

[4] **Berg, L.** : *Corrections to "Inclusion theorems for non-linear difference equations with applications", from [3].* J. Differ. Equations Appl. **11** (2), 181-182 (2005)

[5] **Stević, S.** : *A generalization of the Copson's theorem concerning sequences which satisfy a linear inequality.* Indian J. Math. **43** (3), 277-282 (2001)

[6] **Stević, S.** : *A note on bounded sequences satisfying linear inequality.* Indian J. Math. 43 (2), 223-230 (2001)

[7] **Stević, S.** : *On the recursive sequence* $x_{n+1} = -\frac{1}{x_n} + \frac{A}{x_{n-1}}$. Int. J. Math. Math. Sci. **27** (1), 1-6 (2001)

[8] **Stević, S. :** *A global convergence result.* Indian J. Math. **44** (3), 361-368 (2002)

[9] **Stević, S. :** *A global convergence result with applications to periodic solutions.* Indian J. Pure Appl. Math. **33** (1), 45-53 (2002)

[10] **Stević, S. :** *On the recursive sequence* $x_{n+1} = A/\prod_{i=0}^{k} x_{n-i} + 1/\prod_{j=k+2}^{2(k+1)} x_{n-j}$. Taiwanese J. Math. **7** (2), 249-259 (2003)

[11] **Xianyi, L.**, and **Deming, Z. :** *Global asymptotic stability in a rational equation.* J. Diff. Equations Appl. **9**, 833-839 (2003)

received: August 10, 2004

Authors:

Cengiz Çinar
Mathematics Department
Faculty of Education
Selcuk University
42090, Konya
Turkey

e-mail: ccinar@selcuk.edu.tr

Ibrahim Yalçinkaya
Mathematics Department
Faculty of Education
Selcuk University
42090, Konya
Turkey

e-mail: iyalcinkaya@selcuk.edu.tr

Stevo Stević
Mathematical Institute of Serbian Academy of Science
Knez Mihailova 35/I
11000 Beograd
Serbia

e-mail: sstevic@ptt.yu

e-mail: sstevo@matf.bg.ac.yu