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Periodic Character of a Difference Equation

ABSTRACT. In this note we prove that every positive solution of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{p + x_{n-1} + x_n}, \quad n = 0, 1...$$

where $p \in [0, \infty)$ and the initial conditions x_{-1}, x_0 are positive real numbers, converges to a, not necessarily prime, periodic-two solution. This result confirms Conjecture 7.5.2 in [1] (with $q = 1$). Also, we show that the positive solutions of Eq.(1) converge to the corresponding periodic-two solutions geometrically.

KEY WORDS AND PHRASES. Period two solution, difference equation, positive solution, asymptotics

1 Introduction

In this note we consider the periodic character of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{p + x_{n-1} + x_n}, \quad n = 0, 1, \ldots \tag{1}$$

where $p \in [0, \infty)$ and the initial conditions x_{-1}, x_0 are positive real numbers. In fact we consider the case $p \in (0, 1)$ since when $p \geq 1$ the zero equilibrium of Eq.(1) is obviously global attractor of all positive solutions of Eq.(1), see [1, Theorem 7.4.1 *(a)*]. The case $p = 0$ was considered, for example, in [1, p. 61, (ii)].

Our motivation here stems from Conjecture 7.5.2 in [1]:

Conjecture 1 *Assume that*

$$p < 1$$

Show that every positive solution of Eq.(1) *converges to a, not necessarily prime, periodic-two solution.*

Note that when $p < 1$ all prime period-two solutions of Eq.(1) are given by

$$...\phi, 1-p-\phi, \phi, 1-p-\phi, ...$$

with

$$0 \le \phi \le 1-p \quad \text{and} \quad \phi \ne \frac{1-p}{2},$$

see, [1, p. 134].

Recently there has been a great interest in studying the periodic nature of nonlinear difference equations. For some recent results concerning, among other problems, the periodic nature of scalar nonlinear difference equations see for example, [1, 2], [4]-[9] and references therein.

Our aim in this note is to confirm Conjecture 1. Also, we show that the positive solutions of Eq.(1) converge to the corresponding periodic-two solutions geometrically and we look for their asymptotics.

2 Main results

In this section we prove the main results in this note.

Theorem 1 *Consider the difference Eq.(1) where $p \in (0,1)$ and initial conditions x_{-1}, x_0 are positive real numbers. Then every positive solution of Eq.(1) converges to a, not necessarily prime, periodic-two solution (ρ_0, ρ_1), such that $p+\rho_0+\rho_1 = 1$. If $p + x_0 + x_{-1} > 1$ the sequences $x_{2n+i}, (i = 1, 2)$ are decreasing, if $p + x_0 + x_{-1} < 1$ the sequences $x_{2n+i}, (i = 1, 2)$ are increasing, and if $p + x_0 + x_{-1} = 1$ the sequence x_n is a periodic-two solution of Eq.(1).*

Proof: By the change of variables $x_n = \frac{1}{z_n}$, Eq.(1) becomes

$$z_{n+1} = \frac{z_n + z_{n-1} + pz_n z_{n-1}}{z_n}. \tag{2}$$

From (2) we have

$$\begin{aligned} z_{n+1} - z_{n-1} &= \frac{z_n + z_{n-1} + pz_n z_{n-1} - z_n z_{n-1}}{z_n} \\ &= \frac{z_n + z_{n-1} + pz_n z_{n-1} - z_{n-1} - z_{n-2} - pz_{n-1} z_{n-2}}{z_n} \\ &= \frac{(pz_{n-1}+1)(z_n - z_{n-2})}{z_n} \end{aligned}$$

and consequently

$$z_{n+1} - z_{n-1} = (z_1 - z_{-1}) \prod_{i=1}^{n} \frac{pz_{i-1}+1}{z_i}. \tag{3}$$

From (3) we obtain that the signum of $z_{n+1} - z_{n-1}$ remains invariant for $n \in \mathbf{N}$ and that the sequences (z_{2n+i}), $i = 0, 1$, are nondecreasing or nonincreasing at the same time which implies that the sequences (x_{2n+i}), $i = 0, 1$, are nonincreasing or nondecreasing at the same time. Since

$$z_1 - z_{-1} = \frac{p + x_0 + x_{-1} - 1}{x_{-1}}$$

we see from (3) that if $p + x_0 + x_{-1} < 1$, then the sequences (x_{2n+i}), $i = 0, 1$ are increasing, if $p + x_0 + x_{-1} > 1$, the sequences (x_{2n+i}), $i = 0, 1$ are decreasing and if $p + x_0 + x_{-1} = 1$, then $(x_{-1}, x_0, x_{-1}, x_0, \ldots)$ is a periodic-two solution of Eq.(1).

First suppose that the sequences (x_{2n+i}), $i = 0, 1$ are decreasing, that is $p + x_0 + x_{-1} > 1$. Then there are finite limits

$$\lim_{n\to\infty} x_{2n+i} = \rho_i, \quad i = 0, 1\,.$$

It is clear that (ρ_0, ρ_1) is a two cycle of Eq.(1). Suppose that both of them are equal to zero. Since (x_{2n+i}), $i = 0, 1$ are decreasing from (1) we obtain

$$p + x_{n-1} + x_n > 1, \quad n = 0, 1, \ldots\,. \tag{4}$$

Letting $n \to \infty$ in (4) we obtain $p \geq 1$ which is a contradiction. Hence $(\rho_0, \rho_1) \neq (0, 0)$ and as we mentioned above it is a two cycle of Eq.(1).

Without loss of generality we may assume that $\rho_1 \neq 0$. Then letting $n \to \infty$ in the equation

$$x_{2n+1} = \frac{x_{2n-1}}{p + x_{2n-1} + x_{2n}}$$

we obtain the equality $p + \rho_0 + \rho_1 = 1$.

Now suppose that the sequences (x_{2n+i}), $i = 0, 1$ are increasing, that is $p + x_0 + x_{-1} < 1$. Then there are finite or infinite limits

$$\lim_{n\to\infty} x_{2n+i} = \rho_i, \; i = 0, 1\,.$$

By a result of L. Berg [2, p. 1070] all solutions of Eq.(1) are bounded, hence $\rho_i < \infty$, $i = 0, 1$. On the other hand, since (x_{2n+i}), $i = 0, 1$ are increasing $\rho_0 > x_0 > 0$ and $\rho_1 > x_1 > 0$. Similarly as above we obtain that (ρ_0, ρ_1) is a two cycle of Eq.(1) and $p + \rho_0 + \rho_1 = 1$.

Finally, for the initial conditions $x_{-1} = x_0 = (1-p)/2$, we have $x_n = (1-p)/2, n \geq -1$, which shows that there is a solution which converges to a not prime period-two solution.

Remark 1 Note that the condition $p + x_0 + x_{-1} > 1$, (e.g. condition (4) for $n = 0$) implies (4) for all greater n, that is, for $n \geq 1$, moreover the sequence $u_n = p + x_{n-1} + x_n$ is also decreasing.

Also, the condition $p+x_0+x_{-1}<1$ and (1) imply that the sequence $u_n = p+x_{n-1}+x_n$ is increasing and

$$p+x_{n-1}+x_n<1, \quad n=0,1,\dots .$$

From this and by Theorem 1 it follows that the distance from the point (x_{n-1}, x_n) to the limit line $p+x+y=1$, i.e.,

$$d_n = \frac{p+x_n+x_{n-1}-1}{\sqrt{2}},$$

also converges monotonously to zero (we use here Hesse's normal form).

For the readers who are interested in this area we leave the following open problem.

Open Problem 1 *Let*

$$\dots, \rho_0, 1-p-\rho_0, \rho_0, 1-p-\rho_0, \dots$$

be a positive two cycle of Eq.(1). Find the basin of attraction of this two cycle.

The following result gives an estimation of the convergence rate of the positive solutions of Eq.(1).

Theorem 2 *Every positive solution of Eq.(1) converges to the corresponding periodic-two solution (ρ_0, ρ_1) geometrically, that is, there is an $M>0$ and $q\in(0,1)$ such that*

$$|x_{2n}-\rho_0|+|x_{2n+1}-\rho_1| \le Mq^{2n}, \quad n\ge 0.$$

Proof: As we have seen in the proof of Theorem 1, using the change $x_n = \frac{1}{z_n}$ we obtain

$$z_{n+1}-z_{n-1} = \frac{(pz_{n-1}+1)(z_n-z_{n-2})}{z_n}.$$

If we go back to the sequence x_n we have

$$\frac{p+x_n+x_{n-1}-1}{x_{n-1}} = \frac{p+x_{n-1}}{x_{n-1}} x_n \frac{p+x_{n-1}+x_{n-2}-1}{x_{n-2}},$$

that is,

$$d_n = (p+x_{n-1})\frac{x_n}{x_{n-2}} d_{n-1},$$

where $d_n = \frac{p+x_n+x_{n-1}-1}{\sqrt{2}}$, and consequently

$$d_n = (p+x_{n-1})\frac{x_n}{x_{n-2}}(p+x_{n-2})\frac{x_{n-1}}{x_{n-3}} d_{n-2}. \tag{5}$$

Let $\varepsilon \in (0, (1-(1+p)^2/4))$. Since the sequences (x_{2n+i}), $i = 0, 1$, are convergent, from (5) we have that for such chosen ε there is an $n_0 \in \mathbf{N}$ such that

$$|d_n| \leq (\,(p+\rho_0)(1-\rho_0)+\varepsilon\,)|d_{n-2}| \leq \left(\left(\frac{1+p}{2}\right)^2+\varepsilon\right)|d_{n-2}|, \tag{6}$$

for every $n \geq n_0$.

In view of the choice of ε we see that $r = \left(\frac{1+p}{2}\right)^2 + \varepsilon < 1$. From this and (6), using the following equality

$$|d_{2n+1}| = \frac{|x_{2n+1}-\rho_1+x_{2n}-\rho_0|}{\sqrt{2}} = \frac{|x_{2n+1}-\rho_1|+|x_{2n}-\rho_0|}{\sqrt{2}}$$

we see that for $q = \sqrt{r}$ we can obtain the result easily. Note that in the last equality we have used the fact that the sequences x_{2n+i}, $i = 0, 1$, converge monotonously to ρ_i, $i = 0, 1$.

Corollary 1 *The distance d_n from the point (x_{n-1}, x_n) to the limit line $p + x + y = 1$, converges to zero monotonously and geometrically.*

3 The case of nonnegative solutions of Eq.(1)

If $x_{-1} = 0$ or $x_0 = 0$, from (1) we obtain $x_{2n-1} = 0$ or $x_{2n} = 0$, for all $n \geq 0$. Further, if $x_{-1} = 0$ then Eq.(1) becomes

$$x_{2n} = \frac{x_{2n-2}}{p + x_{2n-2}}.$$

This is a Riccati equation (see [1, Section 1.6]) for x_{2n} with the elementary solution

$$x_{2n} = \frac{x_0(1-p)}{x_0 + (1-p-x_0)p^n}, \quad n \geq 0\,. \tag{7}$$

From (7) we see that $\lim_{n\to\infty} x_{2n} = 1-p$, so far as x_0 is different from 0. Similarly we can treat the case $x_0 = 0, x_{-1} \neq 0$. The case $x_0 = x_{-1} = 0$ yields the constant solution $x_n = 0$ for all $n \geq -1$.

We believe that only these solutions satisfy the condition $\rho_0\rho_1 = 0$, where as before ρ_i, $i = 0, 1$ denote the limits $\lim_{n\to\infty} x_{2n+i}$. Hence we leave the following conjecture:

Conjecture 1 ([3]) *For positive initial values x_{-1} and x_0 there are no solutions of Eq.(1) such that $\rho_0\rho_1 = 0$.*

4 Asymptotically two-periodic solutions

Theorem 2 motivated us to study the asymptotics of the solutions of Eq.(1), as well as the corresponding ones for the sequence d_n.

Let $u_n = x_{2n-1}$ and $v_n = x_{2n}$, then (1) can be written as the following system

$$u_{n+1} = \frac{u_n}{p + u_n + v_n}$$
$$v_{n+1} = \frac{v_n}{p + u_{n+1} + v_n}. \qquad (8)$$

We expect that the asymptotically two-periodic solutions have the following form (see [2, p.1066])

$$u_n = \rho + \sum_{k=1}^{\infty} a_k c^k t^{nk}, \quad \text{and} \quad v_n = 1 - \rho - p + \sum_{k=1}^{\infty} b_k c^k t^{nk}, \qquad (9)$$

where $t \in (0,1)$ is unknown and c an arbitrary real number.

Substituting (9) into system (8) and comparing the coefficients we obtain

$$a_1 t^n = a_1 t^{n+1} + \rho(a_1 + b_1)t^n, \quad \text{and} \quad b_1 t^n = b_1 t^{n+1} + (1 - \rho - p)(a_1 t^{n+1} + b_1 t^n),$$

which implies

$$(1 - t - \rho)a_1 = \rho b_1, \quad \text{and} \quad t(1 - \rho - p)a_1 = (\rho + p - t)b_1\,. \qquad (10)$$

This system has a nontrivial solution a_1, b_1 if and only if its determinant vanishes, i.e

$$t^2 - (\,1 + p + \rho(1 - \rho - p)\,)t + (1 - \rho)(\rho + p) = 0\,. \qquad (11)$$

The only solution of (11) with t contained in $(0,1)$ is $t = (1-\rho)(\rho+p)$ and the corresponding solution of system (10) is $a_1 = \rho$, $b_1 = (1 - \rho)(1 - \rho - p)$ up to a constant factor c which already appears in the series (9). Therefore

$$u_n = \rho + \rho c t^n + \mathcal{O}(t^{2n}), \quad \text{and} \quad v_n = 1 - \rho - p + (1 - \rho)(1 - \rho - p)c t^n + \mathcal{O}(t^{2n})\,. \qquad (12)$$

The asymptotic formulas (12) for u_n and v_n remain valid in the limit cases $\rho = 0$ and $\rho = 1 - p$, with $t = p$, where they express the asymptotic behaviour of the explicitly known solutions with one vanishing initial value (see Section 3). Note that the asymptotic formulas (12) can also be obtained in the case $u_n = x_{2n}$ and $v_n = x_{2n+1}$. We leave the following conjecture:

Conjecture 2 *Let $p \in (0,1)$ and (x_n) be a nonnegative solution of Eq.(1) such that $(x_{2n-1}, x_{2n}) \to (\rho, 1 - \rho - p)$, as $n \to \infty$. Then*

(a) $x_{2n-1} = \rho + \rho c t^n + \mathcal{O}(t^{2n})$;

(b) $x_{2n} = 1 - \rho - p + (1-\rho)(1-\rho-p)ct^n + \mathcal{O}(t^{2n})$;

where $t = (1-\rho)(\rho+p)$ *and the constant* c *depends on initial values* x_{-1} *and* x_0.

If this conjecture is true, then it follows:

Corollary 2 *Let* $p \in (0,1)$ *and* (x_n) *be a nonnegative solution of Eq.(1) such that* $(x_{2n-1}, x_{2n}) \to (\rho, 1-\rho-p)$, *as* $n \to \infty$. *Then the distance* d_n *from the point* (x_{n-1}, x_n) *to the limit line* $p + x + y = 1$, *has the following asymptotics*

$$d_n = \frac{c}{\sqrt{2}} e_n (1-t)(\sqrt{t})^n + \mathcal{O}(t^n),$$

where $t = (1-\rho)(\rho+p)$, $e_{2n} = 1, e_{2n+1} = 1-\rho$, *for* $n \geq 0$, *and the constant* c *depends on initial values* x_{-1} *and* x_0.

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