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Manfred Krüppel

# On the Zeros of an Infinitely Often Differentiable Function and their Derivatives

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ABSTRACT. In this paper, we investigate the structure of an infinitely often differentiable real function $f$ defined on the interval $[0,1]$. We show that for such a function the set $\{t : \exists n \in \mathbb{N}_0 : f^{(n)}(t) = 0,\ f^{(n+1)}(t) \neq 0\}$ is at most countable, and if $f$ is not a polynomial then the set $\{t : f^{(n)}(t) \neq 0,\ \forall n \in \mathbb{N}_0\}$ has the power $\mathbf{c}$.

KEY WORDS. $C^\infty$-functions, derivatives of higher order, Cantor sets, Theorem of Cantor-Bendixsohn, sets of first category.

In this paper we investigate real functions $f$ on $[0,1]$ which are infinitely often differentiable, where in the endpoints we consider the one-side derivatives. For such a given $f$ we define the sets

$$E = \{t : \exists n \in \mathbb{N}_0 : f^{(n)}(t) = 0\} \tag{1}$$

and their complement

$$D = \{t : f^{(n)}(t) \neq 0,\ \forall n \in \mathbb{N}_0\}, \tag{2}$$

i.e. $E \cup D = [0,1]$. Obviously, if $f$ is a polynomial then $E = [0,1]$. But it holds also the conversion:

**Theorem 1 ([3], [5])** *Let $f$ be an infinitely often differentiable real function over $[0,1]$. If $E = [0,1]$ then $f$ is a polynomial.*

Obviously, for a polynomial $f$ the set $D$ from (2) is empty, so that $D = \emptyset$ if and only if $f$ is a polynomial according to Theorem 1. In this paper we investigate the case $D \neq \emptyset$ and prove a general assertion concerning the structure of an infinitely often differentiable real function (Proposition 3). Theorem 1 is an immediately consequence of Proposition 3. The main results of this note are Theorem 6 and 7 which are proved by means of Proposition 3. In order to prove Proposition 3 we need some preparations.

**Lemma 2** *Every closed set $F \subseteq [0,1]$ has a unique representation as union of three disjoint sets*

$$F = A_0 \cup B_0 \cup C_0 \tag{3}$$

*where $A_0$ is an open set, $B_0$ is a nowhere dense perfect set and $C_0$ is at most countable, where $A_0$, $B_0$ and $C_0$ can be empty.*

**Proof:** We assume that the closed set $F$ is not countable. Then, owing to the Theorem of Cantor-Bendixsohn, cf. [6], p. 55, it is representable in the form

$$F = P_0 \cup Q_0$$

where $P_0$ is a nonempty perfect set and where $Q_0$ is at most countable. If $P_0$ is nowhere dense then it follows (3) with $A_0 = \emptyset$, $B_0 = P_0$ and $C_0 = Q_0$. Assume that $P_0$ is dense in the intervals $[a_n, b_n]$ $(n \in \mathbb{N}_0)$ where these intervals are maximal then we put

$$A_0 = \bigcup_n (a_n, b_n) \tag{4}$$

which is an open set with $A_0 \subseteq P_0$ since $P_0$ is closed. Consequently, the set $F_1 = P_0 \setminus A_0$ is nowhere dense and closed, and it holds $A_0 \cap F_1 = \emptyset$. If $F_1$ is countable then (3) is valid with $A_0$ from (4), $B_0 = \emptyset$ and $C_0 = F_1 \cup Q_0$. If the closed set $F_1$ is not countable then, again by the Theorem of Cantor-Bendixsohn, it is representable as

$$F_1 = P_1 \cup Q_1$$

where $P_1$ is a nonempty perfect set and where $Q_1$ is at most countable. In this case (3) is valid with $A_0$ from (4), $B_0 = P_1$ and $C_0 = Q_0 \cup Q_1$.

Assume that besides of (3) for $F$ there exist a further representation

$$F = A_1 \cup B_1 \cup C_1. \tag{5}$$

If $A_0 \neq A_1$ then we can assume that there exist a point $x_0 \in A_0 \setminus A_1$. This means that there exist an interval $(\alpha, \beta) \subset A_0 \setminus A_1$. Since $F \setminus A_1 = B_1 \cup C_1$ is a set of first category and $(\alpha, \beta)$ is a set of second category by a Theorem of Baire, cf. e.g. [4], the relation $(\alpha, \beta) \subseteq F \setminus A_1$ is impossible. This implies that the case $A_0 \neq A_1$ cannot be. In the case $A_0 = A_1$ the set $P = F \setminus A_0 = F \setminus A_1$ is closed. Therefore it holds $B_0 = B_1$ since this set is exactly equal to the set of all points of condensation of $P$, cf. [6]. Finally, it follows $C_0 = C_1$, too ■

On the structure of an infinitely often differentiable function we have the

**Proposition 3** *Let $f$ be an infinitely often differentiable real function over $[0,1]$. Then the set $E$ of all points $t$ for which there exists an integer $n \in \mathbb{N}_0$ such that $f^{(n)}(t) = 0$ has a unique representation as union of three disjoint sets*

$$E = A \cup B \cup C \tag{6}$$

*which have the following form: $A$ is an open set, i.e.*

$$A = \bigcup_j (\alpha_j, \beta_j), \tag{7}$$

*$B$ is the union of at most countably many nowhere dense perfect sets $B_n$ with $B_n \subseteq B_{n+1}$, and $C$ is at most countable, where $A$, $B$ and $C$ can be empty. In the case $A \neq \emptyset$ the function $f$ is a polynomial on each interval $[\alpha_j, \beta_j]$.*

**Proof:** Obviously, $E$ is the union of the sets $E_n = \{t : f^{(n)}(t) = 0\}$ $(n \in \mathbb{N}_0)$, which are closed owing to the continuity of $f^{(n)}$. Hence, according to Lemma 2 for each $n \in \mathbb{N}_0$ the set $E_n$ is representable as union of three disjoint sets

$$E_n = A_n \cup B_n \cup C_n \tag{8}$$

where $A_n$ is an open set, $B_n$ is a nowhere dense perfect set and $C_n$ is at most countable, where $A_n$, $B_n$ and $C_n$ can be empty. Hence, for the union $E$ of all $E_n$ is representable as (6) where $A$ and $B$ are the union of all $A_n$, $B_n$, respectively, and

$$C = \bigcup_n C_n \setminus (A \cup B)$$

is at most countable. Thus $A$ is an open set which has the form (7) where the components $(\alpha_i, \beta_i)$ are pairwise disjoint, and $A \cap C = B \cap C = \emptyset$.

For $t \in A_n$ and $t \in B_n$ we have $f^{(n+1)}(t) = 0$ so that $A_n \subseteq A_{n+1}$ and $B_n \subseteq B_{n+1}$, respectively. Hence, $A_n \cap B_n = \emptyset$ for all $n$ implies that $A \cap B = \emptyset$, too.

The sets $A_n$, $B_n$ and $C_n$ are unique determined according to Lemma 2. This implies the uniqueness of $A$, $B$ and $C$ in (6).

Finally let be $A \neq \emptyset$. We remember that $A_m \subseteq A_n$ for $n > m$. Assume that $I_n = (a_n, b_n)$ and $I_m = (a_m, b_m)$ are components of $A_n$ and $A_m$, respectively, then either $I_n = I_m$ or $\overline{I}_n \cap \overline{I}_m = \emptyset$. This follows from the fact that $f^{(n-1)}(t) = c \neq 0$ for $t \in \overline{I}_n$ and $f^{(n-1)}(t) = 0$ for $t \in \overline{I}_m$. Consequently, $f$ is a polynomial on each interval $[\alpha_j, \beta_j]$. ■

**Remarks 4** **1.** In case $E = [0,1]$ we have $A = (0,1)$, $C = \{0,1\}$, and $f$ is a polynomial on $[0,1]$ so that Theorem 1 is a consequence of Proposition 3.

**2.** In case $A \neq \emptyset$ the endpoints of each component $(\alpha_i, \beta_i)$ belong to $E$. Between two intervals $(\alpha_i, \beta_i)$, $(\alpha_j, \beta_j)$ of $A$ there exists at least one point $t_0 \notin E$. If namely $(\alpha_i, \beta_j) \subseteq E$ where $\alpha_i < \alpha_j$ then, owing to Theorem 1, the function $f$ is equal to a polynomial of degree $n$. Hence, $(\alpha_i, \beta_j) \subseteq A_n$ which is impossible in view of the unique representation of $A_n$ according to Proposition 3.

Let us consider some examples for the different possibilities of the sets $E$, $A$, $B$, $C$ in Proposition 3. Obviously, if $f$ is a polynomial then $E = [0,1]$, $A = (0,1)$, $B = \emptyset$ and $C = \{0,1\}$, but also the case $E = \emptyset$ is possible, e.g. for $f(t) = e^t$. For further possibilities let us consider the homogeneous integral-functional equation

$$\phi(t) = b \int\limits_{at-a+1}^{at} \phi(\tau) d\tau \qquad \left(b = \frac{a}{a-1}\right) \tag{9}$$

with the real variable $t$ and a parameter $a > 1$, cf. [1], [2]. The solutions of (9) were studied for $a = 3$ in Wirsching [9], for $a = 2$ in Schnabl [7] and Volk [8], and for $a > \frac{3}{2}$ in Wirsching [10]. In [1] it was shown that for $a > 1$ equation (9) has a $C^\infty$-solution with the support $[0,1]$ which is uniquely determined by the normalization

$$\int\limits_0^1 \phi(t) dt = 1. \tag{10}$$

In case $a = 2$ the solution $\phi$ has the property $\phi^{(n)}(t) = 0$ if and only if $t = \frac{k}{2^n}$ with $k \in 0, 1, \ldots, 2^n$, cf. [2], formula (4.8), so that in this case we have $A = B = \emptyset$ and $C$ is the countable set of all dyadic rational numbers in $[0,1]$. In case $a > 2$ the solution $\phi$ is a polynomial on each component of an open Cantor set $G$ with Lebesgue measure $|G| = 1$, and the set of all $t \notin G$ with $\phi^{(n)}(t) = 0$ with a certain $n \in \mathbb{N}$ is countable, cf. formula (4.7) in [2]. Hence, in this case we have $A = G$, i.e. $\overline{A} = [0,1]$, $B = \emptyset$ and $C$ is the set of all endpoints of the components of $G$.

The following example shows that also the case $B \neq \emptyset$ is possible.

**Example 5** Let $f_0$ be any infinitely often differentiable function over $[0,1]$ with $f_0(t) > 0$ for $0 < t < 1$ and $f_0^{(k)}(0) = f_0^{(k)}(1) = 0$ for all $k \in \mathbb{N}_0$, e.g.

$$f_0(t) = e^{\frac{1}{t(1-t)}}. \tag{11}$$

For a given nowhere dense perfect set $B_0 \subseteq [0,1]$ with $0, 1 \in B_0$ the open complement $G = [0,1] \setminus B_0$ is representable as union of pairwise disjoint intervals $(a_j, b_j)$ $(j \in \mathbb{N})$. We define a function $f$ by $f(t) = 0$ for $t \in B_0$ and by

$$f(t) = c_j f_0 \left(\frac{t - a_j}{b_j - a_j}\right)$$

for $a_j < t < b_j$, and

$$c_j = \frac{1}{j\,M_j} \tag{12}$$

where

$$M_j = \max_{k\in\{0,\dots,j\}} \max_{a_j<t<b_j} \frac{1}{(b_j-a_j)^k \min(t-a_j, b_j-t)} \left| f_0^{(k)}\left(\frac{t-a_j}{b_j-a_j}\right)\right|. \tag{13}$$

The number $M_j$ exists in view of the continuity of $f_0^{(k)}$ and $f_0^{(k+1)}(0) = f_0^{(k+1)}(1) = 0$ so that $c_j > 0$ for all $j$. Consequently, it holds $E_0 = B_0$. Obviously, for $a_j < t < b_j$ and $k \in \mathbb{N}_0$ it holds

$$f^{(k)}(t) = \frac{c_j}{(b_j-a_j)^k} f_0^{(k)}\left(\frac{t-a_j}{b_j-a_j}\right). \tag{14}$$

We show by induction with respect to $k$ that $f^{(k)}(t) = 0$ for $t \in B_0$. This is true for $k = 0$ according to the definition of $f$. Assume that this is true for a fixed $k$. Let $t_0 \in B_0$ and $t_n \neq t_0$ a sequence which converges to $t_0$. If $t_n \in B_0$ then

$$\frac{f^{(k)}(t_n) - f^{(k)}(t_0)}{t_n - t_0} = 0.$$

Hence, it suffices to consider the case that $t_n \in [0,1] \setminus B_0$ for all $n \in \mathbb{N}$, i.e. $t_n \in (a_{j_n}, b_{j_n})$. Obviously, we need to investigate only two cases: **1.** the sequence $j_n$ is bounded and **2.** $j_n \to \infty$ as $n \to \infty$. The first case is only possible if for $n \geq n_0$ all $t_n$ belong to the same interval $(a_j, b_j)$ and $t_0$ is an endpoint of $(a_j, b_j)$. Then we have

$$\lim_{n\to\infty} \frac{f^{(k)}(t_n) - f^{(k)}(t_0)}{t_n - t_0} = 0$$

in view of $f_0^{(k+1)}(0) = f_0^{(k+1)}(1) = 0$. In the second case we can choose an integer $n_0$ such that $j_n \geq k$ for $n \geq n_0$. From (14) and $f^{(k)}(t_0) = 0$ we obtain

$$\left|\frac{f^{(k)}(t_n) - f^{(k)}(t_0)}{t_n - t_0}\right| = \frac{c_{j_n}}{(b_{j_n} - a_{j_n})^k\,|t_n - t_0|} \left| f_0^{(k)}\left(\frac{t_n - a_{j_n}}{b_{j_n} - a_{j_n}}\right)\right|.$$

Since $|t_0 - t_n| \geq \min(t_n - a_{j_n}, b_{j_n} - t_n)$ we get for $n \geq n_0$ in view of (12), (13) and $k \leq j_n$ that

$$\left|\frac{f^{(k)}(t_n) - f^{(k)}(t_0)}{t_n - t_0}\right| \leq \frac{1}{j_n} \to 0$$

for $n \to \infty$. Altogether we obtain $f^{(k+1)}(t_0) = 0$. According to Proposition 3 it holds $B_0 \subseteq B$ so that here we have an example for an infinitely often differentiable function $f$ with $B \neq \emptyset$.

**Theorem 6** *Let $f$ be an infinitely often differentiable real function over $[0,1]$. Then the set $M = \{t : \exists n \in \mathbb{N}_0 : f^{(n)}(t) = 0, f^{(n+1)}(t) \neq 0\}$ is at most countable.*

**Proof:** For $n \in \mathbb{N}_0$ let $M_n$ the set of all points $t \in [0,1]$ with $f^{(n)}(t) = 0$ and $f^{(n+1)}(t) \neq 0$. Hence, $M_n \subseteq E_n$ with the notations of Proposition 3, cf. (8), where $A_n$ is an open set. Let $(\alpha, \beta)$ be a component of $A_n$ then $f$ is a polynomial of degree $m$. Hence, for $n < m$ the number of points $t \in (\alpha, \beta)$ with $f^{(n)}(t) = 0$ is finite and for $n \geq m$ there is no point with $f^{(n+1)}(t) \neq 0$. It follows that $M_n \cap A_n$ is at most countable. For $t \in B_n$ we have $f^{(n+1)}(t) = 0$ so that $M_n \cap B_n = \emptyset$, i.e. $M_n \subseteq A_n \cap C_n$. It follows that $M$ is at most countable ■

Obviously, for a polynomial $f$ the set $D = \{t : f^{(n)}(t) \neq 0,\ \forall n \in \mathbb{N}_0\}$ is empty.

**Theorem 7** *Let $f$ be an infinitely often differentiable real function over $[0,1]$. If $f$ is not a polynomial then the set $D = \{t : f^{(n)}(t) \neq 0,\ \forall n \in \mathbb{N}_0\}$ has the power* $\mathbf{c}$.

**Proof:** Let $D$ be a nonempty set. We apply Proposition 3 with the introduced notations. Obviously, the set $D$ is the complement of $E$ so that $E \subset [0,1]$ since $D \neq \emptyset$. We consider two cases:

**1.** Assume that there exists an interval $I = (a,b)$ without points of $A$. Then according to Proposition 3 it holds the disjoint decomposition

$$I = (I \cap B) \cup (I \cap C) \cup (I \cap D)$$

where the first and the second set on the right-hand side are sets of first category. Consequently, $I \cap D$ is a set of second category and so $D$ has the power $\mathbf{c}$, cf. [4], 10.12.

**2.** Assume that $[0,1] \setminus A$ is nowhere dense in $[0,1]$, i.e. $\overline{A} = [0,1]$ where because of $D \neq \emptyset$ the case $A = (0,1)$ is excluded in view of Remark 4.1. It follows from Proposition 3 that $A$ is the union of countably many open intervals $(\alpha_i, \beta_i)$ which are pairwise disjoint, cf. (7). Hence, the set $[0,1] \setminus A$ is a nowhere dense perfect set. Then there exists a continuous increasing function $g$ with $g(0) = 0$, $g(1) = 1$ and $g(t) = g_i$ for $t \in (\alpha_i, \beta_i)$ with $g_i \neq g_j$ for $i \neq j$ where the countable set $g(A)$ of all $g_i$ is dense in $[0,1]$, cf. Cantor's stair function. For the set

$$A^* = \bigcup_i [\alpha_i, \beta_i]$$

we have $g(A^*) = g(A) = \{g_i\}$ and the restriction of $g$ to $[0,1] \setminus A^*$ is even strictly increasing and has the following property:

**(i)** The map $g : ([0,1] \setminus A^*) \mapsto [0,1] \setminus g(A^*)$ is bijektive.

According to Remark 4 the set $D$ from (2) is a subset of $[0,1] \setminus A^*$. Next we show that for all $n$ the sets $g(B_n)$ are nowhere dense. Assume that there exists an $n$ such that $g(B_n)$ is dense in an interval $(g_i, g_j)$ with $i \neq j$ then $[g_i, g_j] \subseteq g(B_n)$ since $g(B_n)$ is closed in view of the continuity of $g$. This implies owing to **(i)** that all points of the set $(\alpha_i, \beta_j) \setminus A$ belong to $B_n \subseteq E$ which is impossible, cf. Remark 4. Consequently, $g(B_n)$ is nowhere dense so that $g(B)$ is a set of first category. This is true also for the union $g(A) \cup g(B) \cup g(C)$ since $g(A)$ and $g(C)$ are at most countable sets. This implies that $g(D)$ is a set of second category so that it has the power $\mathbf{c}$, cf. [4]. Since $D \subseteq [0,1] \setminus A^*$ it follows from **(i)** that also the set $D$ has the power $\mathbf{c}$ ■

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**Author:**

Manfred Krüppel
Universität Rostock
Institut für Mathematik
18051 Rostock
Germany

e-mail: manfred.krueppel@uni-rostock.de