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Anti-Maximum Principle for a Schrödinger Equation in $\mathbb{R}^N$, with a non radial potential

ABSTRACT. Anti-maximum for the Schrödinger equation $-\Delta u + q(x)u - \lambda u = f(x)$ in $L^2(\mathbb{R}^N)$ is extended to potentials q non necessarily radial. The anti-maximum is proved in the following form: Let φ_1 denote the positive eigenfunction associated with the principal eigenvalue λ_1 of the Schrödinger operator $\mathcal{A} = -\Delta + q(x)\bullet$ in $L^2(\mathbb{R}^N)$. Assume the potential $q(x)$ grows fast enough near infinity, and the function f satisfy $f \not\equiv 0$ and $0 \le f/\varphi_1 \le C \equiv$ *const* a.e. in $\mathbb{R}^N$. Then there exists a positive number δ (depending upon f) such that, for every $\lambda \in (\lambda_1, \lambda_1 + \delta)$, the inequality $u \le -c\varphi_1$ holds a.e. in $\mathbb{R}^N$, where c is a positive constant depending upon f and λ.

KEY WORDS. Positive or negative solutions; pointwise bounds; principal eigenvalue; positive eigenfunction; strong maximum and anti-maximum principles

1 Introduction

The anti-maximum for the Dirichlet Laplacian defined in a regular bounded domain $\Omega \subset \mathbb{R}^N$ is an important result established first by Ph. Clément and L. A. Peletier [4] and extended to several types of elliptic operators or systems defined in a bounded domain, see e.g. G. Sweers [13], P. Takáč [14],G. Fleckinger et al.[5, 6]. The case of the Schrödinger operator on $\Omega = \mathbb{R}^N$ is more difficult. Indeed, for maximum and anti-maximum in unbounded domain the works of B. Alziary and P. Takáč [3], B. Alziary, G. Fleckinger and P. Takáč [1, 2] and Y. Pinchover [8, 9] show that, on must always take into account the growth of the solution near the infinity.

We investigate here, anti-maximum for a linear partial differential equation with the Schrödinger operator,

$$-\Delta u + q(x)u - \lambda u = f(x) \quad \text{in } \mathbb{R}^N, \tag{1}$$

Here, f is a given function satisfying $0 \le f \not\equiv 0$ in $\mathbb{R}^N$ $(N \ge 1)$, and λ stands for the spectral parameter. As usual, the Schrödinger operator takes the form $\mathcal{A} = -\Delta + q(x)\bullet$ in

$L^2(\mathbb{R}^N)$ where Δ and $q(x)\bullet$, respectively, denote the selfadjoint Laplace operator and the pointwise multiplication operator by the potential q in $L^2(\mathbb{R}^N)$. Let φ_1 denote the positive eigenfunction of $\mathcal{A}$ associated with the principal eigenvalue λ_1. We recall the definition of φ_1-positivity and φ_1-negativity.

Definition 1.1 *A function $u \in L^2(\mathbb{R}^N)$ is called* φ_1-positive *if there exists a constant $c > 0$ such that*

$$u \geq c\varphi_1 \quad \textit{almost everywhere in } \mathbb{R}^N. \tag{2}$$

Analogously, $u \in L^2(\mathbb{R}^N)$ is called φ_1-negative *if there exists a constant $c > 0$ such that*

$$u \leq -c\varphi_1 \quad \textit{almost everywhere in } \mathbb{R}^N. \tag{3}$$

To obtain anti-maximun for the Schrödinger operator on $\Omega = \mathbb{R}^N$, we need to assume f in the strongly ordered Banach space X introduced in Alziary and Takáč [3]:

$$X = \{u \in L^2(\mathbb{R}^N)\colon u/\varphi_1 \in L^\infty(\mathbb{R}^N)\} \tag{4}$$

endowed with the ordered norm

$$\|u\|_X = \inf\{C \in \mathbb{R}\colon |u| \leq C\varphi_1 \text{ almost everywhere in } \mathbb{R}^N\}. \tag{5}$$

The ordering "$\leq$" on X is the natural pointwise ordering of functions. This means that X is an ordered Banach space whose positive cone X_+ has nonempty interior $\mathring{X}_+$. Taking $N \geq 2$, The necessity of such a restriction for the Schrödinger operator in $L^2(\mathbb{R}^N)$ has been justified in [2, Example 4.1 p. 377]. B. Alziary, G. Fleckinger and P. Takáč construct a counterexample to the anti-maximum principle (3) for a positive, radially symmetric function $f \in L^2(\mathbb{R}^N) \setminus X$.

The validity of (2) for a "sufficiently smooth" solution u to Equation (1) is established in Alziary and Takáč [3, Theorem 2.1, p. 284] for a nonnegative function $f \not\equiv 0$ in $L^2(\mathbb{R}^N)$. The inequality (3) is shown in Alziary, Fleckinger and Takáč [1, 2] under considerably more restrictive hypotheses on q and f, since they consider only radially symmetric potentials and they establish the anti-maximum only for f from a Banach space $X^{\alpha,2}$ that contains "sufficiently smooth" perturbations of radially symmetric functions of X.

In the present work we are able to extend this results to some non radial potential and for $f \in X_+ \setminus \{0\}$ and $f \in C^{0,\alpha}(\mathbb{R}^N)$. For either (2) or (3) to be valid, it is necessary and sufficient that the potential $q(x)$, which is assumed to be strictly positive and locally bounded, have a superquadratic growth as $|x| \to \infty$. In particular, $q(x)$ must grow faster than $|x|^2$ as $|x| \to \infty$; the growth like $|x|^{2+\varepsilon}$ with any constant $\varepsilon > 0$ is sufficient. Thus, both (2) and (3) are in general false for the harmonic oscillator, i.e., for $q(x) = |x|^2$ in $\mathbb{R}^N$; see [1], Examples 4.1 and 4.2. As it seems to be inevitable in the theory of Schrödinger operators, we assume that $q(x)$ is a "relatively small" perturbation of a radially symmetric function, $q(x) = q_1(|x|) + q_2(x)$ for $x \in \mathbb{R}^N$.

Y. Pinchover in [8, 9] prove the inequalities (2) and (3) for any solution $f \in X_+ \setminus \{0\}$,but imposing certain growth conditions on the first derivatives of $q(x)$, and assuming the solution u is already in X. Our method combine a comparison result from B. Alziary and P. Takáč [3, Theorem 2.2, p. 285] in the exterior domain $\Omega_R = \{ x \in \mathbb{R}^N : |x| > R \}$, for $0 < R < \infty$ and the approach of Y. Pinchover in the proof of [8, theorem 5.3, p.23]. We study the behavior of the principle eigencurve of a certain two parameter eigenvalue problem and prove the anti-maximum principle using a fixed point argument.

This article is organized as follows. In Section 2 we state our main result, Theorem 2.1. There, the inequality (3) for $\lambda_1 < \lambda < \lambda_1 + \delta$ is stated for the solution u of (1). In Section 3 we first recall the comparison result we will used and then give the proof of the main result.

2 The Main Result

Notation. We denote by $\mathbb{R}^N$ the N-dimensional Euclidean space ($N \geq 2$) endowed with the inner product $x \cdot y$ and the norm $|x| = (x \cdot x)^{1/2}$, for $x, y \in \mathbb{R}^N$. We write $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}^N_+ = (\mathbb{R}_+)^N \subset \mathbb{R}^N$. For a set $M \subset \mathbb{R}^N$, we denote by ∂M ($\overline{M}$, and $\mathring{M}$, respectively) the boundary (closure, and interior) of the set M in $\mathbb{R}^N$. We use analogous notation for sets in all Banach spaces.

Given a set $\Omega \subset \mathbb{R}^N$ and $1 \leq p \leq \infty$, we use the following standard Banach spaces of functions $f : \Omega \to \mathbb{R}$ (or $\mathbb{C}$), see e.g. Gilbarg and Trudinger [7, Chapt. 7]:

$L^p(\Omega)$, where Ω is Lebesgue measurable, is the Lebesgue space of all (equivalence classes of) Lebesgue measurable functions $f : \Omega \to \mathbb{R}$ with the norm

$$\|f\|_p \equiv \|f\|_{L^p(\Omega)} \stackrel{\text{def}}{=} \begin{cases} \left(\int_\Omega |f(x)|^p \, dx\right)^{1/p} < \infty & \text{if } 1 \leq p < \infty; \\ \operatorname*{ess\,sup}_{x \in \Omega} |f(x)| < \infty & \text{if } p = \infty. \end{cases}$$

The space $W^{k,p}(\Omega)$, where $k \geq 1$ is an integer and Ω open in $\mathbb{R}^N$, is the Sobolev space of all functions $f \in L^p(\Omega)$ whose all partial derivatives of order $\leq k$ also belong to $L^p(\Omega)$. The norm $\|f\|_{k,p} \equiv \|f\|_{W^{k,p}(\Omega)}$ in $W^{k,p}(\Omega)$ is defined in a natural way.

The local Lebesgue and Sobolev spaces $L^p_{\text{loc}}(\Omega)$ and $W^{k,p}_{\text{loc}}(\Omega)$ are defined analogously.

The holder spaces $\mathcal{C}^{k,\alpha}(\mathbb{R}^N)$ are defined as the subspaces of $\mathcal{C}^k(\mathbb{R}^N)$ consisting of functions whose K-th order partial derivatives are locally Hölder continuous with exponent α.

Finally, for Ω open in $\mathbb{R}^N$, $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ is the space of all infinitely many times differentiable functions $f\colon \Omega \to \mathbb{R}$ with compact support. It is well-known that $\mathcal{D}(\mathbb{R}^N)$ is a dense linear subspace of both $L^p(\mathbb{R}^N)$ and $W^{k,p}(\mathbb{R}^N)$ for $1 \le p < \infty$.

In order to formulate our hypothesis on the potential $q(x)$, $x \in \mathbb{R}^N$, we first introduce the following class of auxiliary functions $Q(r)$ of $r \equiv |x|$, $R_0 \le r < \infty$, for some $R_0 > 0$:

$$\begin{cases} Q(r) > 0, \quad Q \text{ is locally absolutely continuous,} \\ Q'(r) \ge 0, \qquad \text{and there exists a constant } \beta \text{ with} \\ 0 < \beta < \frac{1}{2} \text{ and } \int_{R_0}^\infty Q(r)^{-\beta}\, dr < \infty. \end{cases} \tag{6}$$

We assume that the potential q takes the form

$$q(x) = q_1(|x|) + q_2(x), \quad x \in \mathbb{R}^N,$$

where $q_1(r)$ and q_2 are Lebesgue measurable functions satisfying the following hypothesis, with some auxiliary function $Q(r)$ which obeys (6):

Hypothesis (**H1**) The potential $q\colon \mathbb{R}_+ \to \mathbb{R}$ is locally essentially bounded, $q(r) \ge \text{const} > 0$ for $r \ge 0$, and there exists a constant $c_1 > 0$ such that

$$c_1 Q(r) \le q(r) + \frac{(N-1)(N-3)}{4r^2} \quad \text{for} \quad R_0 \le r < \infty. \tag{7}$$

(**H2**) The potential $q_2\colon \mathbb{R}_+ \to \mathbb{R}$ is locally essentially bounded, $q(x) = q_1(|x|) + q_2(x) \ge \text{const} > 0$ for $r \ge 0$, and there exists a constant $c_2 > 0$ such that

$$|q_2(x)| \le c_2 Q(|x|)^{\frac{1}{2}-\beta} \quad \text{for } x \in \mathbb{R}^N. \tag{8}$$

Notice that the fraction $(N-1)(N-3)/4r^2$ in the inequality (7) is not essential and has been added for convenience in later applications; it can be left out.

Next we introduce the quadratic form

$$(v, w)_q \stackrel{\text{def}}{=} \int_{\mathbb{R}^N} (\nabla v \cdot \nabla w + q(x) v w)\, dx \tag{9}$$

defined for every pair

$$v, w \in V_q \stackrel{\text{def}}{=} \{f \in L^2(\mathbb{R}^N)\colon (f,f)_q < \infty\}. \tag{10}$$

Notice that V_q is a Hilbert space with the inner product $(v,w)_q$ and the norm $\|v\|_{V_q} = ((v,v)_q)^{1/2}$. The set $\mathcal{D}(\mathbb{R}^N)$ is a dense linear subspace of V_q. By the Lax-Milgram theorem, the Schrödinger operator

$$\mathcal{A} = -\Delta + q(x)\,\bullet \quad \text{in} \quad L^2(\mathbb{R}^N) \tag{11}$$

is defined to be the selfadjoint operator in $L^2(\mathbb{R}^N)$ satisfying

$$\int_{\mathbb{R}^N} (\mathcal{A}v)w\,dx = (v,w)_q \quad \text{for all} \;\; v,w \in \mathcal{D}(\mathbb{R}^N). \tag{12}$$

We denote by $\mathcal{D}(\mathcal{A})$ its domain. The Banach space $\mathcal{D}(\mathcal{A})$ endowed with the graph norm is compactly embedded into $L^2(\mathbb{R}^N)$, by Rellich's theorem combined with $q(x) \to \infty$ as $|x| \to \infty$.

It is well-known that $\mathcal{A}$ possesses an infinite sequence of positive eigenvalues, $\lambda_1 < \lambda_2 < \cdots \lambda_n \cdots$, and the first one, denote by λ_1, is given by

$$\lambda_1 = \inf\left\{(f,f)_q \colon f \in V_q \;\text{ with }\; \|f\|_{L^2(\mathbb{R}^N)} = 1\right\}, \quad \lambda_1 > 0.$$

The eigenvalue λ_1 is simple with the eigenspace spanned by an eigenfunction $\varphi_1 \in \mathcal{D}(\mathcal{A})$ satisfying $\varphi_1 > 0$ throughout $\mathbb{R}^N$. We normalize φ_1 by the condition $\|\varphi_1\|_{L^2(\mathbb{R}^N)} = 1$. Since $q(x) \equiv q(|x|)$ for $x \in \mathbb{R}^N$, we must have also $\varphi_1(x) \equiv \varphi_1(|x|)$ for $x \in \mathbb{R}^N$. Furthermore, if $u \in \mathcal{D}(\mathcal{A})$ and $\mathcal{A}u = f \in L^2(\mathbb{R}^N)$ with $f \in L^p_{\text{loc}}(\mathbb{R}^N)$ for some p with $2 \le p < \infty$, then the local L^p-regularity theory yields $u \in W^{2,p}_{\text{loc}}(\mathbb{R}^N)$, see Gilbarg and Trudinger [7, Theorem 9.15, p. 241]. In particular, if $p > N$ then $u \in C^1(\mathbb{R}^N)$, by the Sobolev imbedding theorem [7, Theorem 7.10, p. 155]. It follows that also $\varphi_1 \in C^1(\mathbb{R}^N)$.

The following theorem about φ_1-negativity of u is our main result:

Theorem 2.1 *Let the hypotheses* (**H1**) *and* (**H2**) *be satisfied and q be locally Hölder continuous. Assume that $u \in \mathcal{D}(\mathcal{A})$, $\mathcal{A}u - \lambda u = f \in L^2(\mathbb{R}^N)$, $\lambda \in \mathbb{R}$. Let $f \in X \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^N)$ be a nonnegative function with $f > 0$ in some set of positive Lebesgue measure. Then there exists a positive number δ (depending upon f) such that, for every $\lambda \in (\lambda_1, \lambda_1 + \delta)$, the inequality*

$$u \le -c\varphi_1 \quad \text{in} \;\; \mathbb{R}^N \tag{13}$$

is valid with a constant $c > 0$ (depending upon f and λ).

If we choose $\delta < \lambda_2 - \lambda_1$, for any $\lambda_1 < \lambda < \lambda_1 + \delta$, the solution of the equation, $\mathcal{A}u - \lambda u = f \in L^2(\mathbb{R}^N)$, always exists and is unique. So it suffices to show the existence of a φ_1-negative solution for $\lambda_1 < \lambda < \lambda_1 + \delta$ as in Y. Pinchover [8, 9]. Y. Pinchover proved that for any $f \in X_+ \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^N)$, $f \not\equiv 0$, there exists a positive number δ such that, for every $\lambda \in (\lambda_1, \lambda_1 + \delta)$, any solution u of X is φ_1-negative. Here we prove that $u \in X$. Moreover, his hypothesis on q_1 is much stronger than ours in that he requires that $\log q_1$ be uniformly Lipschitz in $\mathbb{R}^N$ and q_1 itself satisfy $q_1'(r) \ge 0$ and $\int^\infty q_1(r)^{-1/2}dr < \infty$.

3 Proof of the Main Result

We first recall some comparison result and then prove our theorem.

3.1 Preliminary result

The following theorem, proved by B. Alziary and P. Takáč in [3, Theorem 2.2 p. 285], establish a comparison result for positive solution $u(x)$ and $u_1(x)$ of the Schrödinger equation in the exterior domain Ω_R with the potentials $q(x)$ and $q_1(x)$, respectively, and $f \equiv 0$ in Ω_R:

Theorem 3.1 *Let the hypotheses (**H1**) and (**H2**) be satisfied. Furthermore, fix any constant $R \geq R_0$ such that $Q(R)^{\frac{1}{2}+\beta} \geq 2c_2/c_1$. Assume that u and u_1 are two functions of $x \in \mathbb{R}^N$ such that u, $u_1 \in \mathcal{D}(\mathcal{A})$, both u and u_1 are positive and continuous throughout $\overline{\Omega}_R$, for some $R > 0$, and the following equations hold in the sense of distributions over Ω_R,*

$$-\Delta u + q(x)u = 0 \quad \text{in } \Omega_R, \tag{14}$$

$$-\Delta u_1 + q_1(|x|)u_1 = 0 \quad \text{in } \Omega_R. \tag{15}$$

Then there exists a positive constant γ (depending only upon the potential q) such that:

$$\gamma^{-1}\frac{m_u}{u_1(R)}u_1(|x|) \leq u(x) \leq \gamma\frac{M_u}{u_1(R)}u_1(|x|) \text{ for a.e. } x \in \overline{\Omega}_R, \tag{16}$$

with

$$m_u = \min_{|x|=R} u(x) \quad \text{and} \quad M_u = \max_{|x|=R} u(x).$$

3.2 Proof of the Theorem

Since $\mathcal{A}$ has a discrete spectrum, there exists δ_0 such that $(\lambda_1, \lambda_1+\delta_0) \cap \sigma(\mathcal{A}) = \emptyset$. Therefore, it is enough to show that there exists $\delta \leq \delta_0$ such that, for every $\lambda \in (\lambda_1, \lambda_1+\delta)$ the equation $\mathcal{A}u - \lambda u = f$ admits a negative solution u_λ, satifying $-u_\lambda \geq c\varphi_1$ with a positive constant c.

Set $w_\lambda = -u_\lambda$, the equation becomes

$$(\mathcal{A} + f(x)/w_\lambda - \lambda)w_\lambda = (-\Delta + q(x) + f(x)/w_\lambda - \lambda)w_\lambda = 0 \text{ in } \mathbb{R}^N. \tag{17}$$

Now, we need to prove that the equation (17) has a positive solution w_λ, satisfying $w_\lambda \geq c\varphi_1$ with a positive constant c.

First, for $\lambda_1 < \lambda \leq \lambda_1 + 1$, we introduce the following set of functions:

$$\begin{aligned} Y_\lambda = \{\, u \in \mathcal{D}(\mathcal{A}), \quad & u > 0,\ u(0) = \varphi_1(0),\ \text{and} \\ & \exists V \in \mathcal{C}^{0,\alpha}(\mathbb{R}^N), \quad 0 \leq V \leq 1,\ \text{s.t.} \quad (\mathcal{A} - \lambda + V)u = 0 \,\} \end{aligned} \tag{18}$$

First we prove that Y_λ *is a nonempty convex compact set.*

(i) Y_λ *is nonempty:* Indeed, for $V_\lambda = \lambda - \lambda_1$, we have $0 \leq V_\lambda \leq 1$ and the eigenfunction φ_1 is solution of the equation $(\mathcal{A} - \lambda + V_\lambda)\varphi_1 = 0$. Therefore $\varphi_1 \in Y_\lambda$.

(ii) Y_λ *is convex:* Let u_1 and u_2 be two functions of Y_λ. This functions u_1 and u_2 satisfy respectively the equations $(\mathcal{A} - \lambda + V_1)u_1 = 0$ and $(\mathcal{A} - \lambda + V_2)u_2 = 0$, with $0 \leq V_1, V_2 \leq 1$. Let $0 < t < 1$ and denote $u_t = tu_1 + (1-t)u_2$. We check easily that u_t is solution of $(\mathcal{A} - \lambda + V_t)u_t = 0$, with $0 \leq t\frac{u_1}{u_t}V_1 + (1 - t\frac{u_1}{u_t})V_2 \leq 1$. So $u_t \in Y_\lambda$.

(iii) Let us prove now that *there exists* $C > 0$ *such that*

$$C^{-1}\varphi_1(x) \leq u(x) \leq C\varphi_1(x) \text{ for all } x \in \mathbb{R}^n$$

for every $u \in Y_\lambda$ *and* $\lambda_1 \leq \lambda \leq \lambda_1 + 1$.

We introduce now ψ_1 the radial eigenfunction corresponding to the eigenvalue Λ_1 of the Schrödinger operator $-\Delta + q_1(|x|)\bullet$.

Notice that, since $\lambda_1 < \lambda \leq \lambda_1 + 1$ and $0 \leq V \leq 1$, we have

$$q - \lambda_1 - 1 \leq q + V - \lambda \leq q - \lambda_1 + 1.$$

The potential q goes to $+\infty$ as $|x|$ goes to ∞, so there exists R_1 such that $0 < const < q(x) - \lambda_1 - 1 \leq q(x) + V(x) - \lambda \leq q(x) - \lambda_1 + 1$ for all $|x| \geq R_1$. Thus principal eigenvalues corresponding to those potentials on Ω_{R_1} are all positive. We choose R_1 large enough, so that we could apply theorem 3.1 with the potentials $q(x) - \lambda_1 + 1$ and $q_1(|x|) - \Lambda_1$, $q(x) - \lambda_1$ and $q_1(|x|) - \Lambda_1$, or $q(x) - \lambda_1 - 1$ and $q_1(|x|) - \Lambda_1$.

Let us take any $u \in Y_\lambda$. Now we split our proof of **(iii)** into the cases $x \in \overline{B}_{R_1}$ and $x \in \Omega_{R_1}$.

Case $x \in \Omega_{R_1}$ Denote by $\underline{u}$ and $\overline{u}$ the solutions of the following equations:

$$\begin{cases} -\Delta u + (q + V - \lambda)u = 0 & \text{in } \Omega_{R_1} \\ -\Delta \underline{u} + (q - \lambda_1 + 1)\underline{u} = 0 & \text{in } \Omega_{R_1} \\ -\Delta \overline{u} + (q - \lambda_1 - 1)\overline{u} = 0 & \text{in } \Omega_{R_1} \\ \underline{u}(x) = \overline{u}(x) = u(x) & \text{on } \partial\Omega_{R_1} \end{cases} \tag{19}$$

Since $q - \lambda_1 - 1 \leq Q + V - \lambda \leq q - \lambda_1 + 1$, by the weak maximum principle on Ω_{R_1}, we have:

$$\underline{u} \leq u \leq \overline{u} \text{ in } \overline{\Omega}_{R_1} \tag{20}$$

For the eigenfunctions φ_1 and ψ_1, the following equations hold for all $R > 0$,

$$\begin{cases} -\Delta\varphi + (q(x) - \lambda_1)\varphi_1 = 0 & \text{in } \Omega_R, \\ -\Delta\psi_1 + (q_1(|x|) - \Lambda_1)\psi_1 = 0 & \text{in } \Omega_R. \end{cases} \tag{21}$$

So applying the theorem 3.1 on Ω_{R_1} for φ_1 and ψ_1, there exists a positive constant γ, (depending only upon the potential q) such that :

$$\gamma^{-1}\frac{m_{\varphi_1}}{\psi_1(R_1)}\psi_1(|x|) \leq \varphi(x) \leq \gamma\frac{M_{\varphi_1}}{\psi_1(R_1)}\psi_1(|x|) \text{ for a.e. } x \in \overline{\Omega}_{R_1}, \tag{22}$$

with

$$m_{\varphi_1} = \min_{|x|=R_1} \varphi_1(x) \quad \text{and} \quad M_{\varphi_1} = \max_{|x|=R_1} \varphi_1(x).$$

More clearly, there exists a constant $C_1 > 0$ (depending only on q) such that

$$C_1^{-1}\psi_1(|x|) \leq \varphi_1(x) \leq C_1\psi_1(|x|) \text{ for a.e. } x \in \overline{\Omega}_{R_1}. \tag{23}$$

We apply now the theorem 3.1 on Ω_{R_1} for $\overline{u}$ and ψ_1 and for $\underline{u}$ and ψ_1. So there exist two constants $\overline{\gamma}$ and $\underline{\gamma}$ (depending only on q) such that

$$\overline{\gamma}^{-1}\frac{m_{\overline{u}}}{\psi_1(R_1)}\psi_1(|x|) \leq \overline{u}(x) \leq \overline{\gamma}\frac{M_{\overline{u}}}{\psi_1(R_1)}\psi_1(|x|) \text{ for a.e. } x \in \overline{\Omega}_{R_1}, \tag{24}$$

with

$$m_{\overline{u}} = \min_{|x|=R_1} \overline{u}(x) = \min_{|x|=R_1} u(x) \quad \text{and} \quad M_{\overline{u}} = \max_{|x|=R_1} \overline{u}(x) = \max_{|x|=R_1} u(x),$$

and

$$\underline{\gamma}^{-1}\frac{m_{\underline{u}}}{\psi_1(R_1)}\psi_1(|x|) \leq \underline{u}(x) \leq \underline{\gamma}\frac{M_{\underline{u}}}{\psi_1(R_1)}\psi_1(|x|) \text{ for a.e. } x \in \overline{\Omega}_{R_1}, \tag{25}$$

with

$$m_{\underline{u}} = \min_{|x|=R_1} \underline{u}(x) = \min_{|x|=R_1} u(x) \quad \text{and} \quad M_{\underline{u}} = \max_{|x|=R_1} \underline{u}(x) = \max_{|x|=R_1} u(x).$$

Combining (20) ,(23), (24) and (25), we arrive for a.e. $x \in \overline{\Omega}_{R_1}$ at

$$\frac{\underline{\gamma}\, C_1^{-}1}{\psi(R_1)} m_u\, \varphi_1(x) \leq u(x) \leq \frac{\overline{\gamma}\, C_1}{\psi(R_1)} M_u\, \varphi_1(x), \tag{26}$$

with

$$m_u = \min_{|x|=R_1} u(x) \quad \text{and} \quad M_u = \max_{|x|=R_1} u(x).$$

Case $x \in \overline{B}_{R_1}$. By the Harnack inequality on B_{2R_1} (see Gilbarg and Trudinger [7, Corollary 9.25, p.250]), we gate

$$\sup_{B_R} u(x) \leq C_2 \inf_{B_R} u(x) \text{ for all } R < 2R_1.$$

with a constant C_2 depending only on q and R. Then using the condition $u(0) = \varphi_1(0)$ for $u \in Y_\lambda$, we obtain for $R_1 < R < 2R_1$

$$\begin{aligned} M_u \leq \sup_{B_R} u(x) \leq C_2 \inf_{B_R} u(x) \leq C_2\varphi_1(0), \\ \varphi_1(0) \leq \sup_{B_R} u(x) \leq C_2 \inf_{B_R} u(x) \leq C_2 m_u. \end{aligned} \tag{27}$$

Then for a.e. $x \in B_{R_1}$,

$$\frac{C_2^{-1}\varphi_1(0)}{max_{B_{2R_1}}\varphi_1(x)}\varphi_1(x) \le \inf_{B_R} u \le u(x) \le \sup_{B_R} u \le \frac{C_2\varphi_1(0)}{min_{B_{2R_1}}\varphi_1(x)}\varphi_1(x). \tag{28}$$

Finally, by (28), (27) and (26), we deduce **(iii)**.

(iv) Y_λ *is compact in* $\mathcal{C}^0(\mathbb{R}^N)$*:* Let $(u_n)_{n\in\mathbb{N}} \in Y_\lambda$ be a sequence. By **(iii)**, we know that the functions $(u_n)_{n\in\mathbb{N}}$ are bounded in $L^\infty(\mathbb{R}^N)$ and by the regularity theory, we know that they are continuous.

For $R > 0$, we denote by $u_n^{(R)}$ the restriction of u_n to $\overline{B}_R(0)$. This restriction satisfy

$$(-\Delta + q + V_n - \lambda)u_n^{(R)} = 0 \text{ in } B_R(0) \tag{29}$$

Using the Schauder estimate it follows that $u_n^{(R)} \in \mathcal{C}^{2,\alpha}(B_R(0))$ and that

$$\|u_n^{(R)}\|_{2,\alpha} \le C\|u_n^{(R)}\|_\infty \tag{30}$$

where $C = C(N, R, q)$ (see Gilbarg and Trudinger [7, Theorem 6.13, p.106 and Theorem 6.2 p.90]). By (30) and **(iii)**, we deduce that $(u_n^{(R)})_{n\in\mathbb{N}}$ and $(\nabla u_n^{(R)})_{n\in\mathbb{N}}$ are bounded in $\mathcal{C}^0(B_R(0))$. So, using theorem of Ascoli, one can extract a subsequence $(u_{n_k}^{(R)})$ such that:

$$\begin{cases} u_{n_k}^{(R)} & \to & u^{(R)} & \text{strongly in } \mathcal{C}^0(B_R(0)), \\ \nabla u_{n_k}^{(R)} & \to & \nabla u^{(R)} & \text{strongly in } \mathcal{C}^0(B_R(0)), \\ \Delta u_{n_k}^{(R)} & \to & \Delta u^{(R)} & \text{strongly in } \mathcal{C}^{0,\alpha'}(B_R(0)) \text{ for some } 0 < \alpha' < \alpha. \end{cases} \tag{31}$$

Then, taking the diagonal subsequence $(u_{n_n}^{(n)})_{n\in\mathbb{N}}$, we construct a subsequence of $(u_n)_{n\in\mathbb{N}}$ wich converge, strongly in $\mathcal{C}^{2,\alpha}(B_R(0))$ for all $R > 0$, to a continuous function u satisfying

$$C^{-1}\varphi_1(x) \le u(x) \le C\varphi_1(x) \text{ for all } x \in \mathbb{R}^n.$$

Thus the subsequence $(u_{n_n}^{(n)})_{n\in\mathbb{N}}$ converge to u strongly in $\mathcal{C}^0(\mathbb{R}^N)$. Indeed, by **(iii)**,

$$\forall\varepsilon > 0, \quad \exists n_0 > 0 \quad \text{such that } \forall x \in \overline{\Omega}_{n_0} \quad \forall n \ge n_0 \quad |u_{n_n}^{(n)}(x) - u(x)| \le \varepsilon,$$

and by the strong convergence of $u_{n_n}^n$ to u in $\mathcal{C}^0(B_{n_0}(0))$,

$$\exists n_1 > 0 \quad \text{such that } \forall n \ge n_1, \quad \forall x \in B_{n_0}(0), \quad |u_{n_n}^{(n)}(x) - u(x)| \le \varepsilon.$$

To finish the proof of the compactness of Y_λ, we have to check that u belongs to Y_λ. Since $V_{n_n} = \frac{\Delta u_{n_n}^{(n)}}{u_{n_n}^{(n)}} - q + \lambda$, it follows that $V_{n_n} \to V$ locally in $\mathcal{C}^{0,\alpha}(\mathbb{R}^N)$, where $0 \le V \le 1$. Hence u satisfy the equation

$$(\mathcal{A} - \lambda + V)u = 0 \quad \text{in } \mathbb{R}^N,$$

and $u \in Y_\lambda$.

Now, for every nonzero, nonnegative, bounded function V and any $t > 0$, we define the operator $\mathcal{A}_t$,

$$\mathcal{A}_t := -\Delta + q + tV.$$

The potential $q_t = q + tV$ has the same properties as q, so the operator $\mathcal{A}_t$ has the same properties than $\mathcal{A}$. This operator $\mathcal{A}_t$ possesses an infinite sequence of positive eigenvalues, and the first one, denote by $\lambda_V(t)$, is given by

$$\lambda_V(t) = \inf\left\{\int_{\mathbb{R}^N} |\nabla u|^2 + q_t(x)|u|^2\,dx \colon u \in V_q \ \text{ with } \ \|u\|_{L^2(\mathbb{R}^N)} = 1\right\}. \tag{32}$$

The eigenvalue $\lambda_V(t) > 0$ is simple with the eigenspace spanned by an eigenfunction $\varphi_{V,t} \in \mathcal{D}(\mathcal{A}_t)$ satisfying $\varphi_{V,t} > 0$ throughout $\mathbb{R}^N$ and $\|\varphi_{V,t}\|_{L^2(\mathbb{R}^N)} = 1$. The following properties of the curve $\{(t, \lambda_V(t))\,|\,t > 0\}$ are easy to check with the characterization (32). The function $\lambda(t)$ is a continuous increasing concave function of t such that $\lambda_V(t) \to \lambda_1$ as $t \to 0$. Furthermore, if $V_1 \leq V \leq V_2$, then

$$\lambda_{V_1}(t) \leq \lambda_V(t) \leq \lambda_{V_2}(t). \tag{33}$$

Fix $f \in X \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^N)$, $f \geq 0$, by **(iii)**,

$$V_1 := C^{-1}\frac{f}{\varphi_1} \leq \frac{f}{u} \leq V_2 := C\frac{f}{\varphi_1}, \tag{34}$$

for every $u \in Y_\lambda$ and $\lambda_1 < \lambda \leq \lambda_1 + 1$.

It follows, from the properties of the function $\lambda_V(t)$, that there exists δ_0, such that for every $u \in Y_\lambda$ with $\lambda_1 < \lambda \leq \lambda_1 + \delta_0$, there exist a unique t_λ and a unique eigenfunction φ of the equation

$$\mathcal{A}_{t_\lambda}\varphi - \lambda\varphi = (-\Delta + q + t_\lambda\frac{f}{u} - \lambda)\varphi = 0,$$

wich satisfy $\varphi(0) = \varphi_1(0)$. We define then the mapping T_λ by $T_\lambda(u) = \varphi$.

We prove now that there exists $\delta > 0$ (depending only on f) such that for every $\lambda \in (\lambda_1, \lambda_1 + \delta)$ we have $T_\lambda \colon Y_\lambda \to Y_\lambda$. By (34) we know that there exists some $\varepsilon > 0$ such that

$$|t| \leq \varepsilon \quad \Rightarrow \quad t\frac{f}{u} \leq tV_2 \leq 1.$$

Since the function $\lambda_{V_1}(t)$ is invertible, with a continuous inverse, there exists $\delta > 0$ such that

$$0 < \lambda_{V_1}(t) - \lambda_1 < \delta \quad \Rightarrow 0 < t < \varepsilon.$$

Using (33), $\lambda_{V_1}(t_\lambda) \leq \lambda_V(t_\lambda) = \lambda$, so if $0 < \lambda - \lambda_1 < \delta$ then $t_\lambda \leq \varepsilon$. Thus $T_\lambda(u) = \varphi \in Y_\lambda$.

The mapping T_λ is continuous. If a sequence $(u_n)_{n\in\mathbb{N}} \in Y_\lambda$ converge to $u \in Y_\lambda$ in $\mathcal{C}^0(\mathbb{R}^N)$, the corresponding sequence $(v_n = T_\lambda(u_n))_{n\in\mathbb{N}}$ converge to $v = T_\lambda(u)$ in $\mathcal{C}^0(\mathbb{R}^N)$. Indeed, the sequence $(v_n)_{n\in\mathbb{N}}$ is in the compact set Y_λ and any convergent subsequence clearly converges to $v = T_\lambda(u)$.

Applying the Schauder-Tychonoff fixed point theorem to the operator T_λ, we conclude that there exist $t_\lambda > 0$ and $u_\lambda \in Y_\lambda$ such that u_λ is a positive solution of the equation

$$(\mathcal{A} - \lambda + t_\lambda \frac{f}{u_\lambda})u_\lambda = 0 \text{ in } \mathbb{R}^N.$$

So the function $u = -\frac{u_\lambda}{t_\lambda}$ is the negative solution of the equation

$$-\Delta u + q(x)u - \lambda u = f \text{ in } \mathbb{R}^n$$

and this function satisfy the φ_1-negativity,

$$u \leq -\frac{C^{-1}}{t_\lambda}\varphi_1.$$

■

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