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On the Dynamics of $x_{n+1} = (bx_{n-1}^2)(A + Bx_{n-2})^{-1}$

ABSTRACT. We investigate the boundedness, the global stability, and the periodic nature of the nonnegative solutions of the equation in the title with nonnegative parameters

KEY WORDS AND PHRASES. difference equations, boundedness, global asymptotic stability, semi-cycles

1 Introduction and Preliminaries

In this paper we consider the third order nonlinear rational difference equation

$$x_{n+1} = \frac{bx_{n-1}^2}{A + Bx_{n-2}}, \quad n = 0, 1, ... \tag{1}$$

where the parameters A, B, and b and the initial conditions x_{-2}, x_{-1} and x_0 are arbitrary non-negative real numbers. We investigate the boundedness, the global stability and the periodic nature of the solutions of the Eq. (1).

Recently there has been a great interest in studying rational [1, 2, 5, 7] and nonrational nonlinear difference equations [3, 5, 8, 9, 10, 11, 12], see also the references therein. Some of the results recently obtained in this field can be applied in studying some mathematical biology models, population dynamic etc., see [3, 4, 12].

Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, x_{n-2}) \tag{2}$$

with $x_{-2}, x_{-1}, x_0 \in I$ (where I is some interval of real numbers).

The *linearized equation* of Eq. (2) about an equilibrium $\bar{x}$ is the linear difference equation

$$y_{n+1} = c_1 y_n + c_2 y_{n-1} + c_3 y_{n-2}, \quad n = 0, 1, ... \tag{3}$$

where

$$c_1 = \frac{\partial f}{\partial x}(\bar{x}, \bar{x}, \bar{x}), \quad c_2 = \frac{\partial f}{\partial y}(\bar{x}, \bar{x}, \bar{x}), \quad c_2 = \frac{\partial f}{\partial z}(\bar{x}, \bar{x}, \bar{x})$$

The characteristic equation of Eq. (3) is

$$\lambda^3 - c_1\lambda^2 - c_2\lambda - c_3 = 0. \tag{4}$$

Theorem A ([6, Theorem 1]) (*Linearized Stability Theorem*) *The following statements are true.*

a) *If all roots of Eq. (4) have modulus less than one, then the equilibrium $\bar{x}$ of Eq. (2) is locally asymptotically stable.*

b) *If at least one of the roots of Eq. (4) has modulus greater than one, then the equilibrium $\bar{x}$ of Eq. (2) is unstable.*

A necessary and sufficient condition for all roots of Eq. (4) to have modulus less than one is the following:

$$|c_1 + c_3| < 1 - c_2, \quad |c_1 - 3c_3| < 3 + c_2, \quad \text{and} \quad c_3^2 - c_2 - c_1c_3 < 1.$$

In this case, the locally asymptotically stable equilibrium $\bar{x}$ is called a *sink*.

The equilibrium $\bar{x}$ of Eq. (2) is called a *saddle point* if there exits a root of Eq. (4) with absolute value less than one and a root of Eq. (4) with absolute value greater than one. In particular a saddle point equilibrium is unstable.

2 The case $AB = 0$

In this section we shortly discuss the case when one of the parameters in the Eq. (1) is zero, where we have the following two nontrivial cases:

$$x_{n+1} = \frac{bx_{n-1}^2}{Bx_{n-2}}, \quad n = 0, 1, ... \tag{5}$$

$$x_{n+1} = \frac{b}{A}x_{n-1}^2, \quad n = 0, 1, ... \tag{6}$$

In each of the above equations we assume that all parameters in the equations are positive. Equations (5) and (6) are nonlinear third and second order respectively, and the change of variables $x_n = e^{y_n}$ reduce the equations (5) and (6) to a third and second order linear difference equation respectively, which can be solved. The details we leave to the reader.

3 Main Results

In this section we investigate the dynamics of Eq. (1) under the assumption that all parameters in Eq. (1) are positive and the initial conditions are nonnegative.

The change of variables $x_n = \frac{A}{B} y_n$ reduces Eq. (1) to the difference equation

$$y_{n+1} = \frac{r y_{n-1}^2}{1 + y_{n-2}}, \quad n = 0, 1, ... \tag{7}$$

where $r = \frac{b}{B}$.

It is easy to see that $\bar{y}_1 = 0$ is always an equilibrium point and when $r > 1$ we have also a positive equilibrium point $\bar{y}_2 = \frac{1}{r-1}$.

3.1 An Oscillation Result

Lemma 1 *Assume that* $r > 1$ *and let* $\{y_n\}_{n=-2}^{\infty}$ *be a solution of Eq.* (7) *such that either*

$$y_{-2}, y_0 \geq \bar{y}_2 \quad \textit{and} \quad y_{-1} < \bar{y}_2 \tag{8}$$

or

$$y_{-2}, y_0 < \bar{y}_2 \quad \textit{and} \quad y_{-1} \geq \bar{y}_2 \tag{9}$$

then $\{y_n\}_{n=-2}^{\infty}$ *oscillates about* $\bar{y}_2$ *with semi-cycle of length one.*

Proof: We will assume that (8) holds. The case where (9) holds is similar and will be omitted. From (7) we obtain

$$y_1 = \frac{r y_{-1}}{1 + y_{-2}} < \frac{r \bar{y}_2}{1 + \bar{y}_2} = \bar{y}_2 \quad \text{and} \quad y_2 = \frac{r y_0}{1 + y_{-1}} > \frac{r \bar{y}_2}{1 + \bar{y}_2} = \bar{y}_2.$$

Using induction the result follows.

3.2 Existence of Prime Period-Two Solutions

In this subsection, we show that Eq. (7) has prime period-two solutions.

Theorem 2 *Eq.* (7) *has eventually nonnegative prime period-two solutions if and only if either*

$$y_{-1} = 0 \quad \textit{and} \quad y_0 = \frac{1}{r} \tag{10}$$

or

$$y_0 = 0 \quad \textit{and} \quad \frac{y_{-1}^2}{1 + y_{-2}} = \frac{1}{r^2}, \tag{11}$$

the period-two solution must be in the form

$$..., 0, \frac{1}{r}, 0, \frac{1}{r}, ... \,. \tag{12}$$

Proof: Assume that

$$..., \phi, \psi, \phi, \psi, ...$$

is a nonnegative prime period-two solution of Eq. (7).
Then

$$\phi = \frac{r\phi^2}{1+\psi} \quad \text{and} \quad \psi = \frac{r\psi^2}{1+\phi} . \tag{13}$$

Hence $\phi - \psi = r(\phi^2 - \psi^2)$, and consequently

$$\phi + \psi = \frac{1}{r}. \tag{14}$$

From Eqs.(13) and (14) we get the period-two solution in form (12). If $y_{2k+1} = 0$ for some $k \in \mathbf{N}$ then from (7), it follows that $y_{2n-1} = 0$, $n = 0, 1, \ldots$, $y_{2n} = 1/r$, $n = 1, 2, \ldots$, and y_{-2} is arbitrary. If $y_{2l} = 0$ for some $l \in \mathbf{N}$, then $y_{2n} = 0$, $n = 1, 2, \ldots$, $y_{2n-1} = 1/r$, $n = 1, 2, \ldots$, and $\frac{ry_{-1}^2}{1+y_{-2}} = y_1 = \frac{1}{r}$, as desired.

3.3 Local and Global Stability

As we have already noted $\bar{y}_1 = 0$ is always an equilibrium solution of Eq. (7). Furthermore when $r > 1$, Eq. (7) also possesses the positive equilibrium $\bar{y}_2 = \frac{1}{r-1}$.

Theorem 3 *Consider Eq. (7). Then the following results hold:*

(i) *The zero equilibrium point is locally asymptotically stable.*

(ii) *Assume that $r > 1$ then the equilibrium point $\bar{y}_2 = \frac{1}{r-1}$ is unstable. In particular $\bar{y}_2$ is a saddle point.*

Proof: The linearized equation associated with Eq. (7) about $\bar{y}_i$, $i = 1, 2$, has the form

$$z_{n+1} - \frac{2r\bar{y}_i}{1+\bar{y}_i} z_{n-1} + \frac{r\bar{y}_i^2}{(1+\bar{y}_i)^2} z_{n-2} = 0 \quad , \qquad n = 0, 1, \ldots \ .$$

So the linearized equation of Eq. (7) about $\bar{y}_1 = 0$ is $z_{n+1} = 0$, $n = 0, 1, \ldots$, and the characteristic equation about $\bar{y}_1 = 0$ is $\lambda^3 = 0$ so proof of (i) follows immediately from Theorem A.

The linearized equation of Eq. (7) about $\bar{y}_2 = \frac{1}{r-1}$ is $z_{n+1} = 2z_{n-1} - \frac{1}{r} z_{n-2}$, $\quad n = 0, 1, \ldots$, and the characteristic equation is

$$\lambda^3 - 2\lambda + \frac{1}{r} = 0, \quad \text{with} \quad r > 1.$$

Set

$$f(\lambda) = \lambda^3 - 2\lambda + \frac{1}{r}. \tag{15}$$

Then $f(1) = -1 + \frac{1}{r} < 0$ and $\lim_{\lambda \to +\infty} f(\lambda) = +\infty$, so $f(\lambda)$ has at least a zero in $(1, \infty)$ and the product of the moduli of the zeros of the function f is $\frac{1}{r} < 1$, hence there exists a root in the unit disk. This completes the proof.

Theorem 4 *The zero equilibrium point of Eq. (7) is globally asymptotically stable relative to the set*

$$S = [0, \infty) \times [0, 1/r]^2 \setminus A. \tag{16}$$

where

$$A = \{(x, y, z) | (y, z) = (0, 1/r) \quad or \quad \big(y^2/(1 + x), z\big) = (1/r^2, 0)\},$$

with $(y_{-2}, y_{-1}, y_0) \in S$.

Proof: By Theorem 3 we know that $\bar{y}_1 = 0$ is locally asymptotically stable equilibrium point of Eq. (7), and so it suffices to show that $\bar{y}_1 = 0$ is a global attractor of Eq. (7) relative to S. So let $\{y_n\}_{n=-2}^{\infty}$ be a solution of Eq. (7), such that $(y_{-2}, y_{-1}, y_0) \in S$. We show that $\lim_{n\to\infty} y_n = 0$. We have

$$y_1 = \frac{ry_{-1}^2}{1 + y_{-2}} \le ry_{-1}^2 \le y_{-1} \le \frac{1}{r}$$
$$y_2 = \frac{ry_0^2}{1 + y_{-1}} \le ry_0^2 \le y_0 \le \frac{1}{r}.$$

By induction we obtain

$$0 \le y_{n+1} = \frac{ry_{n-1}^2}{1 + y_{n+2}} \le ry_{n-1}^2 \le y_{n-1} \le \frac{1}{r}, \quad n = 0, 1, ...,$$

that is, $0 \le y_n \le \frac{1}{r}$, $n = -1, 0, 1, ...,$ and $\{y_{2n}\}_{n=-1}^{\infty}$ and $\{y_{2n-1}\}_{n=0}^{\infty}$ are non-increasing and bounded. Hence, there are finite limits

$$\lim_{n\to\infty} y_{2n} = M \quad \text{and} \quad \lim_{n\to\infty} y_{2n-1} = L,$$

moreover, in view of (16), we have

$$M, L \in [0, 1/r). \tag{17}$$

Letting $n \to \infty$ in (7) we obtain

$$M = \frac{rM^2}{1 + L} \quad \text{and} \quad L = \frac{rL^2}{1 + M}.$$

Now, we want to prove that $M = L = 0$. We consider the following cases:
(i) If $M = 0$ and $L \neq 0$ then $L = \frac{1}{r}$, which is a contradiction to (17).
(ii) If $M \neq 0$ and $L = 0$, then $M = \frac{1}{r}$, a contradiction.
(iii) If $M \neq 0$ and $L \neq 0$, then we have

$$1 + L = rM \quad \text{and} \quad 1 + M = rL$$

which implies $L - M = r(M - L)$. Hence $M = L = 1/(r - 1)$, which is a contradiction. Thus $L = M = 0$, as desired.

3.4 Existence of Unbounded Solutions

In this subsection we show that when $r > 1$ Eq. (7) possesses unbounded solutions.

Theorem 5 *Assume that $r > 1$. Then Eq. (7) possesses unbounded solution. In particular, every solution of Eq. (7) which oscillate about the equilibrium $\bar{y}_2 = \frac{1}{r-1}$ with semi-cycle of length one is unbounded.*

Proof: We prove that every solution $\{y_n\}_{n=-2}^{\infty}$ of Eq. (7) which oscillates with semi-cycles of length one is unbounded (see Lemma 1 for the existence of such solutions). Let $r > 1$ and without loss of generality that $\{y_n\}_{n=-2}^{\infty}$ is such that

$$y_{2n-1} < \bar{y}_2 \quad \text{and} \quad y_{2n} > \bar{y}_2 \quad \text{for} \quad n \geq 0.$$

Then

$$y_{2n+2} = \frac{ry_{2n}^2}{1 + y_{2n-1}} > \frac{ry_{2n}^2}{1 + \frac{1}{r-1}} = (r-1)y_{2n}^2 > y_{2n}.$$

and

$$y_{2n+3} = \frac{ry_{2n+1}^2}{1 + y_{2n-2}} < \frac{ry_{2n+1}^2}{1 + \frac{1}{r-1}} = (r-1)y_{2n+1}^2 < y_{2n+1}$$

from which it follows that there are $\lim_{n\to\infty} y_{2n} = M$ and $\lim_{n\to\infty} y_{2n+1} = m \in [0, \bar{y}_2)$. If $M = \infty$, there is nothing to prove. Hence, assume that $M < \infty$. As in the proof of Theorem 3 we can see that $m \neq 0$ and $M < \infty$ is impossible. If $m = 0$ and $M < \infty$ then $M = \frac{1}{r} < \frac{1}{r-1} = \bar{y}_2$, a contradiction. Hence $M = \infty$, from which the result follows.

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