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Stability of Mann Iterative Process with Random Errors for the Fixed Point of Strongly-Pseudocontractive Mapping in Arbitrary Banach Spaces[1]

ABSTRACT. Suppose that X is a arbitrary real Banach space and $T : X \to X$ is a Strongly pseudocontractive mapping. It is proved that certain Mann iterative process with random errors for the fixed point of T is stable(almost stable) with respect to T with(without) Lipschitz condition. And, two related results are obtained that deals with stability(or almost stability) of Mann iterative process for solution of nonlinear equations with strongly accretive mapping. Consequently, the corresponding results of Osilike are improved.

KEY WORDS. strongly pseudocontractive mapping, strongly accretive mapping, Mann iterative process with random errors, stable, almost stable

To set the framework, we recall some basic notations as follows.

Let X be a real Banach space and $K \subset X$ a nonempty subset.

(a) A mapping $T : K \to X$ is said to be strongly pseudocontractive if for any $x, y \in K$ we have

$$\|x - y\| \leq \|x - y + r[(I - T - kI)x - (I - T - kI)y]\| \tag{1}$$

for all $r > 0$, where I is the identity mapping on X and the constant $k \in (0,\ 1)$. A mapping $A : K \to X$ is said to be strongly accretive if $I - A$ is strongly pseudocontractive. Hence, the mapping theory for accretive mappings is intimately connected with the fixed point theory for pseudocontractive mappings.

(b) Let $T : X \to X$ be a mapping. For any given $x_0 \in X$ the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1 - a_n)x_n + a_nTx_n + c_nu_n \qquad (n \geq 0) \tag{2}$$

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is called Mann[1] iteration sequence with random errors, where $u_n \in X(n \geq 0)$ is a bounded and random error term, and $\{a_n\}$ and $\{c_n\}$ are two real sequences in $(0,\ 1)$ satisfying some conditions. By the way, Xu, one of authors introduced another definition of Mann iteration process with random errors on a nonempty convex subset of Banach space in 1998(see, Xu [2]).

(c) Let K be a nonempty convex subset of X and T be a selfmapping of K. Assume that $x_0 \in K$ and $x_{n+1} = f_n(T,\ x_n)$ define an iterative process which yields a sequence of points $\{x_n\}_{n=0}^{\infty}$ in K. Suppose $F(T) = \{x \in K :\ Tx = x\} \neq \emptyset$ and $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point $q \in F(T)$. For any $\{y_n\}_{n=0}^{\infty} \subset K$, let $\varepsilon_n = \|y_{n+1} - f_n(T,\ y_n)\|$. If $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ implies that $\lim y_n = q$ then the iterative process defined by $x_{n+1} = f_n(T,\ x_n)$ is said to be almost T-stable. Furthermore, If $\lim_{n\to\infty} \varepsilon_n = 0$ implies that $\lim y_n = q$ then the iterative process defined by $x_{n+1} = f_n(T,\ x_n)$ is said to be T-stable(see, Zhang [3]).

In recent, some stability results have been established(see [4]-[8]), for example, Osilike[6] showed that the Mann and Ishikawa iterative processes are stable with respect to Lipschitz strongly pseudocontractive mapping T in p-uniformly smooth Banach space. Then, he extended the results to arbitrany real Banach spaces in [7]. Since the consideration of error terms is an important part of any iteration methods and many mappings without Lipschitz condition, therefore, we introduced the Mann iterative process with random errors and to prove that the iterative process is stable(almost stable) with respect to T with(without) Lipschitz condition where T is a strongly pseudocontractive mapping in arbitrary Banach space. And, two related results are obtained that deals with stability(or almost stability) of Mann iterative process for solution of nonlinear equations with Strongly accretive mapping. Consequently, the corresponding results of Osilike are improved.

Now, we prove the following theorems.

Theorem 1 *Suppose that $T : X \to X$ be a Lipschitz strongly pseudocontractive mapping. If q is a fixed point of T and for arbitrary $x_0 \in X$, the Mann iteration sequence with random errors defined by* (2) *satisfying*

$$0 < a \leq a_n \leq k[2(L^2 + 3L + 3)]^{-1} \qquad \textit{and} \qquad \lim_{n\to\infty} c_n = 0 \tag{1.1}$$

where $L > 1$ is Lipschitz constant of T and $a > 0$ is a constant. Then

(1) *$\{x_n\}$ converges strongly to unique fixed point q of T;*

(2) *Let $\{y_n\}$ be any sequence in X. Then y_n converges strongly to q if and only if ε_n converges to* 0.

Proof: Let $\sup\{\|u_n\| : \; n = 0, 1, 2, \cdots\} = M$. Using (2), we have

$$\begin{aligned} x_n - q &= (x_{n+1} - q) - a_n(Tx_n - q) + a_n(x_n - q) - c_n u_n \\ &= (1 + a_n)(x_{n+1} - q) + a_n[(I - T - kI)x_{n+1} - (I - T - kI)q] \\ &\quad + a_n(Tx_{n+1} - Tx_n) - (2 - k)a_n^2(Tx_n - q) - (1 - k)a_n(x_n - q) \\ &\quad + (2 - k)a_n^2(x_n - q) - [1 + (2 - k)a_n]c_n u_n \end{aligned}$$

for all $n \geq 0$. Furthermore,

$$\begin{aligned} \|x_n - q\| &\geq (1 + a_n)\|x_{n+1} - q + \tfrac{a_n}{1+a_n}[(I - T - kI)x_{n+1} - (I - T - kI)q]\| \\ &\quad - a_n\|Tx_{n+1} - Tx_n\| - (2 - k)a_n^2\|Tx_n - q\| - (1 - k)a_n\|x_n - q\| \\ &\quad - (2 - k)a_n^2\|x_n - q\| - [1 + (2 - k)a_n]c_n\|u_n\| \end{aligned} \tag{3}$$

for all $n \geq 0$. by virtue of (1), we have

$$\begin{aligned} \|x_n - q\| &\geq (1 + a_n)\|x_{n+1} - q\| - La_n\|x_{n+1} - x_n\| - (1 - k)a_n\|x_n - q\| \\ &\quad - 2(L + 1)a_n^2\|x_n - q\| - 3Mc_n \\ &\geq (1 + a_n)\|x_{n+1} - q\| - (1 - k)a_n\|x_n - q\| \\ &\quad - (L + 1)(L + 2)a_n^2\|x_n - q\| - (3 + L)Mc_n \end{aligned} \tag{4}$$

for all $n \geq 0$. It follows from (4) and the condition (1.1) that

$$\begin{aligned} \|x_{n+1} - q\| &\leq (1 - a_n + a_n^2)\|x_n - q\| + (1 - k)a_n\|x_n - q\| \\ &\quad + (L + 1)(L + 2)a_n^2\|x_n - q\| + (3 + L)Mc_n \\ &\leq (1 - ka_n)\|x_n - q\| + (L^2 + 3L + 3)a_n^2\|x_n - q\| + (3 + L)Mc_n \\ &\leq (1 - ka_n/2)\|x_n - q\| + (3 + L)Mc_n \\ &\quad + a_n[a_n(L^2 + 3L + 3) - k/2]\|x_n - q\| \\ &\leq (1 - ka/2)\|x_n - q\| + (3 + L)Mc_n \end{aligned} \tag{5}$$

for all $n \geq 0$. Putting

$$\alpha = 1 - ka/2, \quad t_n = \|x_n - q\| \;\; \textit{and} \quad \beta_n = (3 + L)Mc_n \qquad (n \geq 0).$$

Hence, the inequality (5) reduces to

$$t_{n+1} \leq \alpha t_n + \beta_n \qquad (n \geq 0).$$

It follows from the inequality of Q. H. Liu (see Lemma of [9]) that $\lim_{n\to 0}\|x_n - q\| = 0$. I.e., $\{x_n\}$ converges strongly to fixed point q of T. If q' also is a fixed point of T, putting $r = 1$ in (1) we obtain $\|q - q'\| \leq (1 - k)\|q - q'\|$. It implies that $q = q'$.

We now prove part (2). Suppose $\lim_{n\to\infty}\varepsilon_n = 0$. Then

$$\begin{aligned} \|y_{n+1} - q\| &= \|y_{n+1} - (1 - a_n)y_n - a_nTy_n - c_nu_n + (1 - a_n)y_n + a_nTy_n - q + c_nu_n\| \\ &\leq \varepsilon_n + \|(1 - a_n)(y_n - q) + a_n(Ty_n - q) + c_nu_n\| \\ &\leq \|(1 - ka/2)\|y_n - q\| + (3 + L)Mc_n + \varepsilon_n \\ &\leq \alpha\|y_n - q\| + \beta_n + \varepsilon_n \end{aligned}$$

for all $n \geq 0$.
By virtue of the inequality of Q. H. Liu again, we obtain that $y_n \to q$ $(as\ n \to \infty)$. I.e., the iterative process defined by $x_{n+1} = f_n(T,\ x_n)$ is T-stable.
On the contrary, if $\lim_{n\to\infty} y_n = q$ then

$$\begin{aligned}\varepsilon_n &= \|y_{n+1} - (1-a_n)y_n - a_nTy_n - c_nu_n\| \\ &\leq \|y_{n+1} - q\| + (1-a_n)\|y_n - q\| + La_n\|y_n - q\| + Mc_n \to 0(as\ \ n \to \infty).\end{aligned}$$

This implies that $\lim_{n\to\infty} \varepsilon_n = 0$. The Proof is completed.

From this Theorem we can prove

Corollary 1 *Suppose that $A : X \to X$ be a Lipschitz strongly accretive mapping. Let x^* be a solution of $Ax = f$ where f is any given and $Sx = f + x - Ax\ \ \forall\ x \in X$. For arbitrary $x_0 \in X$, if Mann iteration sequence with random errors defined by*

$$x_{n+1} = (1-a_n)x_n + a_nSx_n + c_nu_n \qquad (n \geq 0) \tag{6}$$

satisfying

$$0 < a \leq a_n \leq k[2(L_*^2 + 3L_* + 3)]^{-1} \quad \textit{and} \quad \lim_{n\to\infty} c_n = 0$$

where $L_* > 1$ is Lipschitz constant of S. Then

(1) *$\{x_n\}$ converges strongly to unique solution of $Ax = f$;*

(2) *It is S-stable to approximate the solution of $Ax = f$ by* (6) *(Mann iteration sequence with random errors).*

In fact, from $Sx = f + x - Ax$, it is easy to see that x^* is unique solution of $Ax = f$ if and only if x^* is unique fixed point of S. Since S is a Lipschitz strongly pseudocontractive mapping, by virtue of theorem 1, we know the conclusions of corollary 1 are true.

Theorem 2 *Suppose that $T : X \to X$ be an uniformly continuous strongly pseudocontractive mapping with bounded range. If q is a fixed point of T and for arbitrary $x_0 \in X$, the Mann iteration sequence with random errors defined by* (2) *satisfying*

$$\sum_{n=0}^{\infty} a_n = \infty, \quad \sum_{n=0}^{\infty} a_n^2 < \infty \quad \textit{and} \quad \sum_{n=0}^{\infty} c_n < \infty,$$

then

(1) *$\{x_n\}$ converges strongly to unique fixed point of T;*

(2) *Let* $\{y_n\}$ *be any sequence in* X. *Then* $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ *implies that* y_n *converges strongly to* q;

(3) y_n *converges strongly to* q *implies that* $\lim_{n\to\infty} \varepsilon_n = 0$.

Proof: Putting

$$\begin{aligned} c &= \sup\{\|Tx - q\| : x \in X\} + \|x_0 - q\| \\ d &= \sup\{\|u_n\| : n \geq 0\}. \end{aligned}$$

For any $n \geq 0$, using induction, we obtain

$$\|x_n - q\| \leq c + d\sum_{i=0}^{n-1} c_i \leq c + d\sum_{i=0}^{+\infty} c_i.$$

Hence, we set

$$M = c + d\sum_{i=0}^{+\infty} c_i.$$

Since $\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|a_n(Tx_n - x_n) + c_n u_n\| = 0$, therefore,

$$e_n := \|Tx_{n+1} - Tx_n\| \to 0 (as \;\; n \to \infty)$$

by the uniform continuity of T. From (3) and using (1), we have

$$\begin{aligned} \|x_n - q\| &\geq (1 + a_n)\|x_{n+1} - q + \tfrac{a_n}{1+a_n}[(I - T - kI)x_{n+1} - (I - T - kI)q]\| \\ &\quad -a_n e_n - (2-k)Ma_n^2 - (1-k)a_n\|x_n - q\| \\ &\quad -(2-k)a_n^2\|x_n - q\| - [1 + (2-k)a_n]c_n\|u_n\| \\ &\geq (1 + a_n)\|x_{n+1} - q\| - (1-k)a_n\|x_n - q\| \\ &\quad -a_n e_n - (2-k)a_n^2\|x_n - q\| - 2Ma_n^2 - 3Mc_n \end{aligned} \tag{7}$$

for all $n \geq 0$. It follows from (7) that

$$\begin{aligned} \|x_{n+1} - q\| &\leq (1 - a_n + a_n^2)\|x_n - q\| + (1-k)a_n\|x_n - q\| \\ &\quad +a_n e_n + (2-k)a_n^2\|x_n - q\| + 2Ma_n^2 + 3Mc_n \\ &\leq (1 - ka_n)\|x_n - q\| + a_n e_n + 5Ma_n^2 + 3Mc_n \end{aligned} \tag{8}$$

for all $n \geq 0$. Putting

$$\alpha_n = ka_n, \quad t_n = \|x_n - q\|, \quad a_n e_n = O(\alpha_n) \quad and \quad \beta_n = 5Ma_n^2 + 3Mc_n \qquad (n \geq 0).$$

Hence, the inequality (8) reduces to

$$t_{n+1} \leq (1 - \alpha_n)t_n + O(\alpha_n) + \beta_n \qquad (n \geq 0).$$

It follows from the inequality of L. S. Liu(see Lemma 2 of [10]) that $\lim_{n\to 0}\|x_n - q\| = 0$. So, $\{x_n\}$ converges strongly to unique fixed point q of T.
We now prove part (2) and (3). Suppose $\sum_{n=0}^{\infty}\varepsilon_n \leq \infty$. Observe

$$\begin{aligned}\|y_{n+1} - q\| &\leq \varepsilon_n + \|(1-a_n)(y_n - q) + a_n(Ty_n - q) + c_n u_n\| \\ &\leq (1-\alpha_n)\|y_n - q\| + O(\alpha_n) + \beta_n + \varepsilon_n\end{aligned}$$

for all $n \geq 0$.
By virtue of the inequality of L. S. Liu again, we obtain that $y_n \to q$ $(as\ n \to \infty)$. I.e., the iterative process defined by $x_{n+1} = f_n(T,\ x_n)$ is almost T-stable.
On the contrary, if $\lim_{n\to\infty} y_n = q$ then

$$\begin{aligned}\varepsilon_n &= \|y_{n+1} - (1-a_n)y_n - a_n T y_n - c_n u_n\| \\ &\leq \|y_{n+1} - q\| + (1-\alpha_n)\|y_n - q\| + O(\alpha_n) + \beta_n \to 0(as\ \ n \to \infty).\end{aligned}$$

This implies that $\lim_{n\to\infty}\varepsilon_n = 0$. The proof is completed.

From Theorem 2 we can prove

Corollary 2 *Suppose that $A : X \to X$ is an uniformly continuous strongly accretive mapping and the range of $I - A$ is bounded. Let x^* be a solution of $Ax = f$ where f is any given. for arbitrary $x_0 \in X$, if Mann iteration sequence with random errors defined by* (6) *satisfying*

$$\sum_{n=0}^{\infty} a_n = \infty, \quad \sum_{n=0}^{\infty} a_n^2 < \infty \quad and \quad \sum_{n=0}^{\infty} c_n < \infty,$$

then

(1) $\{x_n\}$ *converges strongly to unique solution of $Ax = f$;*

(2) *It is almost S-stable to approximate the solution of $Ax = f$ by* (6) *(Mann iteration sequence with random errors).*

In fact, from $Sx = f + x - Ax$, it is easy to see that x^* is unique solution of $Ax = f$ if and only if x^* is unique fixed point of S. Since S is an uniformly continuous strongly pseudocontractive mapping, by virtue of theorem 2, we know the conclusions of corollary 2 are true.

Remark The iterative parameters $\{\alpha_n\}$ and $\{c_n\}$ do not depend on any geometric structure of space X and on any property of the mappings, but, the selection of the parameters is deal with the convergence rate of the iterative sequence. In Theorem 2 and Corollary 2, a prototype of iteration parameters is

$$a_n = \frac{1}{n+1} \quad and \quad c_n = \frac{1}{(n+1)^2} \qquad \forall\ \ n \geq 0.$$

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