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An Integral Expression and Some Inequalities of Mathieu Type Series

ABSTRACT. Let $r > 0$ and $a = \{a_k > 0, k \in \mathbb{N}\}$ such that the series $g(x) = \sum_{k=1}^{\infty} e^{-a_k x}$ converges for $x > 0$, then the Mathieu type series $\sum_{k=1}^{\infty} \frac{a_k}{\left(a_k^2+r^2\right)^2} = \frac{1}{2r} \int_0^{\infty} x g(x) \sin(rx) \, \mathrm{d}x$.

If $a = \{a_k > 0, k \in \mathbb{N}\}$ is an arithmetic sequence, then some inequalities of Mathieu type series $\sum_{k=1}^{\infty} \frac{a_k}{\left(a_k^2+r^2\right)^2}$ are obtained for $r > 0$.

KEY WORDS AND PHRASES. Mathieu type series, integral expression, Laplace transform, inequality

1 Introduction

In 1890, Mathieu defined $S(r)$ in [14] as

$$S(r) = \sum_{k=1}^{\infty} \frac{2k}{(k^2+r^2)^2}, \quad r > 0, \tag{1}$$

and conjectured that $S(r) < \frac{1}{r^2}$. We call formula (1) Mathieu's series.

In [3, 13], Berg and Makai proved

$$\frac{1}{r^2+\frac{1}{2}} < S(r) < \frac{1}{r^2}. \tag{2}$$

H. Alzer, J. L. Brenner and O. G. Ruehr in [2] obtained

$$\frac{1}{r^2+\frac{1}{2\zeta(3)}} < S(r) < \frac{1}{r^2+\frac{1}{6}}, \tag{3}$$

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where ζ denotes the zeta function and the number $\zeta(3)$ is the best possible.

The integral form of Mathieu's series (1) was given in [7, 8] by

$$S(r) = \frac{1}{r}\int_0^\infty \frac{x}{e^x - 1}\sin(rx)\,\mathrm{d}x. \tag{4}$$

Recently, the following results were obtained in [16, 18]:

(1) Let Φ_1 and Φ_2 be two integrable functions such that $\frac{x}{e^x-1} - \Phi_1(x)$ and $\Phi_2(x) - \frac{x}{e^x-1}$ are increasing. Then, for any positive number r, we have

$$\frac{1}{r}\int_0^\infty \Phi_2(x)\sin(rx)\,\mathrm{d}x \le S(r) \le \frac{1}{r}\int_0^\infty \Phi_1(x)\sin(rx)\,\mathrm{d}x. \tag{5}$$

(2) For positive number $r > 0$, we have

$$S(r) \le \frac{(1+4r^2)\left(e^{-\pi/r} - e^{-\pi/(2r)}\right) - 4\,(1+r^2)}{(e^{-\pi/r}-1)\,(1+r^2)\,(1+4r^2)}. \tag{6}$$

(3) For positive number $r > 0$, we have

$$S(r) < \frac{1}{r}\int_0^{\pi/r} \frac{x}{e^x-1}\sin(rx)\,\mathrm{d}x < \frac{1+\exp(-\frac{\pi}{2r})}{r^2+\frac{1}{4}}. \tag{7}$$

Remark 1 For $0 < r < 0.83273\cdots$, inequality (6) is better than the right hand side inequality in (3). If $r > 1.57482\cdots$, inequality (7) is better than (6). When $r < 1.574816\cdots$, inequality (7) is not better than (6). When $0 < r < 0.734821\cdots$, inequality (7) is better than the corresponding one in (3).

In [11, 16, 18], the following open problem was proposed by B.-N. Guo and F. Qi respectively: Let

$$S(r,t,\alpha) = \sum_{n=1}^\infty \frac{2n^{\alpha/2}}{(n^\alpha + r^2)^{t+1}} \tag{8}$$

for $t > 0$, $r > 0$ and $\alpha > 0$. Can one obtain an integral expression of $S(r,t,\alpha)$? Give some sharp inequalities for the series $S(r,t,\alpha)$.

In [20], the open problem stated above was considered and an integral expression of $S(r,t,2)$ was obtained: Let $\alpha > 0$ and $p \in \mathbb{N}$, then

$$\begin{aligned} S(\alpha,p,2) = \sum_{n=1}^\infty \frac{2n}{(n^2+\alpha^2)^{p+1}} &= \frac{2}{(2\alpha)^p p!}\int_0^\infty \frac{t^p\cos(\frac{p\pi}{2} - \alpha t)}{e^t - 1}\,\mathrm{d}t \\ &- 2\sum_{k=1}^p \frac{(k-1)(2\alpha)^{k-2p-1}}{k!(p-k+1)}\binom{-(p+1)}{p-k}\int_0^\infty \frac{t^k\cos[\frac{\pi}{2}(2p-k+1) - \alpha t]}{e^t-1}\,\mathrm{d}t. \end{aligned} \tag{9}$$

Using the quadrature formulas, some new inequalities of Mathieu series (1) were established in [9]. By the help of Laplace transform, the open problem mentioned above was partially solved, for example, among other things, an integral expression for $S\big(r, \frac{1}{2}, 2\big)$ was given as follows:

$$S\Big(r, \frac{1}{2}, 2\Big) = \frac{2}{r} \int_0^\infty \frac{t J_0(rt)}{e^t - 1}\, \mathrm{d}t, \tag{10}$$

where J_0 is Bessel function of order zero.

There has been a rich literature on the study of Mathieu's series, for example, [5, 6, 13, 19, 21, 22, 23], also see [4, 12, 15].

In this paper, we are about to investigate the following Mathieu type series

$$S(r, a) = \sum_{k=1}^\infty \frac{a_k}{(a_k^2 + r^2)^2}, \tag{11}$$

where $a = \{a_k > 0, k \in \mathbb{N}\}$ is a sequence satisfying $\lim_{k\to\infty} a_k = \infty$, and obtain an integral expression and some inequalities of $S(r, a)$ under some suitable conditions.

2 An integral expression of Mathieu type series (11)

Using Laplace transform of $x \sin(rx)$ we can immediately establish an integral expression of Mathieu type series (11).

Theorem 1 *Let $r > 0$ and $a = \{a_k > 0, k \in \mathbb{N}\}$ be a sequence such that the series*

$$g(x) \triangleq \sum_{k=1}^\infty e^{-a_k x} \tag{12}$$

converges for $x > 0$ and $xg(x)$ is Lebesgue integrable in $[0, \infty)$. Then we have

$$S(r, a) = \frac{1}{2r} \int_0^\infty x g(x) \sin(rx)\, \mathrm{d}x. \tag{13}$$

Proof: In [1] and [10, p. 559], Laplace transform of $t \sin(\alpha t)$ is given by

$$\int_0^\infty t \sin(\alpha t) e^{-st}\, \mathrm{d}t = \frac{2\alpha s}{(s^2 + \alpha^2)^2}, \tag{14}$$

where $s = \sigma + i\omega$ is a complex variable, α a complex number, and $\sigma > |\operatorname{Im} \alpha|$.

Applying (14) to the case $\alpha = r > 0$ and $s = a_k$, summing up, and interchanging between integral and summation produces

$$\begin{aligned} \int_0^\infty x g(x) \sin(rx)\, \mathrm{d}x &= \sum_{k=1}^\infty \int_0^\infty x \sin(rx) e^{-a_k x}\, \mathrm{d}x \\ &= 2r \sum_{k=1}^\infty \frac{a_k}{(a_k^2 + r^2)^2} = 2r S(r, a) \end{aligned} \tag{15}$$

according to Lebesgue's dominanted convergence theorem. The proof is complete. □

Remark 2 If $a_k = k$ in (13), then we can easily obtain the formula (4) in [8].

Corollary 1 *Let $a = \{a_k > 0, k \in \mathbb{N}\}$ be a sequence with $a_k = kd - c$ and $d > 0$. Then for any positive real number $r > 0$, we have*

$$S(r,a) = \frac{1}{2r}\int_0^\infty \frac{xe^{cx}}{e^{dx}-1}\sin(rx)\,\mathrm{d}x. \tag{16}$$

Proof: Since $a = \{a_k, k \in \mathbb{N}\}$ is an arithmetic sequence with difference $d > 0$, then $\{e^{-a_k x}\}_{k=1}^\infty$ is a positive geometric sequence with constant ratio $e^{-dx} < 1$ for $x > 0$, thus

$$g(x) = \sum_{k=1}^\infty e^{-a_k x} = e^{cx}\sum_{k=1}^\infty e^{-kdx} = \frac{e^{cx}}{e^{dx}-1}. \tag{17}$$

Then formula (16) follows from combination of (13) and (17) in view of $d > c$. □

Remark 3 In fact, in Corollary 1 and the following Theorem 2, Theorem 3 and Theorem 4, it suffices to consider the case $d = 1$, since from this one the general case arises by replacing c and r by $\frac{c}{d}$ and $\frac{r}{d}$, respectively, and dividing $S(r,a)$ by d^3.

3 Some inequalities of Mathieu type series (16)

The following result was obtained in [16, 18].

Lemma 1 ([16, 18]) *For a given positive number T, let $\phi(x)$ be an integrable function such that $\phi(x) = -\phi(x+T)$ and $\phi(x) \ge 0$ for $x \in [0,T]$, and let $f(x)$ and $g(x)$ be two integrable functions on $[0,2T]$ such that*

$$f(x) - g(x) \ge f(x+T) - g(x+T) \tag{18}$$

on $[0,T]$. Then

$$\int_0^{2T} \phi(x)f(x)\,\mathrm{d}x \ge \int_0^{2T} \phi(x)g(x)\,\mathrm{d}x. \tag{19}$$

Now we give a general estimate of Mathieu type series (16) as follows.

Theorem 2 *Let $a = \{a_k > 0, k \in \mathbb{N}\}$ such that $a_k = kd - c$ and $d > 0$. If Φ_1 and Φ_2 are two integrable functions such that $\frac{xe^{cx}}{e^{dx}-1} - \Phi_1(x)$ and $\Phi_2(x) - \frac{xe^{cx}}{e^{dx}-1}$ are increasing, then for $r > 0$,*

$$\frac{1}{2r}\int_0^\infty \Phi_2(x)\sin(rx)\,\mathrm{d}x \le S(r,a) \le \frac{1}{2r}\int_0^\infty \Phi_1(x)\sin(rx)\,\mathrm{d}x. \tag{20}$$

Proof: The function $\phi(x) = \sin(rx)$ has a period $\frac{2\pi}{r}$, and $\phi(x) = -\phi\left(x + \frac{\pi}{r}\right)$.

Since $f(x) = \frac{xe^{cx}}{e^{dx}-1} - \Phi_1(x)$ is increasing, for any $\alpha > 0$, we have $f(x+\alpha) \geq f(x)$. Therefore, from Lemma 1, we obtain

$$\int_{2k\pi/r}^{2(k+1)\pi/r} \frac{xe^{cx}}{e^{dx}-1} \sin(rx)\,\mathrm{d}x \leq \int_{2k\pi/r}^{2(k+1)\pi/r} \Phi_1(x)\sin(rx)\,\mathrm{d}x. \tag{21}$$

Then, from formula (16), we have

$$\begin{aligned} S(r,a) &= \frac{1}{2r}\sum_{k=0}^{\infty}\int_{2k\pi/r}^{2(k+1)\pi/r} \frac{xe^{cx}}{e^{dx}-1}\sin(rx)\,\mathrm{d}x \\ &\leq \frac{1}{2r}\sum_{k=0}^{\infty}\int_{2k\pi/r}^{2(k+1)\pi/r} \Phi_1(x)\sin(rx)\,\mathrm{d}x \\ &= \frac{1}{2r}\int_0^{\infty} \Phi_1(x)\sin(rx)\,\mathrm{d}x. \end{aligned} \tag{22}$$

The right hand side of inequality (20) follows.

Similar arguments yield the left hand side of inequality (20). □

Lemma 2 *For $x > 0$, we have*

$$\frac{1}{e^x} < \frac{x}{e^x - 1} < \frac{1}{e^{x/2}}. \tag{23}$$

Proof: This follows from standard argument of calculus. □

Theorem 3 *Let $a = \{a_k > 0, k \in \mathbb{N}\}$ satisfying $a_k = kd - c$ and $d > 0$. If $d > 2c$, then for $r > 0$, we have*

$$\begin{aligned} &\frac{1}{d}\left\{\frac{1+e^{-\pi(d-c)/r}}{2\left[(d-c)^2+r^2\right]\left(1-e^{-2\pi(d-c)/r}\right)} - \frac{2\left[e^{-\pi(d-2c)/r}+e^{-\pi(d-2c)/(2r)}\right]}{\left[(d-2c)^2+4r^2\right]\left[1-e^{-\pi(d-2c)/r}\right]}\right\} \\ &\leq S(r,a) \\ &\leq \frac{1}{d}\left\{\frac{2\left[1+e^{-\pi(d-2c)/(2r)}\right]}{\left[(d-2c)^2+4r^2\right]\left[1-e^{-\pi(d-2c)/r}\right]} - \frac{e^{-2\pi(d-c)/r}+e^{-\pi(d-c)/r}}{2\left[(d-c)^2+r^2\right]\left(1-e^{-2\pi(d-c)/r}\right)}\right\}. \end{aligned} \tag{24}$$

Proof: For $r > 0$, using (16), by direct calculation, we have

$$S(r,a) = \frac{1}{2r}\sum_{k=0}^{\infty}\left[\int_{2k\pi/r}^{(2k+1)\pi/r} + \int_{(2k+1)\pi/r}^{(2k+2)\pi/r}\right]\frac{xe^{cx}\sin(rx)}{e^{dx}-1}\,\mathrm{d}x. \tag{25}$$

The inequality (23) gives us

$$\begin{aligned} \frac{r\left(1+e^{-\pi(d-c)/r}\right)}{d\left[(d-c)^2+r^2\right]\left(1-e^{-2\pi(d-c)/r}\right)} &= \sum_{k=0}^{\infty}\int_{2k\pi/r}^{(2k+1)\pi/r}\frac{\sin(rx)}{de^{(d-c)x}}\,\mathrm{d}x \\ &\le \sum_{k=0}^{\infty}\int_{2k\pi/r}^{(2k+1)\pi/r}\frac{xe^{cx}\sin(rx)}{e^{dx}-1}\,\mathrm{d}x \\ \le \sum_{k=0}^{\infty}\int_{2k\pi/r}^{(2k+1)\pi/r}\frac{\sin(rx)}{de^{(\frac{d}{2}-c)x}}\,\mathrm{d}x &= \frac{4r\left[1+e^{-\pi(d-2c)/(2r)}\right]}{d\left[(d-2c)^2+4r^2\right]\left[1-e^{-\pi(d-2c)/r}\right]} \end{aligned} \tag{26}$$

and

$$\begin{aligned} -\frac{4r\left[e^{-\pi(d-2c)/r}+e^{-\pi(d-2c)/(2r)}\right]}{d\left[(d-2c)^2+4r^2\right]\left[1-e^{-\pi(d-2c)/r}\right]} &= \sum_{k=0}^{\infty}\int_{(2k+1)\pi/r}^{2(k+1)\pi/r}\frac{\sin(rx)}{de^{(\frac{d}{2}-c)x}}\,\mathrm{d}x \\ &\le \sum_{k=0}^{\infty}\int_{(2k+1)\pi/r}^{2(k+1)\pi/r}\frac{xe^{cx}\sin(rx)}{e^{dx}-1}\,\mathrm{d}x \\ \le \sum_{k=0}^{\infty}\int_{(2k+1)\pi/r}^{2(k+1)\pi/r}\frac{\sin(rx)}{de^{(d-c)x}}\,\mathrm{d}x &= -\frac{r\left(e^{-2\pi(d-c)/r}+e^{-\pi(d-c)/r}\right)}{d\left[(d-c)^2+r^2\right]\left(1-e^{-2\pi(d-c)/r}\right)}. \end{aligned} \tag{27}$$

Substituting (26) and (27) into (25) yields (24). The proof is complete. □

Theorem 4 *Let $a=\{a_k>0, k\in\mathbb{N}\}$ be a sequence such that $a_k=kd-c$ and $d>0$. If $d>2c$, then for any positive number $r>0$, we have*

$$S(r,a)<\frac{1}{2r}\int_0^{\pi/r}\frac{xe^{cx}\sin(rx)}{e^{dx}-1}\,\mathrm{d}x<\frac{2\left[1+e^{\pi(2c-d)/(2r)}\right]}{d[(2c-d)^2+4r^2]}. \tag{28}$$

Proof: It is easy to see that

$$\int_0^{\infty}\frac{xe^{cx}\sin(rx)}{e^{dx}-1}\,\mathrm{d}x=\sum_{k=0}^{\infty}\int_{k\pi/r}^{(k+1)\pi/r}\frac{xe^{cx}\sin(rx)}{e^{dx}-1}\,\mathrm{d}x, \tag{29}$$

and

$$\frac{xe^{cx}}{e^{dx}-1}=\frac{xe^{(c-\frac{d}{2})x}}{2\sinh\left(\frac{dx}{2}\right)}. \tag{30}$$

Since the functions $\frac{\sinh x}{x}$ and $e^{(\frac{d}{2}-c)x}$ are both increasing with $x>0$ for $d>2c$, then the function $\frac{xe^{cx}}{e^{dx}-1}$ is decreasing with $x>0$. Furthermore, $\lim_{x\to\infty}\frac{xe^{cx}}{e^{dx}-1}=0$.

Therefore, the series in (29) is an alternating series whose moduli of the terms are decreasing to zero. As well known, such a series in (29) is always less than its first term $\int_0^{\pi/r}\frac{xe^{cx}\sin(rx)}{e^{dx}-1}\,\mathrm{d}x$. Hence

$$\int_0^{\infty}\frac{xe^{cx}\sin(rx)}{e^{dx}-1}\,\mathrm{d}x<\int_0^{\pi/r}\frac{xe^{cx}\sin(rx)}{e^{dx}-1}\,\mathrm{d}x. \tag{31}$$

Using inequality (23), we have

$$\int_0^{\pi/r} \frac{xe^{cx}\sin(rx)}{e^{dx}-1}\,\mathrm{d}x < \int_0^{\pi/r} \frac{\sin(rx)}{de^{(\frac{d}{2}-c)x}}\,\mathrm{d}x = \frac{4r\left[1+e^{\pi(2c-d)/(2r)}\right]}{d[(2c-d)^2+4r^2]}. \tag{32}$$

Inequality (28) follows from combination of (31) and (32) with (16). □

Remark 4 If taking $a_k = k$ for $k \in \mathbb{N}$ or equivalently $d = 1$ and $c = 0$ in (20), (24) and (28), inequalities (5), (6) and (7) are deduced.

By exploiting a technique presented by E. Makai in [13], we obtain the following inequalities of Mathieu type series (11).

Theorem 5 *Let $a = \{a_k > 0, k \in \mathbb{N}\}$ with $a_k = k - c$. If $r > 0$ satisfies $r^2 + c^2 > c$, then*

$$\frac{1}{2r^2 + 2\left(c-\frac{1}{2}\right)^2 + \frac{1}{2}} < S(r,a) < \frac{1}{2r^2 + 2\left(c-\frac{1}{2}\right)^2 - \frac{1}{2}}. \tag{33}$$

Proof: By standard argument, we obtain

$$\begin{aligned}
&\frac{1}{\left[(k-c)-\frac{1}{2}\right]^2 + r^2 - \frac{1}{4}} - \frac{1}{\left[(k-c)+\frac{1}{2}\right]^2 + r^2 - \frac{1}{4}} \\
&= \frac{2(k-c)}{\left[(k-c)^2 + r^2 - (k-c)\right]\left[(k-c)^2 + r^2 + (k-c)\right]} \\
&> \frac{2(k-c)}{\left[(k-c)^2 + r^2\right]^2 - (k-c)^2} \\
&> \frac{2(k-c)}{\left[(k-c)^2 + r^2\right]^2} \\
&> \frac{2(k-c)}{\left[(k-c)^2 + r^2\right]^2 + r^2 + \frac{1}{4}} \\
&= \frac{2(k-c)}{\left\{\left[(k-c)-\frac{1}{2}\right]^2 + r^2 + \frac{1}{4}\right\}\left\{\left[(k-c)+\frac{1}{2}\right]^2 + r^2 + \frac{1}{4}\right\}} \\
&= \frac{1}{\left[(k-c)-\frac{1}{2}\right]^2 + r^2 + \frac{1}{4}} - \frac{1}{\left[(k-c)+\frac{1}{2}\right]^2 + r^2 + \frac{1}{4}},
\end{aligned} \tag{34}$$

summing up for $k = 1, 2, \ldots$ yields inequalities in (33). □

Remark 5 If letting $c = 0$, inequality (2) is deduced from (33).

Inequalities (24), (28) and (33) for every case do not include each other. This can be verified by using the well known software Mathematica [24].

It is also worthwhile to note that inequality

$$\frac{1}{c^2 + \frac{1}{2}} < \sum_{n=1}^{\infty} \frac{2n^{\alpha/2}}{(n^\alpha + c^2)^2} < \frac{1}{c^2} \tag{35}$$

obtained in [16, 18] and mentioned in [17] is a wrong result.

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