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Coincidence Points for Hybrid Mappings

1 Introduction

There have been several extensions of known fixed point theorems in which a mapping takes each point of a metric space into a closed (resp. closed and bounded) subset of the same (cf. [3, 4, 5, 7, 10, 11]). Hybrid fixed point theory for nonlinear mappings is relatively a recent development within the ambit of fixed point theory of point to set mappings (multivalued mappings) with a wide range of applications (see, for instance, [2, 8, 12, 13, 14, 15, 16]). Recently, in an attempt to improve /generalize certain results of Naidu, Sastry and Prasad [11] and Kaneko [4] and others, Chang [1] obtained some fixed point theorems for a hybrid of multivalued and singlevalued mappings.

However, his main theorem (see Theorem A below) admits a counter example. Our main purpose in this paper is to present a correct version of this result which, in turn, generalizes several known results in this direction.

Let (X, d) be a metric space. We shall use the following notations and definitions:

$$CL(X) = \{A : A \text{ is a nonempty closed subset of } X\},$$
$$CB(X) = \{A : A \text{ is a nonempty closed and bounded subset of } X\},$$
$$N(\epsilon, A) = \{x \in X : d(x, a) < \epsilon \text{ for some } a \in A,\ \epsilon > 0\},\ A \in CL(X),$$
$$E_{A,B} = \{\epsilon > 0 : A \subset N(\epsilon, B),\ B \subset N(\epsilon, A)\},\ A, B \in CL(X),$$
$$H(A, B) = \begin{cases} \inf E_{A,B} & \text{if } E_{A,B} \neq \phi \\ \infty & \text{if } E_{A,B} = \phi, \end{cases}$$
$$D(x, A) = \inf\{d(x, a) : a \in A\}$$

for each $A, B \in CL(X)$, and for each $x \in X$.

H is called the generalized Hausdorff metric for $CL(X)$ induced by d. If $H(A, B)$ is defined for $A, B \in CB(X)$, then H is called the Hausdorff metric induced by d (cf. Nadler [6]).

Definition 1 ([4]) *Mappings $S : X \to CB(X)$ and $I : X \to X$ are called compatible if $ISx \in CB(X)$ for all $x \in X$ and $H(SIx_n, ISx_n) \to 0$, as $n \to \infty$ whenever $\{x_n\}$ is a sequence in X such that $Sx_n \to M \in CB(X)$ and $Ix_n \to t \in M$ as $n \to \infty$.*

Following Singh and Mishra [16] (see also [3], [4] and [9]), we introduce the notion of R-sequentially commuting mappings for a hybrid pair of single-valued and multi-valued maps.

Definition 2 *Let K be a nonempty subset of a metric space X and $I : K \to X$ and $S : K \to CL(X)$ be respectively single-valued and multi-valued mappings. Then I and S will be called R-sequentially commuting on K if for a given sequence $\{x_n\} \subset K$ with $\lim_n Ix_n \in K$, there exists $R > 0$ such that*

$$\lim_n D(Iy, SIx_n) \leq R \lim_n D(Ix_n, Sx_n) \tag{$*$}$$

for each $y \in K \cap \lim_n Sx_n$.

If $x_n = x (x \in K)$ for all $n \in \mathbb{N}$ (naturals), $Ix \in K$ and $(*)$ holds for some $R > 0$, then I and S have been defined to be pointwise R-weakly commuting at $x \in K$ (see [16, Def. 1]). If it holds for all $x \in K$, then I and S are called R-weakly commuting on K. Further, if $R = 1$, we get the definition of weak commutativity of I and S on K due to Hadzic and Gajec [3]. If $I, S : X \to X$, then as mentioned in [16], we recover the definitions of pointwise R-weak commutativity and R-commutativity of single-valued self-maps due to Pant [9] and all the remarks as given in [16] apply.

We now introduce the following.

Definition 3 *Maps $I : K \to X$ and $S : K \to CL(X)$ are to be called sequentially commuting (or s-commuting) at a point $x \in K$ if*

$$I(\lim_n Sx_n) \subset SIx \tag{$**$}$$

whenever there exists a sequence $\{x_n\} \subset K$ such that $\lim_n Ix_n = x \in \lim_n Sx_n \in CL(X)$.

If $x_n = x$ for all $n \in \mathbb{N}$, then the maps I and S will be said to be weakly s-commuting at a point $x \in K$.

The following example shows that s-commutativity of I and S is indeed more general than their R-sequential commutativity (and hence their pointwise R-commutativity and compatibility).

Example 1 Let $X = [0, \infty)$ with the usual metric d and define $I : X \to X$ and $S : X \to CL(X)$ by

$$Ix = \begin{cases} 0, & \text{if } x \in [0,1] \\ x, & \text{if } x \in (1, \infty), \end{cases} \qquad Sx = [x, \infty).$$

Then for the sequence $\{x_n\} \subset X$ defined by $x_n = 1 + \frac{1}{n}$, we have $1 = \lim_n Ix_n \in [1, \infty) = \lim_n Sx_n \in CL(X)$ and $I(\lim_n Sx_n) = \{0\} \cup (1, \infty) \subset [0, \infty) = SI1$. Therefore, I and S are s-commuting but $(*)$ is not satisfied for $y = 1 \in [1, \infty) = \lim_n Sx_n$.

Definition 4 ([1]) *Let $\mathbb{R}^+$ denote the set of all non-negative real numbers, and let $A \subset \mathbb{R}^+$. A function $\varphi : A \to \mathbb{R}^+$ is upper semicontinuous from the right if $\lim_{x \to u+} \sup \varphi(x) \le \varphi(u)$ for all $u \in A$.*

A function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to satisfy (Φ)-conditions if:

(i) *φ is upper semi-continuous from the right on $(0, \infty)$ with $\varphi(t) < t$ for all $t > 0$, and*

(ii) *there exists a real number $s > 0$ such that φ is non-decreasing on $(0, s]$ and $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for all $t \in (0, s]$, where φ^n denotes the composition of φ with itself n times and $\varphi^0(t) = t$.*

Let Γ denote the set of all functions which satisfy the (Φ)-condition.

The following lemmas will be useful in proving our main results.

Lemma 1 *Let (X, d) be a metric space and $I, J : X \to X$ and $S, T : X \to CL(X)$ be such that $S(X) \subset J(X)$ and $T(X) \subset I(X)$ and for all $x, y \in X$,*

$$H(Sx, Ty) \le \varphi(aL(x, y) + (1 - a)N(x, y)), \tag{1}$$

where $a \in [0, 1]$, $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is upper semi-continuous from the right on $(0, \infty)$ with $\varphi(t) < t$ for all $t > 0$, and

$$\begin{aligned} L(x, y) &= \max\{d(Ix, Jy),\ D(Ix, Sx),\ D(Jy, Ty),\ \frac{1}{2}[D(Ix, Ty) + D(Jy, Sx)]\}, \\ N(x, y) &= [\max\{d^2(Ix, Jy),\ D(Ix, Sx)\, D(Jy, Ty),\ D(Ix, Ty)\, D(Jy, Sx), \\ &\qquad \frac{1}{2} D(Ix, Sx)\, D(Jy, Sx),\ \frac{1}{2} D(Jy, Ty)\, D(Ix, Ty)\}]^{1/2}. \end{aligned}$$

Then $\inf_{x \in X} D(Ix, Sx) = 0 = \inf_{x \in X} D(Jx, Tx)$.

Proof: Due to symmetry, we may suppose that

$$\inf_{x \in X} D(Ix, Sx) = \inf_{x \in X} D(Jx, Tx) = \delta.$$

If $\delta > 0$, then $\varphi(\delta) < \delta$. Since φ is upper semi-continuous from the right, there exists $\epsilon > 0$ such that $\varphi(t) < \delta$ for all $t \in [\delta, \delta + \epsilon)$. Pick $x_0 \in X$ such that $D(Ix_0, Sx_0) < \delta + \epsilon$. By $S(X) \subset J(X)$, there exists $x_1 \in X$ such that $Jx_1 \in Sx_0$ and $d(Ix_0, Jx_1) < \delta + \epsilon$.

Consider

$$\delta \le D(Jx_1, Tx_1) \le H(Sx_0, Tx_1) \le \varphi(aL(x_0,x_1) + (1-a)N(x_0,x_1))\,,$$

where

$$\begin{aligned} L(x_0,x_1) &= \max\{d(Ix_0,Jx_1),\ D(Ix_0,Sx_0),\ D(Jx_1,Tx_1),\ \tfrac{1}{2}[D(Ix_0,Tx_1)+D(Jx_1,Sx_0)]\} \\ &= \max\{d(Ix_0,Jx_1),\ D(Jx_1,Tx_1)\} \end{aligned}$$

and

$$\begin{aligned} N(x_0,x_1) &= [\max\{d^2(Ix_0,Jx_1),\ D(Ix_0,Sx_0)\,D(Jx_1,Tx_1),\ D(Ix_0,Tx_1)\,D(Jx_1,Sx_0), \\ &\qquad \tfrac{1}{2}D(Ix_0,Sx_0)\,D(Jx_1,Sx_0),\ \tfrac{1}{2}D(Jx_1,Tx_1)\,D(Ix_0,Tx_1)\}]^{1/2} \\ &\le [\max\{d^2(Ix_0,Jx_1),\ d(Ix_0,Jx_1)D(Jx_1,Tx_1)\}]^{1/2} \\ &\le [\max\{d^2(Ix_0,Jx_1),\ d(Ix_0,Jx_1)D(Jx_1,Tx_1),\ D^2(Jx_1,Tx_1)\}]^{1/2} \\ &\le [\max\{d^2(Ix_0,Jx_1),\ D^2(Jx_1,Tx_1)\}]^{1/2} \\ &= \max\{d(Ix_0,Jx_1),\ D(Jx_1,Tx_1)\}\,. \end{aligned}$$

Hence,

$$\delta \le D(Jx_1,Tx_1) \le \varphi(\max\{d(Ix_0,Jx_1),\ D(Jx_1,Tx_1)\})\,,$$

which is a contradiction, since $\varphi(d(Ix_0,Jx_1)) < \delta$ and $\varphi(D(Jx_1,Tx_1)) < D(Jx_1,Tx_1)$ proving that $\delta = 0$.

Lemma 2 *Let X, I, J, S, T and φ be as defined Lemma 1 such that the inequality (1) holds. If $Ix \in Sx$ for some $x \in X$, then there exists a $y \in X$ such that $Ix = Jy$ and $Jy \in Ty$.*

Proof: Suppose $Ix \in Sx$. Since $S(X) \subset J(X)$, we may choose a $y \in X$ such that $Jy = Ix \in Sx$. By (1), we have

$$D(Jy,Ty) \le H(Sx,Ty) \le \varphi(aL(x,y) + (1-a)N(x,y))\,,$$

where

$$\begin{aligned} L(x,y) &= \max\{d(Ix,Jy),\ D(Ix,Sx),\ D(Ix,Sx),\ D(Jy,Ty),\ \tfrac{1}{2}[D(Ix,Ty)+D(Jy,Sx)]\} \\ &= D(Jy,Ty)\,, \end{aligned}$$

and

$$\begin{aligned} N(x,y) &= [\max\{d^2(Ix,Jy),\ D(Ix,Sx)\,D(Jy,Ty),\ D(Jx,Ty)\,D(Jy,Sx), \\ &\qquad \tfrac{1}{2}D(Ix,Sx)\,D(Jy,Sx),\ \tfrac{1}{2}D(Jy,Ty)\,D(Ix,Ty)\}]^{1/2} \\ &= (1/\sqrt{2})D(Jy,Ty)\,. \end{aligned}$$

Hence

$$D(Jy,Ty) \le \varphi([a+(1-a)/\sqrt{2})]D(Jy,Ty)) < D(Jy,Ty)\,,$$

a contradiction, and so $D(Jy,Ty)=0$, i.e., $Jy \in Ty$.

Remark 1 If the assumptions of Lemma 2 hold, then setting $x_{2n}=x$ and $x_{2n-1}=y$ for all $n \in \mathbb{N}$ and $z = Ix$ we observe that $Ix_{2n} \to z$, $Jx_{2n-1} \to z$, $D(Ix_{2n}, Sx_{2n}) \to 0$ and $D(Jx_{2n-1}, Tx_{2n-1}) \to 0$ as $n \to \infty$.

Lemma 3 ([11]) *Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing function such that*

(i) $\varphi(t+) < t$ *for all* $t>0$ *and* $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ *for all* $t > 0$.

Then there exists a strictly increasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

(ii) $\varphi(t) < \psi(t)$ *for all* $t>0$ *and* $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ *for all* $t > 0$.

Lemma 4 ([1]) *If $\varphi \in \Gamma$, then there exists a function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that:*

(i) ψ *is upper semi-continuous from the right with* $\varphi(t) \le \psi(t) < t$ *for all* $t > 0$,

(ii) ψ *is strictly increasing with* $\varphi(t) < \psi(t)$ *for* $t \in (0,s]$, $s>0$ *and* $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ *for* $t \in (0,s]$.

2 Main Results

The following theorem is the main result of Chang [1, Theorem 1].

Theorem A *Let (X,d) be a complete metric space, let I, J be two functions from X into X, and let $S,T : X \to CB(X)$ be two set-valued functions with $SX \subset JX$ and $TX \subset IX$. If there exists $\varphi \in \Gamma$ such that for all x, y in X,*

$$H(Sx,Ty) \le \varphi\left(\max\left\{d(Ix,Jy),\ D(Ix,Sx),\ D(Jy,Ty),\ \frac{1}{2}[D(Ix,Ty)+D(Jy,Sx)]\right\}\right)\,, \tag{C}$$

then there exists a sequence $\{x_n\}$ in X such that $Ix_{2n} \to z$ and $Jx_{2n-1} \to z$ for some z in X and $D(Ix_{2n}, Sx_{2n}) \to 0$, $D(Jx_{2n-1}, Tx_{2n-1}) \to 0$ as $n \to \infty$. Moreover, if $Iz = z$ and T and J are compatible, then $z \in Sz$ and $Jz \in Tz$. That is, z is a common fixed point of I and S, and z is a coincidence point of J and T.

The following example shows that Theorem A in its present form is incorrect.

Example 2 Let $X = [0,1]$ with absolute value metric d and let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $\varphi(t) = t^2$ for $t \in [0,1)$ and $\varphi(t) = 1/2$ for $t \geq 1$. Define $I = J : X \to X$ and $S = T : X \to CB(X)$ by $Ix = 1 - x$, $x \in X$ and $Sx = \{0, 1/3, 2/3, 1\}$ for all $x \in X$. Then for each $x, y \in X$ and $\varphi \in \Gamma$, we have

$$\begin{aligned} H(Sx,Ty) &= 0 \\ &\leq \psi\left(\max\left\{d(Ix,Jy),\ D(Ix,Sx),\ D(Jy,Ty),\ \frac{1}{2}[D(Ix,Ty)+D(Jy,Sx)]\right\}\right) \end{aligned}$$

and for the sequence $\{x_n\} \subset X$ defined by $x_n = 1/n$ for all $n \in \mathbb{N}$, we have $Sx_n, Tx_n \to \{0, 1/3, 2/3, 1\} = M$, $Ix_n, Jx_n = 1 - 1/n \to 1 \in M \subset X$, $D(Ix_{2n}, Sx_{2n}) \to 0$ and $D(Jx_{2n-1}, Tx_{2n-1}) \to 0$ as $n \to \infty$. Also, $z = 1/2 \in X$ is such that $Iz = z$ and for $\{x_n\}$ as defined above we have $\lim_n H(TJx_n, JTx_n) = 0$, that is, T and J are compatible. Thus, all the conditions of Theorem A are satisfied. Evidently, $z \notin Sz$, $Jz \notin TZ$, that is, $z = 1/2$ is neither a common fixed point of I and S nor it is a coincidence point of J and T.

Before we present a corrected version of Theorem A, we have the following:

Theorem 1 *Let (X,d) be a complete metric space, and let $I, J : X \to X$, $S, T : X \to CL(X)$. Let A be a nonempty subset of X such that $I(A)$ and $J(A)$ are closed subsets of X, and $Tx \subseteq I(A)$ and $Sx \subseteq J(A)$ for all $x \in A$ and there exists a $\varphi \in \Gamma$ such that for all $x, y \in X$, (1) holds. Then*

(i) $F = \{Ix : x \in X \text{ and } Ix \in Sx\} \neq \phi$,

(ii) $G = \{Jx : x \in X \text{ and } Jx \in Tx\} \neq \phi$,

(iii) $F = G$ *if* $A = X$.

Proof: Let ψ be the function satisfying the conclusion of Lemma 4. By (1), we have for any $x, y \in X$, and $Ix \in Ty$,

$$\begin{aligned} D(Ix,Sx) &\leq H(Ty,Sx) \\ &\leq \varphi(aL(x,y) + (1-a)N(x,y)), \end{aligned}$$

where

$$\begin{aligned} L(x,y) &= \max\{d(Ix,Jy),\ D(Ix,Sx),\ D(Jy,Ty),\ \frac{1}{2}[D(Ix,Ty)+D(Jy,Sx)]\} \\ &\leq \max\{d(Ix,Jy),\ D(Ix,Sx),\ [d(Jy,Ix)+D(Ix,Ty)], \\ &\qquad \frac{1}{2}[D(Ix,Ty)+d(Ix,Jy)+D(Ix,Sx)]\} \\ &= \max\{d(Ix,Jy),\ D(Ix,Sx),\ \frac{1}{2}[d(Ix,Jy)+D(Ix,Sx)]\} \\ &= \max\{d(Ix,Jy),\ D(Ix,Sx)\} \end{aligned}$$

and

$$\begin{aligned}
N(x,y) &= [\max\{d^2(Ix,Jy),\ D(Ix,Sx)D(Jy,Ty),\ D(Ix,Ty)D(Jy,Sx),\\
&\qquad \frac{1}{2}D(Ix,Sx)D(Jy,Sx),\ \frac{1}{2}D(Jy,Ty)D(Ix,Ty)\}]^{1/2}\\
&= \max\{d^2(Ix,Jy),\ D(Ix,Sx)D(Jy,Ty),\ \frac{1}{2}(Ix,Sx)D(Jy,Sx)\}^{1/2}\\
&\le [\max\{d^2(Ix,Jy),\ d(Ix,Jy)D(Ix,Sx)\,,\\
&\qquad \frac{1}{2}[d(Ix,Jy)+D(Ix,Sx)]D(Ix,Sx)\}]^{1/2}\\
&\le [\max\{d^2(Ix,Jy),\ d(Ix,Jy)D(Ix,Sx),\ D^2(Ix,Sx)\}]^{1/2}\\
&= [\max\{d^2(Ix,Jy),\ D^2(Ix,Sx)\}]^{1/2}\\
&= \max\{d(Ix,Jy),\ D(Ix,Sx)\}\,.
\end{aligned}$$

Since $D(Ix,Sx) \le \varphi(aD(Ix,Sx)+(1-a)D(Ix,Sx))$ is inadmissible for any $a \in [0,1]$, $D(Ix,Sx) \le \varphi(aD(Ix,Sx)+(1-a)d(Ix,Jy))$ is inadmissible for $a=1$ and $D(Ix,Sx) \le \varphi(ad(Ix,Jy)+(1-a)D(Ix,Sx))$ is inadmissible for $a=0$, it follows that

$$\begin{aligned}
D(Ix,Sx) &\le (ad(Ix,Jy)+(1-a)d(Ix,Jy))\\
&= \varphi(d(Ix,Jy))\,.
\end{aligned}$$

Similarly we can show that

$$D(Jy,Ty) \le \varphi(d(Ix,Jy)) \text{ if } Jy \in Sx\,.$$

Pick $x_0 \in A$ such that $D(Ix_0,Sx_0) < s$. Since $Sx_0 \subseteq J(A)$, there exists $x_1 \in A$ such that $Jx_1 \in Sx_0$. Then we have

$$\begin{aligned}
D(Jx_1,Tx_1) &\le H(Sx_0,Tx_1)\\
&\le \varphi(aL(x_0,x_1)+(1-a)N(x_0,x_1))\\
&\le \psi(aL(x_0,x_1)+(1-a)N(x_0,x_1))\,.
\end{aligned}$$

Since $Tx_1 \subseteq I(A)$, we may choose $x_2 \in A$ such that $Ix_2 \in Tx_2$ and

$$d(Jx_1,Ix_2) \le \psi(aL(x_0,x_1)+(1-a)N(x_0,x_1))\,.$$

Therefore

$$\begin{aligned}
D(Ix_2,Sx_2) &\le H(Sx_2,Tx_1)\\
&\le \varphi(aL(x_2,x_1)+(1-a)N(x_2,x_1))\\
&< \psi(aL(x_2,x_1)+(1-a)N(x_2,x_1))\,.
\end{aligned}$$

Hence we can choose $x_3 \in A$ such that $Jx_3 \in Sx_2$ and $d(Ix_2, Jx_3) \leq \psi(aL(x_2,x_1)+ (1-a)N(x_2,x_1))$. Proceeding in this way, we can construct a sequence $\{x_n\}_{n=0}^{\infty}$ in A such that $Jx_{2n+1} \in Sx_{2n}$, $Ix_{2n+2} \in Tx_{2n+1}$ $(n = 0, 1, 2, \dots)$ and

$$\begin{aligned} d(Ix_{2n}, Jx_{2n+1}) &\leq \psi(aL(x_{2n}, x_{2n-1}) + (1-a)N(x_{2n}, x_{2n-1}))\,, \\ d(Jx_{2n-1}, Ix_{2n}) &\leq \psi(aL(x_{2n-2}, x_{2n-1}) + (1-a)N(x_{2n-2}, x_{2n-1})) \end{aligned}$$

for all $n \in \mathbb{N}$ (naturals). By the construction of $\{x_n\}$ we have

$$\begin{aligned} L(x_{2n}, x_{2n-1}) &\leq \max\{d(Ix_{2n}, Jx_{2n-1}),\ d(Ix_{2n}, Jx_{2n+1})\}\,, \\ N(x_{2n}, x_{2n-1}) &\leq \max\{d(Ix_{2n}, Jx_{2n-1}),\ d(Ix_{2n}, Jx_{2n+1})\}\,, \\ L(x_{2n-2}, x_{2n-1}) &\leq \max\{d(Ix_{2n-2}, Jx_{2n-1}),\ d(Ix_{2n}, Jx_{2n+1})\} \text{ and} \\ N(x_{2n-2}, x_{2n-1}) &\leq \max\{d(Ix_{2n-2}, Jx_{2n-1}),\ d(Ix_{2n-2}, Jx_{2n+1})\} \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Since ψ is strictly increasing on $(0, s]$ and $\psi(t) < t$ for $t > 0$, we have

$$\begin{aligned} d(Ix_{2n}, Jx_{2n+1}) &\leq \psi(ad(Ix_{2n}, Jx_{2n-1}) + (1-a)d(Ix_{2n}, Jx_{2n-1})) \\ &= \psi(d(Ix_{2n}, Jx_{2n-1})) \\ &\leq d(Jx_{2n-1}, Ix_{2n})\psi(ad(Ix_{2n-2}, Jx_{2n-1}) + (1-a)d(Ix_{2n-2}, J_{2n-1})) \\ &= \psi(d(Ix_{2n-2}, Jx_{2n-1})) \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Hence

$$\begin{aligned} d(Ix_{2n}, Jx_{2n+1}) &\leq \psi^{2n}(d(Ix_0, Jx_1)) \text{ and} \\ d(Jx_{2n-1}, Ix_{2n}) &\leq \psi^{2n-1}(d(Ix_0, Jx_1)) \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Set $y_{2n} = Ix_{2n}$ and $y_{2n+1} = Jx_{2n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Then

$$d(y_n, y_{n+1}) \leq \psi^n(d(y_0, y_1)) \text{ for all } n \in \mathbb{N}.$$

Since $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for $t \in (o, s]$ and $d(y_0, y_1) = d(Ix_0, Jx_1) < s$, it follows that $\sum_{n=1}^{\infty} d(y_n, y_{n+1})$ is convergent. Hence by the completeness of X, $\{y_n\}$ converges to z for some $z \in X$. Since $\{y_{2n}\}$ is a sequence in $I(A)$ converging to z and $I(A)$ is closed, it follows that $z \in I(A)$. So there exists a $w \in X$ such that $Iw = z$. Now by (1), we have

$$\begin{aligned} D(Ix_{2n}, Sw) &\leq H(Sw, Tx_{2n-1}) \\ &\leq \varphi(aL(w, x_{2n-1}) + (1-a)N(w, x_{2n-1})) \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Making $n \to \infty$ in the above inequality, we obtain

$$D(z, Sw) \leq \varphi(aD(z, Sw)_+ + (1-a)D(z, Sw)_+) = \varphi(D(z, Sw)_+)\,.$$

By the definition of φ, we have $\varphi(t_+) < t$ for all $t \in (0, \infty)$, it follows that $D(z, Sw) = 0$. Hence $Iw \in Sw$ and so

$$F = \{Ix : x \in X \text{ and } Ix \in Sx\} \neq \phi \,.$$

Similarly

$$G = \{Jx : x \in X \text{ and } Jx \in Tx\} \neq \phi \,.$$

We now suppose that $Sx \subseteq J(X)$ and $Tx \subseteq I(X)$ for all $x \in X$. Pick $u \in X$ such that $Iu \in Su$. Then since $Su \subseteq J(X)$, there exists a $v \in X$ such that $Jv = Iu$. By the inequality (1), we have

$$\begin{aligned} D(Jv, Tv) &\leq H(Su, Tv) \\ &\leq \varphi(aD(Jv, Tv) + (1 - a)D(Jv, Tv)) \\ &< D(Jv, Tv) \,. \end{aligned}$$

Hence $Jv \in Tv$. It follows that $F \subseteq G$. Similarly we can prove that $G \subseteq F$. Hence $F = G$. Further, suppose that $I(X)$ and $J(X)$ are closed. Choose a sequence $\{u_n\}$ in X such that $Iu_n \in Su_n$ for all $n \in \mathbb{N}$ and $\{Iu_n\}$ is convergent. Since $I(X)$ is closed, it follows that $\lim_n Iu_n = Iu$ for some $u \in X$. Since $Iu_n \in Su_n \subseteq J(X)$ for all $n \in \mathbb{N}$ and $J(X)$ is closed, it follows that $Iu \in J(X)$. So there exists a $v \in X$ such that $Iu = Jv$. Again by (1), we have

$$\begin{aligned} D(Iu_n, Tv) &\leq H(Su_n, Tv) \\ &\leq \varphi(aL(u_n, v) + (1 - a)N(u_n, v)) \,. \end{aligned}$$

Making $n \to \infty$ in the above inequality, we obtain

$$D(Jv, Tv) \leq \varphi(aD(Jv, Tv)_+ + [(1 - a)/\sqrt{2}]D(Jv, Tv)_+) \,.$$

Hence $Jv \in Tv$. By (1), we have

$$\begin{aligned} D(Iu, Su) &\leq H(Su, Tv) \\ &\leq \varphi(aD(Iu, Su) + [(1 - a)/\sqrt{2}]D(Iu, Su)) \,. \end{aligned}$$

Hence $Iu \in Su$. It follows that G is closed. #

Remark 2 Theorem 1 of Naidu [7] and Theorem 9 of Sastry, Naidu and Prasad [11] follow as direct corollaries of Theorem 1.

Remark 3 For $a = 1$, Example 10 of Sastry, Naidu and Prasad [11] shows that Theorem 1 fails if $\frac{1}{2}[D(Ix, Ty) + D(Jy, Sx)]$ is replaced by $\max\{D(Ix, Ty), D(Jy, Sx)\}$ even if $S = T$, $I = J = i_d$ (the identity mapping on X) and φ is continuous on $\mathbb{R}^+$.

Remark 4 If (1) is assumed to be valid only for those $x, y \in X$ for which $Ix \neq Jy$, $Ix \notin Sx$ and $Jy \notin Ty$ instead of all $x, y \in X$, then we conclude from Theorem 1 that: either $F = \{Ix : x \in X \text{ and } Ix \in Sx\} \neq \phi$ or $G = \{Jx : x \in X \text{ and } Jx \in Tx\} \neq \phi$.

The following theorem presents a correct version of Theorem A.

Theorem 2 *Let (X, d) be a complete metric space, and let $I, J : X \to X$, $S, T : X \to CL(X)$ be such that $S(X) \subseteq J(X)$ and $T(X) \subseteq I(X)$. If there exists a $\varphi \in \Gamma$ such that for all $x, y \in X$, (1) holds, then there is a sequence $\{x_n\}$ in X such that $Ix_{2n} \to z$ and $Jx_{2n-1} \to z$ for some $z \in X$ and $D(Ix_{2n}, Sx_{2n}) \to 0$, $D(Jx_{2n-1}, Tx_{2n-1}) \to 0$ as $n \to \infty$. Moreover,*

(i) *if $Iz \in Sz$ and $d(Iz, z) \leq D(z, Sx)$ for all $x \in X$, then $z \in Sz$, and if $d(Iz, z) \leq D(z, Tx)$ for all $x \in X$, J and T are weakly s-commuting, then $Jz \in Tz$.*

(ii) *if $Jz \in Tz$ and $d(Jz, z) \leq D(z, Tx)$ for all $x \in X$, then $z \in Tz$; and if $d(Jz, z) \leq D(z, Sx)$ for all $x \in X$, I and S are weakly s-commuting, then $Iz \in Sz$.*

(iii) *if $Iz = z$ and J and T are weakly s-commuting, then $z \in Sz$ and $Jz \in Tz$.*

(iv) *if $Jz = z$ and I and S are weakly s-commuting, then $z \in Tz$ and $Iz \in Sz$.*

Proof: By replacing A with X throughout in the proof of Theorem 1, we can construct a sequence $\{x_n\}_{n=0}^{\infty} \subset X$ such that $Jx_{2n+1} \in Sx_{2n}$, $Ix_{2n+2} \in Tx_{2n+1}$ $(n = 0, 1, 2, \dots)$ and the sequences $\{Ix_{2n}\}$, $\{Jx_{2n-1}\}$ are Cauchy sequences which converge to the same limit $z \in X$ and $D(Ix_{2n}, Sx_{2n}) \to 0$, $D(Jx_{2n-1}, Tx_{2n-1}) \to 0$ as $n \to \infty$. It then follows that $D(z, Sx_{2n}) \to 0$ and $D(z, Tx_{2n-1}) \to 0$ as $n \to \infty$.

(i) Suppose $Iz \in Sz$, since $d(Iz, z) \leq D(z, Sz)$ and J and T are weakly s-commuting. Choose $m \in \mathbb{N}$ such that

$$\sup\{d(Ix_{2n}, z),\ d(Jx_{2n-1}, z),\ D(z, Sx_{2n}),\ D(z, Tx_{2n-1}) : n \geq m\} < s\,.$$

Then for $n \geq m$ we have

$$\begin{aligned} D(z, Sz) &\leq d(z, Ix_{2n}) + D(Ix_{2n}, Sz) \\ &\leq d(z, Ix_{2n}) + H(Sz, Tx_{2n-1}) \\ &\leq d(z, Ix_{2n}) + \varphi(aL(z, x_{2n-1}) + (1-a)N(z, x_{2n-1}))\,, \end{aligned} \tag{2}$$

where

$$\begin{aligned}L(z,x_{2n-1}) &= \max\{d(Iz,Jx_{2n-1}),\ D(Iz,Sz),\ D(Jx_{2n-1},Tx_{2n-1})\,,\\ &\qquad \frac{1}{2}[D(Iz,Tx_{2n-1})+D(Jx_{2n-1},Sz)]\}\\ &\le \max\{d(Iz,Jx_{2n-1}),\ 0,\ d(x_{2n-1},Tx_{2n-1})\,,\\ &\qquad \frac{1}{2}[d(Iz,z)+D(z,Tx_{2n-1}),\ d(Jx_{2n-1},z)+D(z,Tx_{2n-1})]\}\\ &\qquad \to \max\{d(Iz,z),\ 0,\ 0,\ \frac{1}{2}d(Iz,z)\} \text{ as } n\to\infty\,,\end{aligned}$$

i.e.

$$\lim_n L(z,x_{2n-1}) \le D(z,Sz)\,;$$

and

$$N(z,x_{2n-1}) \le [\max\{d^2(Iz,z),\ 0,\ 0,\ 0,\ 0\}]^{1/2} \text{ as } n\to\infty$$

i.e.

$$\lim_n N(z,x_{2n-1}) \le D(z,Sz)\,.$$

Hence making $n\to\infty$ in (2), we obtain

$$D(z,Sz) \le 0+\varphi(aD(z,Sz)+(1-a)D(z,Sz))\,,$$

that is, $D(z,Sz)=0$ and so $z\in Sz$. Choose $z'\in X$ such that $Jz'=z$, then

$$\begin{aligned}D(z,Tz') &\le H(Sz,Tz')\\ &\le \varphi(aL(z,z')+(1-a)N(z,z'))\,,\end{aligned} \tag{3}$$

where

$$\begin{aligned}L(z,z') &= \max\{d(Iz,Jz'),\ D(Iz,Sz), D(Jz',Tz'),\\ &\qquad \frac{1}{2}[D(Iz,Tz')+D(Jz',Sz)]\}\\ &\le \max\{d(Iz,z),\ D(Iz,Sz),\ D(z,Tz'),\\ &\qquad \frac{1}{2}[d(Iz,z)+D(z,Tz')+D(z,Sz)]\}\\ &= \max\{d(Iz,z),\ D(z,Tz')\} \le D(z,Tz')\end{aligned}$$

and

$$N(z,z') \le [\max\{d^2(Iz,z),0,0,0,\ \frac{1}{2}D(z,Tz')[d(Iz,z)+d(z,Tz')]\}]^{1/2}$$
$$\le D(z,Tz')\,.$$

Hence by (3)

$$D(z,Tz') \le \varphi(D(z,Tz'))\,,$$

and so $D(z,Tz')=0$; i.e.,$Jz'=z\in Tz'$.

Since J and T are weakly s-commuting and $Jz'\in Tz'$, we have

$$JJz'\in JTz'\subset TJz'\,,$$

which implies that $Jz\in Tz$.

(ii) The proof is analogous to the proof of (i) due to symmetry.

(iii) Suppose $Iz=z$ and J and T are weakly s-commuting. Choose m as in (i), then for all $n\ge m$

$$\begin{aligned} D(z,Sz) &\le d(z,Ix_{2n})+D(Ix_{2n},Sz) \\ &\le d(z,Ix_{2n})+H(Sz,Tx_{2n-1}) \\ &\le d(z,Ix_{2n})+\varphi(aL(z,x_{2n-1})+(1-a)N(z,x_{2n-1}))\,, \end{aligned} \tag{4}$$

where

$$L(z,x_{2n-1})\to\max\{0,\ D(z,Sz),\ 0,\ \frac{1}{2}D(z,Sz)\}\ \text{as}\ n\to\infty\,,$$

i.e.,

$$\lim_n L(z,x_{2n-1})=D(z,Sz)$$

and

$$N(z,x_{2n-1})\to[\max\{0,0,0,\ \frac{1}{2}D^2(z,Sz),0\}]^{1/2}\ \text{as}\ n\to\infty$$

i.e.,

$$\lim_n N(z,x_{2n-1})=D(z,Sz)\,.$$

Making $n\to\infty$ in (4), we obtain

$$D(z,Sz)\le 0+\varphi(aD(z,Sz)+[(1-a)/\sqrt{2}]D(z,Sz))$$
$$< D(z,Sz)\,,$$

which implies $D(z,Sz)=0$ and so $z\in Sz$. Choose $z'\in X$ such that $Jz'=z$, then

$$D(z,Tz')\le H(Sz,Tz')$$
$$\le\varphi(aL(z,z')+(1-a)N(z,z'))\,,$$

where

$$L(z,z') = \max\{d(Iz,Jz'),\ D(Iz,Sz),\ D(Jz',Tz'),\ \frac{1}{2}[D(Iz,Tz')+D(Jz,Sz)]\}$$
$$= D(z,Tz')$$

and

$$N(z,z') = [\max\{d^2(Iz,Jz'), D(Iz,Sz)D(Jz',Tz'),\ D(Iz,Tz')D(Jz',Sz),$$
$$\frac{1}{2}D(Iz,Sz)D(Jz',Sz),\ \frac{1}{2}D(Jz',Tz')D(Iz,Tz')\}]^{1/2}$$
$$= (1/\sqrt{2})D(z,Tz')\,.$$

Hence

$$D(z,Tz') \le \varphi(aD(z,Tz') + [(1-a)/\sqrt{2}]D(z,Tz'))$$
$$< D(z,Tz')\,.$$

It follows that $D(z,Tz') = 0$ and so $Jz' = z \in Tz'$. Since J and T are weakly s-commuting $Jz' \in Tz'$, we have $JJz' \in JTz'$. Hence $Jz \in Tz$.

(iv) Due to symmetry, the proof is analogous to the proof of (iii).#

Theorem 3 *Suppose that* $\lim_{t\to+\infty}(t-\varphi(t)) = +\infty$, *there are sequences* $\{x_n\}_{n=0}^{\infty}$ *and* $\{y_n\}_{n=0}^{\infty}$ *in* X *such that* $\{Ix_n, Ix_{n+1}\} \subset Sx_n$ *and* $\{Jy_n, Jy_{n+1}\} \subset Ty_n$ $(n = 0,1,2,\dots)$, *and*

$$H(Sx,Ty) \le \varphi(aL_1(x,y) + (1-a)N_1(x,y)) \tag{5}$$

for all $x,y \in X$ *and* $a \in [0,1]$, *where*

$$L_1(x,y) = \max\{D(Ix,Sx),\ D(Jy,Ty),\ \frac{1}{2}[D(Ix,Ty)+D(Jy,Sx)]\}$$

and

$$N_1(x,y) = [\max\{D(Ix,Sx)D(Jy,Ty),\ D(Ix,Ty)D(Jy,Sx),$$
$$\frac{1}{2}D(Ix,Sx)D(Jy,Sx),\ \frac{1}{2}D(Jy,Ty)D(Ix,Ty)\}]^{1/2}\,.$$

Then:

(i) *the sequences* $\{Sx_n\}$ *and* $\{Ty_n\}$ *converge in* $CL(X)$ *to the same limit* A *for some* $A \in CL(X)$.

(ii) $F = \{Ix : x \in X \text{ and } Ix \in Sx\} = I(X) \cap A,$ *and*
$G = \{Jy : y \in y \in X \text{ and } Jy \in Ty\} = J(X) \cap A.$

(iii) $Sx = A$ *whenever* $Ix \in Sx$ *and* $Ty = A$ *whenever* $Jy \in Ty$.

Proof: For a fixed $n \in \mathbb{N}$, let

$$\beta_n = \sup\{H(Sx_i, Ty_j) : 1 \le i, j \le n\}\,.$$

Let $\delta = \max\{H(Ix_0, Sx_1),\ H(Ty_0, Ty_1)\}$.

For $i, j \in \mathbb{N}$, the inequality (5) yields $H(Sx_i, Ty_j) \le \varphi(aL_1(x_i, y_j)) + (1-a)N_1(x_i, y_j)$, where

$$\begin{aligned} L_1(x_i, y_j) &= \max\{D(Ix_i, Sx_j),\ D(Jy_j, Ty_j),\ \frac{1}{2}[D(Ix_i, Ty_j) + D(Jy_j, Sx_i)]\} \\ &\le \frac{1}{2}[H(Sx_{i-1}, Ty_j) + H(Ty_{j-1}, Sx_i)] \\ &\le \max\{H(Sx_i, Ty_j),\ H(Ty_{j-1}, Sx_i)\} \end{aligned}$$

and

$$\begin{aligned} N_1(x_i, y_j) &\le [H(Sx_{i-1}, Ty_j)H(Ty_{j-1}, Sx_1)]^{1/2} \\ &\le \max\{H(Sx_{i-1}, Ty_j),\ H(Ty_{j-1}, Sx_i)\}\,. \end{aligned}$$

Hence for $i, j \in \mathbb{N}$, we have

$$H(Sx_i, Ty_j) \le \varphi(\max\{H(Sx_{i-1}, Ty_j),\ H(Ty_{j-1}, Sx_i)\})\,. \tag{6}$$

It follows that $\beta_n \le \varphi(\beta_n + \delta)$ for all $n = 1, 2, 3, \ldots$. Hence $(\beta_n + \delta) - \varphi(\beta_n + \delta) \le \delta$ for all $n = 1, 2, 3, \ldots$. Since $t - \varphi(t) \to +\infty$ as $t \to +\infty$, it follows that $\{\beta_n\}$ is bounded. Hence $\sup\{H(Sx_i, Ty_j) : i, j \in \mathbb{N}\}$ is finite.

For $n \in \mathbb{N}$,

$$\text{let } \nu_n = \sup\{H(Sx_i, Ty_j) : i, j \ge n\}\,.$$

Then the inequality (6) yields $\nu_n \le (\nu_{n-1})$ for all $n \in \mathbb{N}$. It follows that $\nu_n \le \varphi^n(\nu_0)$ for all $n \in \mathbb{N}$. Since $\varphi(t+) < t$ for all $t \in (0, \infty)$ and $\varphi(0) = 0$, it follows that $\varphi^n(\nu_0) \to 0$ as $n \to \infty$. So $\{\nu_n\}$ converges to zero. Again for all $i, j \in \mathbb{N}$, we have

$$H(Sx_i, Sx_j) \le H(Sx_i, Ty_i) + H(Ty_i, Sx_j)$$

Thus for all $i, j \ge n$ and using the fact that $\nu_n \to 0$ as $n \to \infty$ we have

$$H(Sx_i, Sx_j) \le z\nu_n \to o \text{ as } i, j \to +\infty\,.$$

It follows that $\{Sx_n\}$ is Cauchy. Since $(CL(X), H)$ is complete, $\{Sx_n\}$ is convergent in $CL(X)$. We can similarly show that $\{Ty_n\}$ is also convergent in $CL(X)$. Since $H(Sx_n, Ty_n) \to 0$ as $n \to \infty$, it follows that the sequences $\{Sx_n\}$ and $\{Ty_n\}$ converge in $CL(X)$ to the same limit A for some $A \in CL(X)$.

Suppose $u \in X$ such that $Iu \in A$. Then from the inequality (5) it follows that, for all $n \in \mathbb{N}$,

$$H(Su, Ty_n) \leq \varphi(aL_1(u, y_n) + (1-a)N_1(u, y_n)) \,, \tag{7}$$

where

$$\begin{aligned} L_1(u, y_n) &\leq \max\{H(A, Su),\ \frac{1}{2}[H(A, Ty_n) + H(Ty_n, Su)]\} \\ &\to H(A, Su)_+ \text{ as } n \to \infty \end{aligned}$$

and

$$\begin{aligned} N_1(u, y_n) &\leq [\max\{H(A, Ty_n)H(Ty_n, Su),\ \frac{1}{2}H(A, Su)H(Ty_n, Su)\}]^{1/2} \\ &\to (1/\sqrt{2})H(A, Su)_+ \text{ as } n \to \infty \,. \end{aligned}$$

Hence passing over to limit as $n \to \infty$ in (7), we obtain

$$\begin{aligned} H(Su, A) &\leq \varphi(aH(Su, A)_+ + [(1-a)/\sqrt{2}]H(Su, A)_+) \\ &\leq \varphi(H(Su, A)_+) \,. \end{aligned}$$

Since $\varphi(t_+) < t$ for all $t \in (0, \infty)$, it follows that $H(Su, A) = 0$. Hence $Su = A$. We now suppose that $v \in A$ such that $Iv \in Sv$. Then from the inequality (5), for all $n \in \mathbb{N}$, we have

$$\begin{aligned} H(Sv, Ty_n) &\leq \varphi\big(a \cdot \max\{D(Iv, Sv),\ D(Jy_n, Ty_n),\ \frac{1}{2}[D(Iv, Ty_n) + D(Jy_n, Sv)]\} \\ &\qquad + (1-a) \cdot [\max\{D(Iv, Sv)D(Jy_n, Ty_n), D(Iv, Ty_n)D(Jy_n, Sv)\,, \\ &\qquad \frac{1}{2}D(Iv, Sv)D(Jy, Sv), \frac{1}{2}D(Jy_n, Ty_n)D(Iv, Ty_n)\}]^{1/2}\big) \\ &\leq \varphi(a \cdot H(Sv, Ty_n) + (1-a) \cdot H(Sv, Ty_n)) \\ &= \varphi(H(Sv, Ty_n)) \,. \end{aligned}$$

Passing over to limit as $n \to \infty$ in the above inequality, we obtain $H(Sv, A) \leq \varphi(A(Sv, A)_+)$. Hence $H(Sv, A) = 0$. It follows that $Sv = A$. Thus we have shown that $F = I(X) \cap A$ and $Sx = A$ whenever $Ix \in Sx$. We can similarly show that $G = J(X) \cap A$ and $Ty = A$ whenever $Jy \in Ty$.#

Remark 5 Theorem 3 improves Theorem 2 of Naidu [7].

Corollary 2 *Suppose that* $\lim_{t\to+\infty}[t-\varphi(t)] = +\infty$*,* $S(X)$ *and* $T(X)$ *are closed subsets of* X*,* $Sx \subseteq I(X)$ *and* $Gx \subseteq J(X)$ *for all* $x \in X$ *and the inequality* (5) *holds for all* $x, y \in X$*,* $a \in [0,1]$*. Then:*

(i) $\{Ix : x \in X \text{ and } Ix \in Sx\} = \{Jx : x \in X \text{ and } Jx \in Tx\} = A$ *for some* $A \in CL(X)$*,*

(ii) $Sx = A = Ty$ *for all* $x \in I^{-1}(A)$ *and for all* $y \in J^{-1}(A)$*.*

Proof: The conclusion follows immediately from Theorem 3.#

Theorem 4 *Let* (X,d) *be a complete metric space, and let* $I, J : X \to X$ *and* $S, T : X \to CL(X)$*. Suppose that* $\lim_{t\to+\infty}(t-\varphi(t)) = +\infty$*, there are sequences* $\{x_n\}_{n=0}^{\infty}$ *and* $\{y_n\}_{n=0}$ *in* X *such that* $\{Ix_n, Ix_{n+1}\} \subset Sx_n$ *and* $\{Jy_n, Jy_{n+1}\} \subset Ty_n$ $(n = 0, 1, 2, \dots)$*, and* (5) *holds for all* $x, y \in X$*. If* I*,* J*,* S *and* T *are continuous ,* I*,* S *and* J*,* T *are compatible mappings, then there exists a point* $t \in X$ *such that* $It \in St$ *and* $Jt \in Tt$*, i.e.,* t *is a coincidence point of* I *and* S *and* J *and* T*.*

Proof: Following the proof technique of Theorem 3, we can show that the sequences $\{Sx_n\}$ and $\{Ty_n\}$ converge in $CL(X)$ to the same limit A for some A in $CL(X)$. By (5), for $m \geq n$ $(m, n \in \mathbb{N})$, we have

$$\begin{aligned} d(Ix_n, Jy_m) &\leq D(Ix_n, Sx_n) + D(Jy_m, Sx_n) \\ &\leq D(Ix_n, Sx_n) + H(Sx_n, Ty_m) \\ &\leq D(Ix_n, Sx_n) + \varphi(aL_1(x_n, y_m) + (1-a)N_1(x_n, y_m)), \end{aligned} \tag{8}$$

where

$$\begin{aligned} L_1(x_n, y_m) &= \max\{D(Ix_n, Sx_n),\ D(Jy_m, Ty_m),\ \frac{1}{2}[D(Ix_n, Ty_n) + D(Jy_m, Sx_n)]\} \\ &\leq H(Sx_n, Ty_m) \end{aligned}$$

and

$$\begin{aligned} N_1(x_n, y_m) &= [\max\{D(Ix_n, Sx_n)D(Jy_m, Ty_m),\ D(Ix_n, Ty_m)D(Jy_m, Sx_n), \\ &\qquad \frac{1}{2}D(Ix_n, Sx_n)D(Jy_m, Sx_n),\ \frac{1}{2}D(Jy_m, Ty_n)D(Ix_n, Ty_m)\}]^{1/2} \\ &\leq H(Sx_n, Ty_m). \end{aligned}$$

Making $n \to \infty$ in (8), we obtain

$$\lim_n d(Ix_n, Jy_m) \leq 0 + \varphi(0).$$

It follows that $Ix_n, Jy_n \to t$ as $n \to \infty$ for some $t \in X$, since X is complete, $d(Ix_n, Ix_m) \leq d(Ix_n, Jy_m) + d(Jy_m, Ix_n)$ and $d(Jy_n, Jy_m) \leq d(Jy_n, Ix_m) + d(Ix_m, Jy_m)$. Again since $D(t, A) \leq D(t, Sx_n) + H(Sx_n, A) \to 0$ as $n \to \infty$, it follows that $t \in A$. By continuity of I and S, and since S and I are compatible, we have

$$\begin{aligned} D(It, St) &= \lim_n D(It, SIx_n) \leq \lim_n H(IA, SIx_n) \\ &= \lim_n H(ISx_n, SIx_n) = 0 \end{aligned}$$

Hence $It \in St$. Due to symmetry, we can similarly show that $Jt \in Tt$.#

By applying the same arguments as in the proof of Theorem 3, we can easily prove the following theorems:

Theorem 5 *Let (X, d) be a complete metric space, and $I, J : X \to X$ and $S, T : X \to CL(X)$. Suppose that $\lim_{t\to+\infty}(t - \varphi(t)) = +\infty$, there are sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ in X such that $Ix_{n+1} \in Sx_n$ and $Jy_{n+1} \in Ty_n$ $(n = 0, 1, 2, \dots)$ and*

$$H(Sx, Ty) \leq \varphi\left(\frac{a}{2}[D(Ix, Ty) + D(Jy, Sx)] + (1 - a)[D(Ix, Ty)D(Jy, Sx)]^{1/2}\right) \qquad (5')$$

for all $x, y \in X$ and $a \in [0, 1]$. Then the sequences $\{Sx_n\}$, $\{Ty_n\}$ converge in $CL(X)$ to the same limit A for some $A \in CL(X)$, $\{Ix| \in X$ and $Ix \in Sx\} = I(X) \cap A$ and $\{Jy|y \in X$ and $Jy \in Ty\} = J(X) \cap A$. Further, $Sx = A$ whenever $Ix \in Sx$ and $Ty = A$ whenever $Jy \in Ty$.

Theorem 6 *Let (X, d) be a complete metric space, and let $I, J : X \to X$ and $S, T : X \to CL(X)$. Suppose that $\lim_{t\to+\infty}(t - \varphi(t)) = +\infty$, there are sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$in X such that $Ix_{n+1} \in Sx_n$ and $Jy_{n+1} \in Ty_n$ $(n = 0, 1, 2, \dots)$ and $(5')$ holds for all $x, y \in X$. If I, J, S and T are continuous, I, S and J, T are compatible mappings. Then there exists a point $t \in X$ such that $It \in St$ and $Jt \in Tt$; i.e., t is a coincidence point of I and S and J and T.*

Remark 6 In view of Example 10 of Sastry, Naidu and Prasad [11], the condition $\lim_{t\to\infty}(t - \varphi(t)) = +\infty$ in Theorems 3-6 cannot be dispensed with even if $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty$ for all $t \in \mathbb{R}^+$ with $S = T$ and $I = J = i_d$, the identity mapping on X.

Remark 7 It is not yet known whether the continuity of all four maps I, J, S and T in Theorems 4 and 6 are necessary or not.

Remark 8 Condition (2) of Naidu [7] is implied by condition $(5')$ of Theorem 5, and hence Theorem 2 of Naidu [7] is a direct consequence of Theorem 5 .

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