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Symmetric Proximal Hypertopology

Dedicated to our friend Professor Dr. Harry Poppe on his 70^{th} birthday.

ABSTRACT. In 1988 a new hypertopology, called **proximal (finite) hypertopology** was discovered. It involves the use of a proximity in the upper part but leaves the lower part the same as the lower Vietoris topology. In 1966, the lower Vietoris topology, which involves finitely many open sets, led to another lower topology involving locally finite families of open sets. In this paper, we change the lower hypertopology using a proximity and thus get a "symmetric" proximal hypertopology, which includes the earlier 'finite' topologies.

KEY WORDS AND PHRASES. Proximities, hyperspace, lower proximal hypertopology, upper proximal hypertopology, symmetric proximal hypertopology, Bombay hypertopology.

1 INTRODUCTION

Let (X, d) be a metric space and let $\mathrm{CL}(X)$ denote the family of all nonempty closed subsets of X. In 1914 Hausdorff defined a metric d_H on $\mathrm{CL}(X)$, which is now known as the **Hausdorff metric** ([10]). In 1922 Vietoris defined, for a T_1 space X, a topology $\tau(\mathrm{V})$, now called the **Vietoris or finite topology** on $\mathrm{CL}(X)$ ([24, 25] and [15]).

The Hausdorff metric topology was first generalized to uniform spaces by Bourbaki and later (1966) to the Wijsman topology ([26]), wherein the convergence of distance functionals is *pointwise* rather than uniform as in the Hausdorff case. A base for the Vietoris topology consists of two parts, the lower which involves the intersection of closed sets with finitely many open sets and the upper part involving closed sets contained in one open set.

In 1962 Fell in [9] changed the upper part by considering the complements of compact sets and this was further generalized by Poppe (see [19] and [20]) in 1966 to complements of members of Δ, a subfamily of $\mathrm{CL}(X)$. Also in 1966 Marjanovic in [14] altered the lower part to consist of intersection of closed sets with members of locally finite families of open sets.

In 1988 the upper part of τ(V) was generalized to consist of closed sets that are proximally contained in an open set while the lower part could be finite or locally finite as before ([7]). It was found that even the well known Hausdorff metric topology is essentially a kind of locally finite proximal topology, thus showing the importance of proximities in hyperspaces. Moreover, recently it was shown that, with the use of two proximities in the upper part, all known hypertopologies could be subsumed under one **Bombay topology** ([6], see also [17]). In the present paper we radically change the lower part by using as a base, families of closed sets that are *proximally near* finitely many open sets. This is the first time that proximities are used in the lower part. The use of proximities in both upper and lower parts yield symmetric proximal hypertopologies and we believe that they will play an important role in the literature. In fact, since a topological space X has an infinite spectrum of proximities compatible with its topology it is possible to have an unusually broad range of symmetrical proximal hypertopologies because our constructs are based on proximal relations between families of closed sets and finite collections of open subsets instead of the usual Boolean operations on open and closed subsets of X, namely unions, intersections, set inclusions.

In what follows (X, τ), (or X), always denotes a T_1 topological space. A binary relation δ on the power set of X is a *generalized proximity* iff

(i) $A \ \delta \ B$ implies $B \ \delta \ A$;

(ii) $A \ \delta \ (B \cup C)$ implies $A \ \delta \ B$ or $A \ \delta \ C$;

(iii) $A \ \delta \ B$ implies $A \neq \emptyset$, $B \neq \emptyset$;

(iv) $A \cap B \neq \emptyset$ implies $A \ \delta \ B$.

A generalized proximity δ is a *LO-proximity* iff it satisfies

(LO) $A \ \delta \ B$ and $b \ \delta \ C$ for every $b \in B$ together imply $A \ \delta \ C$.

Moreover, a LO-proximity δ is a *LR-proximity* iff it satisfies

(R) $x \ \underline{\delta} \ A$ (where $\underline{\delta}$ means the negation of δ) implies there exists $E \subset X$ such that $x \ \underline{\delta} \ E$ and $E^c \ \underline{\delta} \ A$.

A generalized proximity δ is an *EF-proximity* iff it satisfies

(EF) $A \ \underline{\delta} \ B$ implies there exists $E \subset X$ such that $A \ \underline{\delta} \ E$ and $E^c \ \underline{\delta} \ B$.

Note that each EF-proximity is a LR-proximity.

Whenever δ is a LO-proximity, $\tau(\delta)$ denotes the topology on X induced by the Kuratowski closure operator $A \to A^\delta = \{x \in X : x\ \delta\ A\}$. The proximity δ is declared compatible with respect to the topology τ iff $\tau = \tau(\delta)$ (see [8], [18] or [27]).

A T_1 topological space X admits always a compatible LO-proximity. A topological space X has a compatible LR- (respectively, EF-) proximity iff it is T_3 (respectively, Tychonoff).

If $A\ \delta\ B$, then we say A is δ-near to B; if $A\ \underline{\delta}\ B$ we say A is δ-far from B. A is declared δ-strongly included in B, written $A \ll_\delta B$, iff $A\ \underline{\delta}\ B^c$. $A\ \underline{\ll}_\delta\ B$ stands for its negation, i.e. $A\ \delta\ B^c$.

We assume, in general, that every compatible proximity δ on X is LO or even LR. These assumptions simplify the results and allow us to display readable statements and makes the theory transparent. In fact, it is a useful fact that in a LO-proximity δ two sets are δ-far iff their closures are δ-far (see [18] or [3]). Moreover, if δ is a compatible LR-proximity, then

(∗) for each $x \notin$ A, with $A = \mathrm{cl}\, A$, there is an open neighbourhood W of x such that $\mathrm{cl}\, W\ \underline{\delta}\ A$.

We use the following notation :

$\mathsf{N}(x)$ denotes the filter of open neighbourhoods of $x \in X$;

$\mathrm{CL}(X)$ is the family of all non-empty closed subsets of X;

$\mathsf{K}(X)$ is the family of all non-empty compact subsets of X.

We set $\Delta \subset \mathrm{CL}(X)$ and assume, without any loss of generality, that it contains all singletons and finite unions of its members.

In the sequel δ, η will denote compatible proximities on X.

The most popular and well studied proximity is the Wallman or fine LO-proximity δ_0 (or η_0) given by

$$A\ \delta_0\ B \Leftrightarrow \mathrm{cl}\, A \cap \mathrm{cl}\, B \neq \emptyset\,.$$

We note that δ_0 is a LR-proximity iff X is regular (see [11, Lemma 2]). Moreover, δ_0 is an EF-proximity iff X is normal (Urysohn Lemma).

Another useful proximity is the discrete proximity η^* given by

$$A\ \eta^*\ B \Leftrightarrow A \cap B \neq \emptyset\,.$$

We note that η^* is not a compatible proximity, unless (X, τ) is discrete.

We point out that in this paper η^* is the only proximity that might be non compatible.

For any open set E in X, we use the notation

$$E^+_\delta = \{F \in \mathrm{CL}(X) : F \ll_\delta E \text{ or equivalently } F \underline{\delta} E^c\} .$$
$$E^+ = E^+_{\delta 0} = \{F \in \mathrm{CL}(X) : F \ll_{\delta 0} E \text{ or equivalently } F \subset E\} .$$
$$E^-_\eta = \{F \in \mathrm{CL}(X) : F \,\eta\, E\} .$$
$$E^- = E^-_{\eta^*} = \{F \in \mathrm{CL}(X) : F \,\eta^*\, E \text{ or equivalently } F \cap E \neq \emptyset\} .$$

Now we have the material necessary to define the basic proximal hypertopologies.

(1.1) The lower $\boldsymbol{\eta}$-proximal topology $\boldsymbol{\sigma(\eta^-)}$ is generated by $\{E^-_\eta : E \in \tau\}$.

(1.2) The upper $\boldsymbol{\delta - \Delta}$-proximal topology $\boldsymbol{\sigma(\delta^+, \Delta)}$ is generated by $\{E^+_\delta : E^c \in \Delta\}$. We omit Δ and write it as $\sigma(\delta^+)$ if $\Delta = \mathrm{CL}(X)$.

We note that the upper Vietoris topology $\tau(\mathsf{V}^+)$ equals $\sigma(\delta_0^+)$ and the lower Vietoris topology $\tau(\mathsf{V}^-)$ can be written as $\sigma(\eta^{*-})$. These show that proximal topologies are generalizations of the classical upper and lower Vietoris topologies.

Moreover, we have:

The **Vietoris** topology $\tau(\mathsf{V}) = \sigma(\eta^{*-}) \vee \sigma(\delta_0^+) = \tau(\mathsf{V}^-) \vee \tau(\mathsf{V}^+)$.

The **$\boldsymbol{\delta}$-proximal topology** $\sigma(\delta) = \sigma(\eta^{*-}) \vee \sigma(\delta^+) = \tau(\mathsf{V}^-) \vee \sigma(\delta^+)$.

The following is a *new* hypertopology, which generalizes the above as well as the Δ-topologies introduced by Poppe ([19], [20]).

(1.3) The $\boldsymbol{\eta - \delta - \Delta}$-symmetric-proximal topology $\pi(\eta, \delta; \Delta) = \sigma(\eta^-) \vee \sigma(\delta^+; \Delta)$.

For references on proximities, we refer to [8], [18] and [27]. For LO-proximities see [18] and [16]. For LR-proximities see [11], [12] and also [3] and [4].
For references on hyperspaces up to 1993, we refer to [1], except when a specific reference is needed.

2 Three Important Lower Proximal Topologies

Let (X, d) be a metric space. The associated metric proximity η is defined by $A\ \eta\ B$ iff $\inf\{d(a,b) : a \in A, b \in B\} = 0$. Note that a metric proximity η is EF and compatible.

First we observe that, unlike the lower Vietoris topology, $\sigma(\eta^-)$ need not be admissible as the following example shows. The reader is referred to the Appendix (section 7) where the admissibility of the symmetrical proximal topology is investigated in details.

Example 2.1 Let $X = [-1, 1]$ with the metric proximity η. Let $A = \{0\}$, $A_n = \{\frac{1}{n}\}$, for all $n \in \mathbb{N}$. Then $\frac{1}{n}$ converges to 0 in X, but A_n does not converge to A in the topology $\sigma(\eta^-)$. Hence the map $i : X \to \mathrm{CL}(X)$, where $i(x) = \{x\}$, is not an embedding.

We recall that if η and η' are proximities on X, then η is declared *coarser* than η' (or equivalently η' is *finer* than η), written $\eta \leq \eta'$, iff $A\ \underline{\eta}\ B$ implies $A\ \underline{\eta'}\ B$.

We now begin to study lower proximal topologies corresponding to proximities η, η_0, η^*. Observe that for any compatible proximity η we have always $\eta \leq \eta_0 \leq \eta^*$ (since η_0 is the finest compatible LO-proximity and η^* is the discrete proximity). In the case of a metric space, we will take η to be the metric proximity and, in the case of a T_3 topological space, we will take η to be a compatible LR-proximity.

Theorem 2.2 *Let (X, τ) be a T_3 topological space and η a compatible LR-proximity. The following inclusions occur:*

(a) $\tau(V^-) = \sigma(\eta^{*-}) \subset \sigma(\eta^-)$,

(b) $\tau(V^-) = \sigma(\eta^{*-}) \subset \sigma(\eta_0^-)$,

i.e. the finest proximity η^* induces a coarser hypertopology.

Proof: We show (a). Suppose that the net $\{A_\lambda\}$ of closed sets converges to a closed set A in the topology $\sigma(\eta^-)$. If $A\ \eta^*\ U$, where $U \in \tau$, then there is an $x \in A \cap U$ and a $V \in \mathsf{N}(x)$, with $x \in V \subset \mathrm{cl}\, V \subset U$ and $V\ \underline{\eta}\ U^c$ (use $(*)$ in Section 1). We claim that eventually A_λ intersects U. For if not, then frequently $A_\lambda \subset U^c$ and so frequently $A_\lambda\ \underline{\eta}\ V$; a contradiction.

The case (b) is similar.

Remark 2.3 In Section 6, namely Theorem 6.7, we characterize proximities for which the above inclusions hold. Furthermore, we will look for transparent conditions under which a finer proximity induces a coarser hypertopology, a natural phenomenon.

This example shows that the assumption of regularity of the base space X cannot be dropped in Theorem 2.2.

Example 2.4 Let $X = [0, +\infty)$ with the topology τ consisting of the usual open sets together with all sets of the form $U = [0, \varepsilon)\backslash B$ where $\varepsilon > 0$ and $B \subset A = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then X is Hausdorff and not regular since $V = [0, \varepsilon')\backslash A \subset U$ for each $0 < \varepsilon' < \varepsilon$, but $\mathrm{cl}\, V \not\subset U$.

Now, set $F = \{0\}$ and for all $n \in \mathbb{N}$, $F_n = \{\frac{1}{n}\}$. Then $\{F_n\}$ does not converge to F in $\sigma(\eta^{*^-})$ (note that $0 \in [0, \varepsilon)\backslash A$ and $\frac{1}{n} \notin [0, \varepsilon)\backslash A$ for all $n \in \mathbb{N}$). However, if $\{0\}\ \eta\ W$, W open, then eventually $\{\frac{1}{n}\}\ \eta\ W$, where η is any compatible LO-proximity. So

$$\sigma(\eta^{*^-}) \not\subset \sigma(\eta^-)\,.$$

Now, we give an example to show that the inclusions in 2.2 (a), (b) are strict except in pathological situations.

Example 2.5 Let $X = [0, 2]$, $A = [0, 1]$, and for each natural number n set $A_n = [0, 1 - \frac{1}{n}]$. Then the sequence $\{A_n\}$ converges to A with respect to the $\sigma(\eta^{*^-})$ topology, but it converges to A neither with respect to the $\sigma(\eta^-)$ topology nor with respect to the $\sigma(\eta_0^-)$ topology.

We note that the space involved is one of the “best“ possible spaces and the sets involved are also compact.

In a UC metric space, i.e. one in which the metric proximity $\eta = \eta_0$, $\sigma(\eta_0^-) = \sigma(\eta^-)$.

We now give an example to show that $\sigma(\eta_0^-) \neq \sigma(\eta^-)$.

Example 2.6 Let $\mathbb{N}$ be the set of all natural numbers, $\mathbb{M} = \{n - \frac{1}{n} : n \in \mathbb{N}\}$, $X = \mathbb{N} \cup \mathbb{M}$ as subspace of the real line. X is not a UC space. Set $A = \mathbb{N}$ and $A_n = \{m \in \mathbb{N} : m < n\}$, for all $n \in \mathbb{N}$.

Then A_n converges to A in the topology $\sigma(\eta_0^-)$. However, $A\ \eta\ \mathbb{M}$ but $A_n\ \underline{\eta}\ \mathbb{M}$ for each $n \in \mathbb{N}$. So, the sequence $\{A_n\}$ does not converge to A in the topology $\sigma(\eta^-)$.

Note that in the above example $\sigma(\eta_0^-) \subset \sigma(\eta^-)$. We show below that this natural inclusion holds in a locally compact space, too.

Theorem 2.7 *Let (X, τ) be a locally compact Hausdorff space. If η and η_0 are respectively a compatible LO-proximity and the Wallman proximity on X, then:*

$$\sigma(\eta_0^-) \subset \sigma(\eta^-)\,.$$

Proof: Let V be an open set and let $A \in V_{\eta 0}^-$. Then there is a $z \in A \cap \mathrm{cl}\, V$. Let U be a compact neighbourhood of z and set $W = U \cap V$. Note that $\mathrm{cl}\, W$ is compact and that a closed set is η-near a compact set iff it is η_0-near it. Hence $A \in W_\eta^- \subset V_{\eta 0}^-$.

Remark 2.8 If the involved proximities η, η_0 are LR and the net of closed sets A_λ is eventually locally finite and converges to A in $\sigma(\eta^-)$, then it also converges in $\sigma(\eta_0^-)$. Hence, the same inclusion (i.e. $\sigma(\eta_0^-) \subset \sigma(\eta^-)$) as in the previous result holds.

We next consider first and second countability of $\sigma(\eta^-)$.
Observe, that the first and second countability of $\sigma(\eta^{*-})$, i.e. the lower Vietoris topology $\tau(V^-)$, are well known results. $(\mathrm{CL}(X), \tau(V^-))$ is first countable if and only if X is first countable and each closed subset of X is separable (cf. Theorem 1.2 in [13]); $(\mathrm{CL}(X), \tau(V^-))$ is second countable if and only if X is second countable (cf. Proposition 1.11 in [13]).
Thus, we study the first and the second countability of $\sigma(\eta^-)$ when η is different from η^*.
The following definitions have a key role.

Definitions 2.9 (see [2]) *Let (X, τ) be a T_1 topological space, η a compatible LO-proximity and $A \in CL(X)$.*

*A family N_A of open sets of X is an **external proximal local base at A with respect to η** (or, briefly a **η-external proximal local base at A**) if for any U open subset of X such that $A\,\eta\,U$, there exists $V \in \mathsf{N}_A$ satisfying $A\,\eta\,V$ and $cl V \subset cl U$.*

*The **external proximal character of A with respect to η** (or, briefly the **η-external proximal character of A**) is defined as the smallest (infinite) cardinal number of the form $|\mathsf{N}_A|$, where N_A is a η-external proximal local base at A, and it is denoted by $E\chi(A, \eta)$.*

*The **external proximal character of CL(X) with respect to η** (or, briefly the **η-external proximal character**) is defined as the supremum of all number $E\chi(A, \eta)$, where $A \in CL(X)$. It is denoted by $E\chi(CL(X), \eta)$.*

Remark 2.10 If X is a T_3 topological space, η is a compatible LR-proximity and the η-external proximal character $\mathrm{E}\chi(\mathrm{CL}(X), \eta)$ is countable, then X is separable .

Theorem 2.11 *Let (X, τ) be a T_3 topological space with a compatible LR-proximity η. The following are equivalent:*

(a) *$(CL(X),\ \sigma(\eta^-))$ is first countable;*

(b) *the η-external proximal character $E\chi(CL(X), \eta)$ is countable.*

Proof: (a)$\Rightarrow$(b). It suffices to show that for each $A \in \mathrm{CL}(X)$ there exists a countable family N_A of open sets of X which is a η-external proximal local base at A. First note a useful fact:

$$\text{For open sets } V, U, \mathrm{cl}\, V \subset \mathrm{cl}\, U \;\Leftrightarrow\; V_\eta^- \subset U_\eta^- . \qquad (\#)$$

Now, let $A \in \mathrm{CL}(X)$ and Z a countable subbase $\sigma(\eta^-)$-neighbourhood system of A. Set $\mathsf{N}_A = \{V : V$ an open set with $V_\eta^- \in \mathrm{Z}\}$. Clearly, N_A is a countable family of open sets with the property that $A\,\eta\,U$, U open, implies there is a $V \in \mathsf{N}_A$ satisfying $A\,\eta\,V$ and $V_\eta^- \subset U_\eta^-$. Thus the result follows from (#).

(b)⇒(a). It is obvious.

Definitions 2.12 (see [2]) *Let (X,τ) be a T_1 topological space with a compatible LO-proximity η.*

*A family N of open sets of X is an **external proximal base with respect to η** (or, briefly a **η-external proximal base**) if for any closed subset A of X and any open subset U of X with $A\,\eta\,U$, there exists $V \in \mathsf{N}$ satisfying $A\,\eta\,V$ and $clV \subset clU$.*

*The **external proximal weight of $CL(X)$ with respect to η** (or, briefly the **η-external proximal weight of $CL(X)$**) is the smallest (infinite) cardinality of its η-external proximal bases and it is denoted by $EW(CL(X),\eta)$.*

Theorem 2.14 *Let (X,τ) be a T_3 topological space with a compatible LR-proximity η. The following are equivalent:*

(a) *$(CL(X), \sigma(\eta^-))$ is second countable;*

(b) *the η-external proximal weight $EW(CL(X),\eta)$ is countable.*

3 Six Hyperspace Topologies

The three lower topologies $\sigma(\eta^{*^-})$, $\sigma(\eta^-)$, $\sigma(\eta_0^-)$ combined with two upper ones $\sigma(\delta^+;\Delta)$, $\sigma(\delta_0^+;\Delta)$ yield six distinct hypertopologies of which the first two are already well known as we remarked before.

$$\pi(\eta^*, \delta_0; \Delta) = \sigma(\eta^{*^-}) \vee \sigma(\delta_0^+; \Delta) = \tau(\Delta)\,, \tag{1}$$

the Δ-topology which equals the Vietoris topology when $\Delta = \mathrm{CL}(X)$ and equals the Fell topology when $\Delta = \mathsf{K}(X)$.

$$\pi(\eta^*, \delta; \Delta) = \sigma(\eta^{*^-}) \vee \sigma(\delta^+; \Delta) = \sigma(\delta; \Delta)\,, \tag{2}$$

the proximal-Δ-topology which equals the proximal topology when $\Delta = \mathrm{CL}(X)$ and equals the Fell topology when $\Delta = \mathsf{K}(X)$ and δ is a LR-proximity.

$$\pi(\eta, \delta_0; \Delta) = \sigma(\eta^-) \vee \sigma(\delta_0^+; \Delta)\,. \tag{3}$$

$$\pi(\eta, \delta; \Delta) = \sigma(\eta^-) \vee \sigma(\delta^+; \Delta)\,. \tag{4}$$

$$\pi(\eta_0, \delta_0; \Delta) = \sigma(\eta_0^-) \vee \sigma(\delta_0^+; \Delta)\,. \tag{5}$$

$$\pi(\eta_0, \delta; \Delta) = \sigma(\eta_0^-) \vee \sigma(\delta^+; \Delta)\,. \tag{6}$$

Theorem 3.1 *The following relationships hold when X is a T_3 topological space, η_0 is the Wallman proximity and η is a LR-proximity. Moreover, for simplicity we consider $\Delta = CL(X)$.*

(a) $\sigma(\delta) \subset \tau(V) \subset \pi(\eta_0, \delta_0)$.

(b) $\sigma(\delta) \subset \tau(V) \subset \pi(\eta, \delta_0)$.

(c) $\sigma(\delta) \subset \pi(\eta_0, \delta) \subset \pi(\eta_0, \delta_0)$.

(d) $\sigma(\delta) \subset \pi(\eta, \delta) \subset \pi(\eta, \delta_0)$.

(e) $\sigma(\delta) \subset \tau(V) \subset \pi(\eta_0, \delta_0) \subset \pi(\eta, \delta_0)$ *when X is locally compact.*

(f) $\sigma(\delta) \subset \pi(\eta_0, \delta) \subset \pi(\eta_0, \delta_0) \subset \pi(\eta, \delta_0)$ *when X is locally compact.*

Let (X, τ) be a T_3 topological space with a compatible LR-proximity δ. Then X is called a **PC space** if $\delta = \delta_0$. In such a space, every continuous function on X to an arbitrary proximity space is proximally continuous. In case X is a metric space with the metric proximity δ, then X is PC if and only if X is UC (i.e. continuous functions on X are uniformly continuous).

Theorem 3.2 *Let (X, τ) be a T_3 topological space. The following are equivalent:*

(a) *X is PC;*

(b) $\sigma(\delta) = \tau(V)$;

(c) $\pi(\eta_0, \delta) = \pi(\eta_0, \delta_0)$;

(d) $\pi(\eta, \delta_0) \subset \pi(\eta, \delta)$.

4 Properties of $(\mathbf{CL}(\mathbf{X}), \pi(\eta, \delta))$

We now study some properties of the topological space $(\mathrm{CL}(X), \pi(\eta, \delta))$.
In general, the map $x \to \{x\}$ from (X, τ) into $(\mathrm{CL}(X), \sigma(\eta^-))$ fails to be an embedding (see Example 2.1). As a result, the topology $\pi(\eta, \delta)$ in general is not admissible. The admissibility of $\pi(\eta, \delta)$ is studied in Appendix (see section 7).
From 2.2 we know that if the space X is regular and the involved proximities are LR, then $(\mathrm{CL}(X), \pi(\eta, \delta))$ is Hausdorff. We begin with some lemmas.

The following is a generalization of the well known result: sets of the form $< U_k^- >= \{E : E \cap U_k \neq \emptyset$ and $E \subset U = \cup U_k\}$ form a base for the Vietoris topology, where $\{U, U_k\}$ is a finite family of open sets.

We recall that given two proximity δ and η on X, $\delta \leq \eta$ iff $A \underline{\delta} B$ implies $A \underline{\eta} B$.

Lemma 4.1 *Let (X, τ) be a T_1 topological space and δ, η compatible proximities. If $\delta \leq \eta$, then all sets of the form*

$$< U_k^-, U^+ >_{\eta,\delta} = \{E : E \; \eta \; U_k, E \ll_\delta U \text{ and } \bigcup U_k \subset U\}$$

form a base for the $\pi(\eta, \delta)$ topology.

Proof: If $A \in \bigcap\{V_{k,\eta}^- : 1 \leq k \leq n\} \cap U_\delta^+$, then $A \underline{\delta} U^c$ and $A \; \eta \; V_k$ for each $k \in \{1, \ldots, n\}$. Thus, we may replace each V_k by $V_k \cap U$. In fact, from $\delta \leq \eta$ we have $A \underline{\eta} U^c$ and thus $A \; \eta \; V_k$ iff $A \; \eta \; (V_k \cap U)$.

Remark 4.2 (i) If $\delta \leq \eta$ and D is a dense subset of X, then the family of all finite subsets of D is dense in $(\mathrm{CL}(X), \pi(\eta, \delta; \Delta))$.

(ii) Note that in the above Lemma and in (i) we need only $\delta \leq \eta$, restricted to the pairs (E, W) where E is closed and W is open.

The following is a generalization of the result: $< \mathrm{cl}\, U_k^- >$, $1 \leq k \leq n$, is closed in $\tau(\mathrm{V})$.

First, we need a definition.

Definition 4.3 *Let (X, τ) be a T_1 topological space and δ, η compatible proximities. The hypertopologies $\pi(\delta, \eta)$ and $\pi(\eta, \delta)$ are called* ***conjugate****.*

Lemma 4.4 *Let (X, τ) be a T_1 topological space with compatible proximities δ, η. If $P =< clU_k^-, clU >_{\eta,\delta}$, $1 \leq k \leq n$, $U = \cup U_k$ then P is closed with respect to its conjugate $\pi(\delta, \eta)$.*

Proof: $A \notin P$ if and only if $A \underline{\ll_\delta} \mathrm{cl}\, U$ for $A \underline{\eta} \mathrm{cl}\, U_k$ for some k which in turn is equivalent to $A \; \delta \; [\mathrm{cl}\, U]^c$ or $A \ll_\eta [\mathrm{cl}\, U_k]^c$.

Remark 4.5 Obviously, if $\eta \leq \delta$ when restricted to the pairs (E, W) where E is closed and W is open, then P is closed also in $\pi(\eta, \delta)$.

We now consider the uniformizability of $(\mathrm{CL}(X), \pi(\eta, \delta))$.

We assume that (X, τ) is Tychonoff, δ is a compatible EF-proximity and η a LO-proximity. Let $\boldsymbol{\mathcal{W}}$ be the unique totally bounded uniformity compatible with δ (see [8] or [18]). It is known that the Hausdorff-Bourbaki or H-B uniformity $\boldsymbol{\mathcal{W}}_{\mathrm{H}}$, which has as a base all sets of the form $W_{\mathrm{H}} = \{(A,B) : A \subset W(B) \text{ and } B \subset W(A)\}$ induces the proximal finite topology $\sigma(\delta) = \sigma(\eta^{*^-}) \vee \sigma(\delta^+)$ (cf. [7]).

By 2.2 we know that $\sigma(\eta^{*^-}) \subset \sigma(\eta^-)$. So, in order to get $\sigma(\eta^-)$ we have to augment a typical entourage $W_{\mathrm{H}} \in \boldsymbol{\mathcal{W}}_{\mathrm{H}}$ by adding sets of the type

$$P_{\{U_{\mathrm{k}}\}} = \{(A,B) \in \mathrm{CL}(X) \times \mathrm{CL}(X) : A\ \eta\ U_{\mathrm{k}} \text{ and } B\ \eta\ U_{\mathrm{k}}\}$$

for a finite family of open sets $\{U_{\mathrm{k}}\}$.

Then, by a routine argument we have.

Theorem 4.6 *If δ is a compatible EF-proximity, η a LO-proximity on a Tychonoff space (X,τ) and $\boldsymbol{\mathcal{W}}$ the unique totally bounded uniformity compatible with δ, then the family*

$$\boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{W}}_{\mathrm{H}} \cup \{W_{\mathrm{H}} \cup P_{\{U_{\mathrm{k}}\}} : W_{\mathrm{H}} \in \boldsymbol{\mathcal{W}}_{\mathrm{H}}, \{U_{\mathrm{k}}\} \textit{ finite family of open sets } \}$$

defines a compatible uniformity on $(CL(X), \pi(\eta,\delta))$.

We next study the first and second countability as well as the metrizability of $\pi(\eta,\delta;\Delta)$, where (X,τ) is a T_3 topological space, η, δ compatible LR-proximities on X. Observe that the first and second countability as well as the metrizability of $\pi(\eta,\delta;\Delta)$ when $\eta = \eta^*$, i.e. $\pi(\eta^*,\delta;\Delta) = \sigma(\delta;\Delta)$, have been studied by Di Maio and Hola in [5].
So, we attack the case $\eta \neq \eta^*$.

Theorem 4.7 *Let (X,τ) be a T_3 topological space, η, δ compatible LR-proximities on X with $\delta \leq \eta$ (cf. Lemma 4.1). The following are equivalent:*

(a) *$(CL(X), \pi(\eta,\delta;\Delta))$ is first countable;*

(b) *$(CL(X), \sigma(\eta^-))$ and $(CL(X), \sigma(\delta^+;\Delta))$ are both first countable.*

Proof: (b)⇒(a). Since $\pi(\eta,\delta;\Delta) = \sigma(\eta^-) \vee \sigma(\delta^+;\Delta)$, this implication is clear.
(a)⇒(b). Let $(\mathrm{CL}(X), \pi(\eta,\delta;\Delta))$ be first countable and take $A \in \mathrm{CL}(X)$.
Let $\mathsf{Z} = \{\mathsf{L} =< S_j^-, V^+ >_{\eta,\delta}$, with $A \ll_\delta V, V^c \in \Delta, S_j$ open, $A\ \eta\ S_j$, $\bigcup\{S_j : j \in J\} \subset V$ and J finite$\}$ be a countable local base of A with respect to the topology $\pi(\eta,\delta)$.
We claim that the family $\mathsf{Z}^+ = \{V_\delta^+ : V_\delta^+$ occurs in some $\mathsf{L} \in \mathsf{Z}\} \cup \{\mathrm{CL}(X)\}$ forms a local base of A with respect to the topology $\sigma(\delta^+;\Delta)$. Indeed, if there is no open subset U with $A \ll_\delta U$, $U^c \in \Delta$, then $\mathrm{CL}(X)$ is the only open set in $\sigma(\delta^+;\Delta)$ containing A. If there is U with $A \ll_\delta U$, $U^c \in \Delta$, then U_δ^+ is a $\pi(\eta,\delta)$-nbhd. of A. Hence, there exists $\mathsf{L} \in \mathsf{Z}$ with $\mathsf{L} \subseteq U_\delta^+$. Note, that L cannot be of the form $\mathsf{L} =< S_j^- >_\eta = \bigcap\{(S_j)_\eta^- : j \in J\}$, otherwise by setting $F = A \cup U^c$, we have $F \in \mathsf{L}$, but $F \notin U_\delta^+$, a contradiction. Thus, L has the form $< S_j^-, V^+ >_{\eta,\delta}$. We claim that $V_\delta^+ \subset U_\delta^+$. Assume not and let $E \in V_\delta^+ \setminus U_\delta^+$. Set $F = E \cup A$,

we have $F \in \mathsf{L} \setminus U_\delta^+$, a contradiction.
We now show, that there is a countable local base of A with respect to the topology $\sigma(\eta^-)$. Without any loss of generality we may suppose that in the expression of every element from Z the family of index set J is non-empty, in fact

$$\{V_\delta^+ : A \ll_\delta V, V^c \in \Delta\} = \{V_\delta^+ : A \ll_\delta V, V^c \in \Delta\} \cap \{V_\eta^-\}$$

By Lemma 4.1 if $\mathsf{L} \in \mathsf{Z}$, then $\mathsf{L} =< S_j^-, V^+ >_{\eta,\delta}= V_\delta^+ \cap \{(S_j)_\eta^- : j \in J\}$ where $A \underline{\delta}\, V^c, V^c \in \Delta, \bigcup\{S_j : j \in J\} \subset V, A\ \eta\ S_j, S_j \in \tau$ for each $j \in J$ and J finite.
Set $\mathsf{Z}^- = \{(S_j)_\eta^- : (S_j)_\eta^-$ occurs in some $\mathsf{L} \in \mathsf{Z}\}$. We claim that the family Z^- forms a local subbase of A with respect to the topology $\sigma(\eta^-)$. Take U open with $A\ \eta\ U$. Then U_η^- is a $\pi(\eta,\delta)$-nbhd. of A. Hence, there exists $\mathsf{L} \in \mathsf{Z}$ with $\mathsf{L} \subset U_\eta^-$. $\mathsf{L} =< S_j^-, V^+ >_{\eta,\delta}= V_\delta^+ \cap \{(S_j)_\eta^- : j \in J\}$. We claim that there exists a $j \in J$ such that $(S_j)_\eta^- \subset U_\eta^-$. It suffices to show that there exists a $j \in J$ such that $S_j \subset U$. assume not and for each $j \in J$ let $x_j \in S_j \setminus U$. The set $F = \{x_j : j \in J\} \in \mathsf{L} \setminus U_\eta^-$, a contradiction.

Definitions 4.8 (see [2]) *Let (X,τ) be a T_1 topological space, δ a compatible LO-proximity, A a closed subset of X and $\Delta \subset CL(X)$ a ring.*

A family L_A of open neighbourhoods of A is a ***local proximal base at A with respect to δ*** *(or briefly a* ***δ-local proximal base at A****) if for any open subset U of X with $U^c \in \Delta$ and $A \ll_\delta U$ there exists $W \in \mathsf{L}_A$ such that $A \ll_\delta W$, $W^c \in \Delta$ and $W \subset U$.*

The ***δ-proximal character of A*** *is defined as the smallest (infinite) cardinal number of the form $|\mathsf{L}_A|$, where L_A is a δ-local proximal base at A and it is denoted by $\chi(A,\delta)$.*

The ***δ-proximal character of CL(X)*** *is defined as the supremum of all number $\chi(A,\delta)$, where $A \in CL(X)$, and it is denoted by $\chi(CL(X),\delta)$.*

By Theorems 4.7 and 2.11 we have.

Theorem 4.9 *Let (X,τ) be a T_3 topological space, η, δ compatible LR-proximities on X with $\delta \le \eta$. The following are equivalent:*

(a) *$(CL(X), \pi(\eta,\delta;\Delta))$ is first countable;*

(b) *the η-external proximal character $E\chi(CL(X),\eta)$ and the δ-proximal character $\chi(CL(X),\delta)$ are both countable.*

For the second countability we have a similar result.

Theorem 4.10 *Let (X,τ) be a T_3 topological space, η, δ compatible LR-proximities on X with $\delta \le \eta$. The following are equivalent:*

(a) $(CL(X), \pi(\eta, \delta; \Delta))$ *is second countable;*

(b) $(CL(X), \sigma(\eta^-))$ *and* $(CL(X), \sigma(\delta^+; \Delta))$ *are both second countable.*

Proof: We omit the proof that is similar to that in Theorem 4.7.

Definitions 4.11 *Let* (X, τ) *be a* T_1 *topological space,* δ *a compatible LO-proximity and* $\Delta \subset CL(X)$ *a ring.*

A family $\mathcal{B}$ *of open sets of* X *is a* ***δ-proximal base with respect to Δ*** *if whenever* $A \ll_\delta V$ *with* $V^c \in \Delta$*, there exists* $W \in \mathcal{B}$ *such that* $A \ll_\delta W$*,* $W^c \in \Delta$ *and* $W \subset V$*.*

The ***δ-proximal weight of CL(X) with respect to Δ*** *(or, briefly the* ***δ-proximal weight with respect to Δ****) is the smallest (infinite) cardinality of its* δ*-proximal base with respect to* Δ *and is denoted by* $W(CL(X); \delta, \Delta)$*.*

Theorem 4.12 *Let* (X, τ) *be a* T_3 *topological space,* η*,* δ *compatible LR-proximities on* X *with* $\delta \leq \eta$*. The following are equivalent:*

(a) $(CL(X), \pi(\eta, \delta; \Delta))$ *is second countable;*

(b) *the* η*-external proximal weight* $EW(CL(X), \eta)$ *and the* δ*-proximal weight* $W(CL(X); \delta, \Delta)$ *with respect to* Δ *are both countable.*

Theorem 4.13 *Let* (X, τ) *be a Tychonoff space,* δ *a compatible EF-proximity,* η *a compatible LR-proximity and* $\delta \leq \eta$*. The following are equivalent:*

(a) $(CL(X), \pi(\eta, \delta))$ *is metrizable;*

(b) $(CL(X), \pi(\eta, \delta))$ *is second countable and uniformizable.*

Proof: (b)⇒(a). It follows from the Urysohn Metrization Theorem.
(a)⇒(b). Observe that since CL(X) is first countable, X is separable (use Theorem 4.7 and Remark 2.10). Thus, $(\mathrm{CL}(X), \pi(\eta, \delta))$ is second countable (see (i) in Remarks 4.2).

Corollary 4.14 *Let* (X, τ) *be a Tychonoff space,* δ *a compatible EF-proximity and* η *a compatible LR-proximity on* X*. The following are equivalent:*

(a) $(CL(X), \pi(\eta, \delta))$ *is metrizable;*

(b) $(CL(X), \sigma(\eta^-))$ *and* $(CL(X), \sigma(\delta^+))$ *are second countable.*

5 Subspace Hypertopologies

It is well known that in the lower Vietoris topology $\tau(\mathrm{V}^-)$ subspace topologies behave nicely, i.e. if $A \in \mathrm{CL}(X)$ then

$$(\mathrm{CL}(A), \tau(\mathrm{V}_A^-)) = (\mathrm{CL}(X), \tau(\mathrm{V}^-)) \cap \mathrm{CL}(A)\,.$$

We now give some examples to show that in case of lower proximal topology $\sigma(\eta^-)$ analogous result is not true and the two topologies are not even comparable.

Example 5.1 Let $X = [0,2] \times [-1,1]$, $Q = [0,2] \times [0,1]$, T the closed triangle with vertices at $(0,-1)$, $(1,0)$, $(2,-1)$.

Let $A = \mathrm{T} \cup [0,2] \times \{0\}$, η the metric proximity on X and $V = (0,2) \times (0,1)$. Let $H = V_\eta^- \cap \mathrm{CL}(A)$ which is an open set in $(\mathrm{CL}(X), \sigma(\eta^-)) \cap \mathrm{CL}(A)$. Then $H = \{\mathrm{CL}([0,2] \times \{0\})\} \cup \{B \in \mathrm{CL}(A) : (1,0) \in B\}$ and is not open in $(\mathrm{CL}(A), \sigma(\eta_A^-))$, where η_A denotes the induced proximity on A. So, $(\mathrm{CL}(X), \sigma(\eta^-)) \cap \mathrm{CL}(A) \not\subset (\mathrm{CL}(A), \sigma(\eta_A^-))$.

Next examples show that even the reverse inclusion in general does not occur.

Examples 5.2

1. This is an example of a Hausdorff non-regular space X, having a closed subset A such that $(\mathrm{CL}(A), \sigma(\eta_A^-)) \not\subset (\mathrm{CL}(X), \sigma(\eta^-)) \cap \mathrm{CL}(A)$.

   The space X is the "Irrational Slope Topology" (Example 75 on Page 93 [**22**]). Let $X = \{(x,y) : y \geq 0, x, y \in Q\}$, θ a fixed irrational number and X endowed with the irrational slope topology τ generated on X by neighbourhoods of the form

   $$N_\varepsilon((x,y)) = \{(x,y)\} \cup B_\varepsilon(x + y/\theta) \cup B_\varepsilon(x - y/\theta)$$

   where $B_\varepsilon(\zeta) = \{r \in Q : |r - \zeta| < \varepsilon\}$, Q being the rationals on the x-axis. Let η_0 be the Wallman proximity. The set $A = \{(x,y) : y > 0, x, y \in Q\}$ is a closed discrete set. Let $\{(x,1) : x \in Q\} = O$ which is clopen in A. Then there is no open neighbourhood H in X such that $[(H_{\eta 0}^- \cap \mathrm{CL}(X)] \subset O^-$ and the claim.

2. This is a less pathological example. Let $X = l_2$ be the space of square summable sequences of real numbers with the usual norm, θ the origin and $\{e_n : n \in \mathbb{N}\}$ the standard basis of unit vectors. Let X be equipped with the Alexandroff proximity η_1 (i.e. $E\ \eta_1\ F$ iff $\mathrm{cl}\,E \cap \mathrm{cl}\,F \neq \emptyset$ or both $\mathrm{cl}\,E, \mathrm{cl}\,F$ are not compact). Since X is not locally compact, η_1 is not an EF-proximity.

   Let $A = \{\theta\} \cup \{e_n : n \in \mathbb{N}\}$. Then $\{\theta\}$ is clopen , $\{\theta\}\eta_1\{\theta\}$.

   Note that $F' \in \mathrm{CL}(A)$ and $F'\eta_1\{\theta\}$ iff $\theta \in F'$.

   Clearly, $A' = (A - \{\theta\}) \in \mathrm{CL}(A)$ and $A'\underline{\eta}_1\{\theta\}$. However, for each open set V in X, $V\eta_1 A'$, showing thereby that $\{\theta\}_{\eta 1}^-$ is not a member of $\sigma(\eta_1^-) \cap \mathrm{CL}(A)$.

   Hence, $(\mathrm{CL}(A), \sigma(\eta_A^-)) \not\subset (\mathrm{CL}(X), \sigma(\eta^-)) \cap \mathrm{CL}(A)$.

6 COMPARISONS

Now we compare two lower proximal topologies. If η is a LO-proximity on X, then for $A \subset X$, we use the notation $\eta(A) = \{E \subset X : E\ \eta\ A\}$ (cf. [23]).

Lemma 6.1 *Let γ and η be compatible LO-proximities on a T_1 topological space (X, τ). The following are equivalent:*

(a) $W_\gamma^- \subset V_\eta^-$ *on* $CL(X)$*;*

(b) $\gamma(W) \subset \eta(V)$*.*

Theorem 6.2 *Let γ and η be compatible LO-proximities on a T_1 topological space (X, τ). The following are equivalent:*

(a) $\sigma(\eta^-) \subset \sigma(\gamma^-)$*;*

(b) *for each $F \in CL(X)$ and $U \in \tau$ with $F\ \eta\ U$ there exists $V \in \tau$ such that $F \in \gamma(V) \subset \eta(U)$.*

Proof: (a)⇒(b). Let $F \in \mathrm{CL}(X)$ and $U \in \tau$ with $F\ \eta\ U$. Then $F \in U_\eta^-$.

By assumption there exists $\mathbf{V} =< \mathrm{V}_1^-, \mathrm{V}_2^-, \ldots, \mathrm{V}_n^- >_\gamma \in \sigma(\gamma^-)$ such that $F \in \mathbf{V} \subset U_\eta^-$. Clearly, $F \in \gamma(V_i)$ for each $i \in \{1, 2, \ldots, n\}$. We claim that there exist $i^* \in \{1, 2, \ldots, n\}$ such that $\gamma(V_{i^*}) \subset \eta(U)$. Assume not. Then for each $i \in \{1, 2, \ldots, n\}$ there exists $T_i \in \mathrm{CL}(X)$ such that $T_i\ \gamma\ V_i$ and $T_i\ \underline{\eta}\ U$. Set $T = \bigcup\{T_i : i \in \{1, 2, \ldots, n\}\}$. Then $T \in \mathrm{CL}(X)$, $T\ \gamma\ V_i$ for each $i \in \{1, 2, \ldots, n\}$ and $T\ \underline{\eta}\ U$. This show that $T \in \mathbf{V} \not\subset U_\eta^-$; a contradiction.

(b)⇒(a) It is obvious.

Definition 6.3 *Let (X, τ) be a T_1 topological space and η a LO-proximity.*

*(X, τ) is **nearly regular** iff whenever $x \in U$ with $U \in \tau$ there exists $V \in \tau$ such that $x \in cl V \subset U$.*

*η is **nearly regular (n-R** for short) iff it satisfies*

(n-R) $x\ \underline{\eta}\ A$ implies there exists $E \subset X$ such that $x\ \eta\ E$ and $E^c\ \underline{\eta}\ A$.

Remarks 6.4 (a) It is easy to verify that each LR-proximity is also an n-R proximity. The converse in general does not occur as (a) in the remark (6.6) shows.

(b) If η is a compatible (n-R)-proximity, then for each $x \in U$ and $U \in \tau$, there is a $V \in \tau$ with $x \in \mathrm{cl}\, V$ and $V\ \underline{\eta}\ U^c$.

(c) If (X, τ) is a T_3 topological space, then (X, τ) is nearly regular but the converse is not true in general as the next examples show.

Examples 6.5 (1) Let $X = \mathbb{R}$ with the topology τ consisting of the usual open sets together with sets of the form $U = (-\varepsilon, \varepsilon) \backslash B$, $\varepsilon > 0$, $B \subset \{1/n : n \in \mathbb{N}\} = A$. Then X is **nearly regular** but not regular since $V = (-\varepsilon', \varepsilon') \backslash B \subset U$ for $0 < \varepsilon' < \varepsilon$, but $\mathrm{cl}\, V \not\subset U$. Take $W = (-\varepsilon', 0)$ for $0 < \varepsilon' < \varepsilon$, then $0 \in \mathrm{cl}\, W \subset (-\varepsilon, \varepsilon) \backslash A$.

(2) The space X, of example 2.4, is Hausdorff but not nearly regular.

Remarks 6.6 (a) If in Examples 6.5 we endow X with the proximity η_0, then in the case (1) η_0 is an n-R proximity but not a LR-proximity; whereas in the case (2) η_0 is not n-R.

(b) It is easy to show that if η is a compatible n-R-proximity on X, then X is nearly-regular. Thus by the above result (a) we can state the following.

A topological space (X, τ) admits a compatible n-R proximity η if and only if the base space X is nearly regular.

(c) The above examples show that the nearly regular property is not hereditary. On the other hand it is easy to show that it is open hereditary.

Next Theorem characterizes those proximities η for which the corresponding lower η topologies $\sigma(\eta^-)$ are finer than the lower Vietoris topology $\tau(V^-) = \sigma(\eta^{*-})$.

Theorem 6.7 *Let (X, τ) be a T_1 topological space, η a compatible LO-proximity and η^* the discrete proximity. The following are equivalent:*

(a) $\sigma(\eta^{*-}) \subset \sigma(\eta^-)$;

(b) *η is an n-R proximity.*

Proof: (a)$\Rightarrow$(b) Let $x \in X$ and $U \in \tau$ with $x \in U$. The result follows from proof (a)$\Rightarrow$(b) of Theorem 6.2 when $F = \{x\}$.

(b)$\Rightarrow$(a) Let $\mathbf{U} = \mathrm{U}^-_{\eta^*}$ be a subbasic element of $\sigma(\eta^{*-})$ and $F \in \mathbf{U}$. Let $x \in U \cap F$. By assumption there exists a $V \in \tau$ such that $x \in \mathrm{cl}\, V$ and $V \underline{\eta}\, U^c$. Set $\mathbf{V} = \mathrm{V}^-_{\eta}$. Then $F \in \mathbf{V} \in \sigma(\eta^-)$ and $\mathbf{V} \subset \mathbf{U} = \mathrm{U}^-_{\eta^*}$.

Theorem 6.8 *Let (X, d) be a metric space, η the associated metric proximity and η_0 the Wallman proximity. The following are equivalent:*

(a) *X is UC;*

(b) $\eta = \eta_0$;

(c) $\sigma(\eta^-) = \sigma(\eta_0^-)$;

(d) $\sigma(\eta^-) \subseteq \sigma(\eta_0^-)$;

(e) *for each* $F \in CL(X)$ *and* $U \in \tau$ *with* $F\,\eta\,U$ *there is* $V \in \tau$ *with* $F \in \eta_0(V) \subset \eta(U)$.

Proof: (a)⇔(b), (b)⇔(c) and (c)⇒(d) are obvious.

(c)⇔(e). It follows by the above Theorem 6.2.

(d)⇒(a). It suffices to show that (not a)⇒(not d). So, suppose X is not UC. Then there exists a pair of sequences $\{x_n\}$, $\{y_n\}$ in X without cluster points, which are parallel (i.e. $\lim_{n\to\infty} d(x_n, y_n) = 0$) (see [21] or [1] on Page 54). Let $A = \{x_n : n \in \mathbb{N}\}$, for each $n \in \mathbb{N}$ set $A_n = \{x_m : m \leq n\}$ and $U = \cup\{\mathrm{S}(y_n, \varepsilon_n) : n \in \mathbb{N}\}$ where $\varepsilon_n = \frac{1}{4}d(x_n, y_n)$.

Clearly , $A\,\eta\,U$ but $A_n\,\underline{\eta}\,U$ for each $n \in \mathbb{N}$ showing that $\{A_n : n \in \mathbb{N}\}$ does not converge to A with respect to the $\sigma(\eta^-)$ topology. On the other hand $\{A_n : n \in \mathbb{N}\}$ converges to A with respect to $\sigma(\eta_0^-)$.

Now, we study comparisons between symmetric proximal topologies.

Theorem 6.9 *Let* α, γ, δ *and* η *be compatible LO-proximities on a* T_1 *topological space* (X, τ) *with* $\alpha \leq \gamma$ *and* $\delta \leq \eta$, *and* Δ *and* Λ *cobases. The following are equivalent:*

(a) $\pi(\eta, \delta; \Delta) \subset \pi(\gamma, \alpha; \Lambda)$;

(b) (1) *for each* $F \in CL(X)$ *and* $U \in \tau$ *with* $F\,\eta\,U$ *there are* $W \in \tau$ *and* $L \in \Lambda$ *such that* $F \in [\gamma(W)\backslash\alpha(L)] \subset \eta(U)$;

(2) *for each* $B \in \Delta$ *and* $W \in \tau$, $W \neq X$ *with* $B \ll_\delta W$, *there exists* $M \in \Lambda$ *such that* $M \ll_\alpha W$ *and* $\delta(B) \subset \alpha(M)$.

Proof: (a)⇒(b) We start by showing (1). So, let $F \in \mathrm{CL}(X)$ and $U \in \tau$ with $F\eta U$. Then U_η^- is a $\pi(\eta, \delta; \Delta)$ neighbourhood of F.

By assumption there is a $\pi(\gamma, \alpha; \Lambda)$ neighbourhood $\boldsymbol{W}$ of F such that $\boldsymbol{W} \subset U_\eta^-$.

$\boldsymbol{W} =< W_1^-, W_2^-, \ldots, W_n^-, W^+ >_{\gamma,\alpha}$, $W_i \in \tau$ for each $i \in \{1, 2, \ldots, n\}$, $\bigcup\{W_i : i \in \{1, 2, \ldots, n\}\} \subset W$ and $W^c \in \Lambda$.

Set $L = W^c$. By construction $F\,\underline{\alpha}\,L$ as well as $F\,\gamma\,W_i$ for each $i \in \{1, 2, \ldots, n\}$. We claim that there exists an $i^* \in \{1, 2, \ldots, n\}$ such that

$$[\gamma(W_{i^*})\backslash\alpha(L)] \subset \eta(U)\,.$$

Assume not. Then for each $i \in \{1, 2, \ldots, n\}$ there exists $T_i \in \mathrm{CL}(X)$ with $T_i \in [\gamma(W_i) \backslash \alpha(L)]$ but $T_i \notin \eta(U)$, i.e. $T_i \, \gamma \, W_i$ as well as $T_i \ll_\alpha W = L^c$ and $T_i \, \underline{\eta} \, U$.

Set $T = \bigcup\{T_i : i \in \{1, 2, \ldots, n\}\}$. By construction $T \in \mathrm{CL}(X)$,
$T \in \boldsymbol{W} =< W_1^-, W_2^-, \ldots, W_n^-, W^+ >_{\gamma,\alpha}$ and $T \notin U_\eta^-$ which contradicts $\boldsymbol{W} \subset U_\eta^-$.

Now we show (2). So, let $B \in \Delta$ and $W \in \tau$, $W \neq X$ with $B \ll_\delta W$.

Set $A = W^c$. Then $A \in \mathrm{CL}(X)$ and $A \in (B^c)_\delta^+ \in \pi(\eta, \delta; \Delta)$. Thus there exists a $\pi(\gamma, \alpha; \Lambda)$-neighbourhood

$$\boldsymbol{O} =< O_1^-, O_2^-, \ldots, O_n^-, O^+ >_{\gamma,\alpha} \text{ such that } A \in \boldsymbol{O} \subset (B^c)_\delta^+ .$$

Note that $\bigcup\{O_i : i \in \{1, 2, \ldots, n\}\} \subset O$ and $O^c \in \Lambda$. Set $M = O^c$. Since $A \in \boldsymbol{O}$, then $M \ll_\alpha A^c = W$. We claim $\delta(B) \subset \alpha(M)$. Assume not. Then there exists $F \in \mathrm{CL}(X)$ such that $F \delta B$ but $F \underline{\alpha} M$. Set $E = A \cup F \in \mathrm{CL}(X)$. Then $E \in \boldsymbol{O}$ but $E \notin (B^c)_\delta^+$, which contradicts $\boldsymbol{O} \subset (B^c)_\delta^+$.

(b)⇒(a) Let $F \in \boldsymbol{U} =< U_1^-, U_2^-, \ldots, U_n^-, U^+ >_{\eta,\delta}$ be a $\pi(\eta, \delta; \Delta)$-neighbourhood of F. Then $F \eta U_i$ for each $i \in \{1, 2, \ldots, n\}$ as well as $F \ll_\alpha U$, with $U_i, U \in \tau, \bigcup\{U_i : i \in \{1, 2, \ldots, n\}\} \subset U$ and $B = U^c \in \Delta$.

By (1) for each $i \in \{1, 2, \ldots, n\}$ there are $W_i \in \tau$ and $L_i \in \Lambda$ with $F \in [\gamma(W_i) \backslash \alpha(L_i)] \subset \eta(U_i)$.

By (2), there exists $M \in \Lambda$ with $M \ll_\alpha F^c$ and $\delta(B) \subset \alpha(M)$.

Set $N = \bigcup\{L_i : i \in \{1, 2, \ldots, n\}\} \cup M \in \Lambda$, $O = N^c$ and for each $i \in \{1, 2, \ldots, n\}$ $O_i = W_i \backslash N$.

Note that $F \underline{\alpha} N$ together with $\alpha \leq \gamma$ imply $F \underline{\gamma} N$. $W_i = O_i \cup (W_i \cap N)$. But $F \gamma W_i$ together with $F \underline{\gamma} N$ imply $F \gamma O_i$ as well as $O_i \neq \emptyset$. Moreover $F \underline{\alpha} N$ implies $F \ll_\alpha O = N^c$. Therefore

$$F \in \boldsymbol{O} =< O_1^-, O_2^-, \ldots, O_n^-, O^+ >_{\gamma,\alpha} \in \pi(\gamma, \alpha; \Lambda) .$$

We claim

$$\boldsymbol{O} =< O_1^-, O_2^-, \ldots, O_n^-, O^+ >_{\gamma,\alpha} \subset \boldsymbol{U} =< U_1^-, U_2^-, \ldots, U_n^-, U^+ >_{\eta,\delta} .$$

Assume not. Then there exists $E \in \boldsymbol{O}$, but $E \notin \boldsymbol{U}$.

Hence either (◇) $E \, \underline{\eta} \, U_i$ for some i or (◇◇) $E \, \delta \, U^c$.

If (◇) occurs, then since $E \gamma O_i$, $O_i \subset W_i$, $E \ll_\alpha O = N^c$ and $L_i \subset N$ we have $E \in [\gamma(W_i) \backslash \alpha(L_i)] \not\subset \eta(U_i)$, which contradicts (1).

If (◇◇) occurs, then since $E \delta B = U^c$, $E \ll_\alpha O = N^c$ and $M \subset N$ we have $E \in \delta(B) \backslash \alpha(M)$, i.e. $\delta(B) \not\subset \alpha(M)$, which contradicts (2).

7 APPENDIX (ADMISSIBILITY)

It is a well known fact, that if (X, τ) is a T_1 topological space, then the lower Vietoris topology $\tau(V^-)$ is an admissible topology, i.e. the map $i : (X, \tau) \to (\mathrm{CL}(X), \tau(V^-))$, defined by $i(x) = \{x\}$, is an embedding. On the other hand (as observed in Example 2.1), if the involved proximity η is different from the discrete proximity η^*, then the map $i : (X, \tau) \to (\mathrm{CL}(X), \sigma(\eta^-)$ is, in general, not even continuous. So, we start to study the behaviour of $i : (X, \tau) \to (\mathrm{CL}(X), \sigma(\eta^-)$, when $\eta \neq \eta^*$. First we give the following Lemma.

Lemma 7.1 *Let (X, τ) be a T_1 topological space, $U \in \tau$ with $clU \neq X$ and $V = (clU)^c$. If $z \in clU \cap clV$, then there exists a net (z_λ) τ-converging to z such that for all λ either*

(i) $z_\lambda \in U$ and $z_\lambda \neq z$, or

(ii) $z_\lambda \in V$ and $z_\lambda \neq z$

Proof: Let $\mathsf{N}(z)$ be the filter of open neighbourhoods of z. For each $I \in \mathsf{N}(z)$, select $w_I \in I \cap V$ and $y_I \in I \cap U$. Then, the net (w_I) is τ-converging to z and $(w_I) \subset V$ as well as the net (y_I) is τ-converging to z and $(y_I) \subset U$.
We claim that for all $I \in \mathsf{N}(z)$ either $w_I \neq z$ or $y_I \neq z$.
Assume not. Then there exist I and $J \in \mathsf{N}(z)$ such that $y_I = z$ and $w_J = z$. As a result $z \in U \cap V \subset \mathrm{cl}\, U \cap V = \emptyset$, a contradiction.

Recall, that a Hausdorff space X is called **extremally disconnected** if for every open set $U \subset X$ the closure $\mathrm{cl}\, U$ of U is open in X (see [8] on page 368).

Proposition 7.2 *Let (X, τ) be a Hausdorff space with a compatible LO-proximity η. The following are equivalent:*

(a) X is extremally disconnected;

(b) the map $i : (X, \tau) \to (CL(X), \sigma(\eta^-))$, defined by $i(x) = \{x\}$, is continuous.

Proof: (a)⇒(b). Let $x \in X$ and (x_λ) a net τ-converging to x. Let $V \subset X$ with V open and $\{x\}\eta V$. Since $\{x\}\eta V$, then $x \in \mathrm{cl}\, V$. By assumption $\mathrm{cl}\, V$ is an open subset of X and the net (x_λ) τ-converges to x. Thus, eventually $x_\lambda \in \mathrm{cl}\, V$.
(b)⇒(a). By contradiction, suppose (a) fails. Then there exists open set $U \subset X$ such that closure $\mathrm{cl}\, U$ is not open in X. Then $\mathrm{cl}\, U \neq X$. Set $V = (\mathrm{cl}\, U)^c$. V is non-empty and open in X.

We claim that $\operatorname{cl} U \cap \operatorname{cl} V \neq \emptyset$. Assume not, i.e. $\operatorname{cl} U \cap \operatorname{cl} V = \emptyset$. Then, $\operatorname{cl} U \subset (\operatorname{cl} V)^c \subset V^c = \operatorname{cl} U$. Thus, $\operatorname{cl} U = (\operatorname{cl} V)^c$, i.e. $\operatorname{cl} U$ is open, a contradiction. Let $z \in \operatorname{cl} U \cap \operatorname{cl} V$. From Lemma 7.1, there exists a net (z_λ) τ-converging to z such that for all λ either (i) $z_\lambda \in U$ and $z_\lambda \neq z$, or (ii) $z_\lambda \in V$ and $z_\lambda \neq z$. In both cases, there exists an open subset W such that $z \in \operatorname{cl}, W$ and $z_\lambda \notin \operatorname{cl}, W$ for all λ (in fact, if (i) holds, then set $W = V$, otherwise set $W = U$). Thus the net (z_λ) τ-converging to z and the open subset W witness that the map $i : (X, \tau) \to (\mathrm{CL}(X), \sigma(\eta^-))$ fails to be continuous.

Now, we investigate when the map $i : (X, \tau) \to (\mathrm{CL}(X), \sigma(\eta^-))$ is open.

Proposition 7.3 *Let (X, τ) be a T_1-topological space with a compatible LO-proximity η. The following are equivalent:*

(a) (X, τ) is nearly regular (cf. Definition 6.3);

(b) the map $i : (X, \tau) \to (CL(X), \sigma(\eta^-))$ is open.

Proof: Left to the reader.

Note that if (X, τ) is a T_1-topological space with a compatible LO-proximity δ, then the map $i : (X, \tau) \to (\mathrm{CL}(X), \sigma(\delta^+; \Delta))$ is always continuous with respect the upper proximal Δ topology $\sigma(\delta^+; \Delta)$. So, we have:

Proposition 7.4 *Let (X, τ) be a T_1-topological space, δ a compatible LO-proximity and $\Delta \subset CL(X)$ a cobase. The following are equivalent:*

(a) the map $i : (X, \tau) \to (CL(X), \sigma(\delta^+; \Delta))$, defined by $i(x) = \{x\}$, is an embedding;

(b) the map $i : (X, \tau) \to (CL(X), \sigma(\delta^+; \Delta))$, defined by $i(x) = \{x\}$ is an open map;

(c) whenever $U \in \tau$ and $x \in U$, there exists a $B \in \Delta$ such that $x \in B^c \subset U$.

Finally, we have the following result concerning with the admissibility of the entire symmetric proximal Δ topology $\pi(\eta, \delta; \Delta)$. Obviously, we investigate just the significant case $\eta \neq \eta^*$ (the standard proximal Δ topology $\sigma(\delta; \Delta) = \pi(\eta^*, \delta; \Delta)$ is always admissible).

Proposition 7.5 *Let (X, τ) be a Hausdorff space, η, δ compatible LO-proximities and $\Delta \subset CL(X)$ a cobase. The following are equivalent:*

(a) the map $i : (X, \tau) \to (CL(X), \pi(\eta, \delta; \Delta))$ *is an embedding;*

(b) $i : (X, \tau) \to (CL(X), \sigma(\eta^-))$ *is continuous and either* $i : (X, \tau) \to (CL(X), \sigma(\eta^-))$ *or* $i : (X, \tau) \to (CL(X), \sigma(\delta^+; \Delta))$ *is open.*

Theorem 7.6 *Let* (X, τ) *be a Hausdorff space,* η, δ *compatible LO-proximities and* $\Delta \subset CL(X)$ *a cobase. The following are equivalent:*

(a) the map $i : (X, \tau) \to (CL(X), \pi(\eta, \delta; \Delta))$ *is an embedding;*

(b) X *is extremally disconnected and either* X *is also nearly regular, or whenever* $U \in \tau$ *and* $x \in U$*, there exists a* $B \in \Delta$ *such that* $x \in B^c \subset U$*.*

We raise the following question.

Question 7.1 There exists an extremally disconnected space X which turns out to be nearly regular, but not regular?

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