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Coincidence Points For Multivalued Mappings

ABSTRACT. In this paper we show some coincidence theorems for contractive type multivalued mappings in compact metric spaces, which extend properly the results of Kubiak and Kubiaczyk.

KEY WORDS. Coincidence point, multivalued mappings, compact metric space.

1 Introduction and preliminaries

Let (X, d) be a metric space. For any nonempty subsets A, B of X we define $D(A, B) = \inf\{d(a, b) : a \in A \text{ and } b \in B\}, \delta(A, B) = \sup\{d(a, b) : a \in A \text{ and } b \in B\}$ and $H(A, B) = \max\{\sup[D(a, B) : a \in A], \sup[D(A, b) : b \in B]\}$. Let $CL(X) = \{A : A$ is a nonempty closed subset of $X\}$ and $CB(X) = \{A : A$ is a nonempty bounded closed subset of $X\}$. It is well known that $(CB(X), H)$ is a metric space. Obviously $CB(X) = CL(X)$ if (X, d) is a compact metric space. Let S be a mapping of X into $CL(X)$, f a selfmapping of X. A point $x \in X$ is called a coincidence point of f and S if $fx \in Sx$.

Kubiak [1] and Kubiaczyk [2] proved some fixed point theorems for contractive type multivalued mappings in compact metric spaces. The purpose of this paper is to extend their results to a more general case.

2 Coincidence theorems

Theorem 2.1 *Let (X,d) be a compact metric space and let S and T be mappings of X into $CL(X)$. Suppose that f and g are selfmappings of X satisfying*

$$\begin{aligned}\delta(Sx,Ty) < \max\Big\{&d(fx,gy), H(fx,Sx), H(gy,Ty),\\ &\frac{1}{2}[D(fx,Ty)+D(gy,Sx)],\\ &H(fx,Sx)H(gy,Ty)/d(fx,gy),\\ &D(fx,Ty)D(gy,Sx)/d(fx,gy)\Big\}\end{aligned}\tag{2.1}$$

for all $x,y\in X$ with $fx\neq gy$. Let $SX\subseteq gX$ and $TX\subseteq fX$. If either f and S or g and T are continuous, then either f and S or g and T have a coincidence point u with $Su=\{fu\}$ or $Tu=\{gu\}$.

Proof: We assume without loss of generality that f and S are continuous. It follows that $H(fx,Sx)$ is a continuous function on X. By the compactness of X, there exists a point $u\in X$ such that $H(fu,Su)=\inf\{H(fx,Sx): x\in X\}$. It is easy to check that there is a point $y\in Su$ with $d(fu,y)=H(fu,Su)$. Since $SX\subseteq gX$, then there exists a point $v\in X$ with $y=gv$. Consequently $d(fu,gv)=H(fu,Su)$ for some $gv\in Su$. Similarly, there are two points $w,x\in X$ such that $d(gv,fw)=H(gv,Tv)$, $d(fw,gx)=H(fw,Sw)$, where $fw\in Tv, gx\in Sw$. We now assert that $H(fu,Su)H(gv,Tv)=0$. Otherwise $H(fu,Su)H(gv,Tv)>0$. Using (2.1) we have

$$\begin{aligned}\delta(Su,Tv) &< \max\Big\{d(fu,gv), H(fu,Su), H(gv,Tv),\\ &\qquad \frac{1}{2}[D(fu,Tv)+D(gv,Su)],\\ &\qquad H(fu,Su)H(gv,Tv)/d(fu,gv),\\ &\qquad D(fu,Tv)D(gv,Su)/d(fu,gv)\Big\}\\ &= \max\Big\{H(fu,Su), H(gv,Tv),\\ &\qquad \frac{1}{2}[d(fu,gv)+H(gv,Tv)]\Big\}\\ &= \max\Big\{H(fu,Su), H(gv,Tv)\Big\}\end{aligned}$$

which implies

$$H(gv,Tv)\le\delta(Su,Tv)<\max\big\{H(fu,Su),H(gv,Tv)\big\}=H(fu,Su)\,. \tag{2.2}$$

Similarly we can show

$$H(fw,Sw)\le\delta(Sw,Tv)<\max\big\{H(gv,Tv),H(fw,Sw)\big\}=H(gv,Tv)\,. \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$H(fw, Sw) < H(gv, Tv) < H(fu, Su) = \inf\Big\{H(fx, Sx) : x \in X\Big\}$$

which is a contradiction and hence $H(fu, Su)H(gv, Tv) = 0$, which implies that $Su = \{fu\}$ or $Tv = \{gv\}$. This completes the proof.

If f and g are the identity mapping on X, Theorem 2.1 reduces to the following.

Corollary 2.2 *Let (X, d) be a compact metric space and let S and T be mappings of X into $CL(X)$ satisfying*

$$\begin{aligned}\delta(Sx, Ty) < \max\Big\{&d(x, y), H(x, Sx), H(y, Ty),\\ &\frac{1}{2}[D(x, Ty) + D(y, Sx)],\\ &H(x, Sx)H(y, Ty)/d(x, y),\\ &D(x, Ty)D(y, Sx)/d(x, y)\Big\} \qquad (2.4)\end{aligned}$$

for all $x, y \in X$ with $x \neq y$. If S or T is continuous, then S or T has a fixed point u with $Su = \{u\}$ or $Tu = \{u\}$.

Remark 2.1 Theorem 4 in [1] and Theorem 4 in [2] are special cases of Corollary 2.2. The following example demonstrates that Corollary 2.2 extends properly Theorem 4 in [1] and Theorem 4 in [2].

Example 2.1 Let $X = \{1, 3, 6, 10\}$, d the ordinary distance, and define S and T by $S1 = \{3, 6\}$, $S3 = \{3, 6, 10\}$, $S6 = S10 = T1 = T6 = T10 = \{6\}$ and $T3 = \{10\}$. Then (X, d) is a compact metric space, S and T are continuous mappings of X into $CL(X)$. It is easy to show that S and T satisfy (2.4). But Theorem 4 in [1] and Theorem 4 in [2] are not applicable since

$$\delta(Sx, Ty) < \max\Big\{d(x, y), H(x, Sx), H(y, Ty), \frac{1}{2}[D(x, Ty) + D(y, Sx)]\Big\}$$

and

$$\begin{aligned}\delta(Sx, Ty) \;<\; &a(x, y)d(x, y) + b(x, y)[H(x, Sx) + H(y, Ty)]\\ &+c(x, y)[D(x, Ty) + D(y, Tx)]\end{aligned}$$

are not satisfied for $x = 1$ and $y = 3$, where a, b and c are functions of $X \times X$ into $[0, \infty)$ with $\sup\{a(x, y) + 2b(x, y) + 2c(x, y) : (x, y) \in X \times X\} \leq 1$.

Theorem 2.3 *Let (X,d) be a compact metric space and let S and T be mappings of X into $CL(X)$. Assume that f and g are selfmappings of X satisfying*

$$\begin{aligned} H(Sx,Ty) < \max\Big\{ & d(fx,gy), D(fx,Sx), D(gy,Ty), \\ & \frac{1}{2}[D(fx,Ty)+D(gy,Sx)], \\ & D(fx,Sx)D(gy,Ty)/d(fx,gy), \\ & D(fx,Ty)D(gy,Sx)/d(fx,gy)\Big\} \end{aligned} \tag{2.5}$$

for all $x, y \in X$ with $fx \neq gy$. Let $SX \subseteq gX$ and $TX \subseteq fX$. If either f and S or g and T are continuous, then either f and S or g and T have a coincidence point.

Proof: We may assume that f and S are continuous on X. Then $D(fx,Sx)$ is continuous and attains its minimum at some $u \in X$. As in the proof of Theorem 2.1, there exist $v, w,\ x \in X$ such that $d(fu,gv) = D(fu,Su)$, $d(gv,fw) = D(gv,Tv)$ and $d(fw,gx) = D(fw,Sw)$, where $gv \in Su$, $fw \in Tv$, $gx \in Sw$. Assume that $D(fu,Su)D(gv,Tv) > 0$. The same argument as that of the proof of Theorem 2.1 shows that $D(fw,Sw) < D(gv,Tv) < D(fu,Su)$, which contradicts the miniality of $D(fu,Su)$. Hence $D(fu,Su)D(gv,Tv) = 0$. That is, $fu \in Su$ or $gv \in Tv$. This completes the proof.

As an immediate consequence of Theorem 2.3 we have the following.

Corollary 2.4 *Let (X,d) be a compact metric space and let S and T be mappings of X into $CL(X)$. Suppose that f and g are selfmappings of X satisfying*

$$\begin{aligned} H(Sx,Ty) < \max\Big\{ & d(fx,gy), D(fx,Sx), D(gy,Ty), \\ & \frac{1}{2}[D(fx,Ty)+D(gy,Sx)]\Big\} \end{aligned} \tag{2.6}$$

for all $x, y \in X$ with $fx \neq gy$. Let $SX \subseteq gX$ and $TX \subseteq fX$. If either f and S or g and T are continuous, then either f and S or g and T have a coincidence point.

Remark 2.2 If f and g are the identity mapping on X, Corollary 2.4 reduces to Theorem 2 in [1] and includes Theorem 3 in [2]. The following example verifies that Corollary 2.4 does indeed generalize Theorem 2 in [1] and Theorem 3 in [2], that not both f, S and g, T of Corollary 2.4 need have a coincidence point and that the coincidence point may not be unique.

Example 2.2 Let $X = \{1,3,6\}$ with the usual metric, and define S, T, f and g by $S1 = S3 = T6 = \{1,3\}$, $S6 = T1 = \{3\}$, $T3 = \{1\}$, $f1 = f6 = 3$, $f3 = 1$, $g1 = g3 = g6 = 6$. It is easy to see that the hypothesis of Corollary 2.4 is satisfied. Clearly f and S have three coincidence points while g and T have none. However, Theorem 2 in [1] and Theorem 3 in [2] are not applicable since

$$\begin{aligned} H(Sx,Ty) &< \max\Big\{d(x,y), D(x,Sx), D(y,Ty), \\ &\qquad \frac{1}{2}[D(x,Ty)+D(y,Sx)]\Big\} \end{aligned}$$

and

$$\begin{aligned} H(Sx,Ty) &< a(x,y)d(x,y) + b(x,y)[D(x,Sx)+D(y,Ty)] \\ &\quad + c(x,y)[D(x,Ty)+D(y,Sx)] \end{aligned}$$

are not satisfied for $x = 1$ and $y = 3$, where a, b and c are functions of $X \times X$ into $[0, \infty)$ with $\sup\{a(x,y) + 2b(x,y) + 2c(x,y) : (x,y) \in X \times X\} \leq 1$.

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