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A Recolouring Problem on Undirected Graphs

ABSTRACT. We consider an algorithm on a graph $G = (V, E)$ with a 2-colouring of V, that is motivated from the computer-aided text-recognition. Every vertex changes simultaneously its colour if more than a certain proportion c of its neighbours have the other colour. It is shown, that by iterating this algorithm the colouring becomes either constant or 2-periodic. For $c = \dfrac{1}{2}$ the presented theorem is a special case of a known result [1], but here developed independently with another motivation and a new proof.

There are algorithms for the computer-aided text recognition, that search in a given pixel pattern for characteristic properties of characters. But often this search becomes very difficult because of certain dirt effect like single white pixels in a large black area. So it seems plausible that we can increase the efficiency of such algorithms by first weeding out such effects. For example, we can change simultaneously the colour of every pixel if in a properly defined neighbourhood the proportion of pixels with the opposite colour exceeds a certain number c with $0 < c < 1$. It is easy to see that iterating this recolouring finally runs into a period, so the question for the length of such a period naturally arises. A similar question in the more general situation of an arbitrary finite number of colours is motivated in [1] by a model of a society, where the pixels correspond to the members of the society, whose opinions are influenced by their neighbours. It is shown there that the period is 1 or 2 for this model.

To reformulate our problem in an explicit graph theoretical context let $G = (V, E)$ be a graph with vertex set $V = \{v_1, v_2, \dots, v_r\}$. We colour the vertices with two colours, say black and white. Now we can change the colouring by the following algorithm: For i from 1 to r, if v_i is black and has more white than black neighbours, it changes its colour to white, and if v_i is white and has more black than white neighbours, it changes its colour to black. It is a well-known problem in mathematical contests, to show that one always gets a constant colouring by iterating this algorithm. Here every vertex v changes its colour if there are more than $\frac{1}{2}\deg(v)$ vertices of the other colour in its neighbourhood. But what happens, if we vary the fraction of opposite-coloured neighbours that is necessary for changing the colour?

Thus the new condition for colour-changing is that there are more than $c\deg(v)$ vertices of the other colour in the neighbourhood for some c, $0 < c < 1$. Furthermore we want to recolour the vertices not one after another, but all simultaneously. The standard-solution of the mentioned contest problem gives a hint, how to tackle this problem, namely by the search for an integer-valued, bounded and monotonous function of the number of steps.

We may assume that G has no isolated vertices, because if there were any, they would never change their colour. Every 2-colouring of the vertices is given by a function $V \to \{-1, 1\}$. Then the series of recolouring steps corresponds to a series of functions $(f_n : V \to \{-1,1\})_{n\in\mathbb{N}}$. To describe the recolouring steps, we define the series of functions:

$$g_n : E \to \{-1, 1\}, \quad \{v, w\} \mapsto f_n(v)f_n(w)$$

So the number of neighbours of v, that are coloured in a different way than v before the n-th recolouring step is $\frac{1}{2}\left(\deg(v) - \sum_{e\in E: v\in e} g_n(e)\right)$, and v changes its colour iff $\frac{1}{2\deg(v)}\left(\deg(v) - \sum_{e\in E: v\in e} g_n(e)\right) > c$. With

$$h_n : V \to \mathbb{Z}, \quad v \mapsto \sum_{e\in E: v\in e} g_n(e) - (1-2c)\deg(v)$$

this is equivalent to $h_n(v) < 0$. Next we introduce another global parameter, for which we will see, that it grows monotonously with n:

$$s_n = \sum_{v\in V} |h_n(v)|$$

Lemma *For every $n \in \mathbb{N}$ we define $V_n^- := \{v \in V \mid h_n(v) < 0\}$, the set of vertices, that change their colours in the n-th step. Furthermore we set $V_n^+ := V \setminus V_n^-$. Then, for all $n \in \mathbb{N}$,*

$$s_{n+1} = s_n + 2\left(\sum_{v\in V_n^-\cap V_{n+1}^+} |h_{n+1}(v)| + \sum_{v\in V_n^+\cap V_{n+1}^-} |h_{n+1}(v)|\right).$$

Proof: We choose $n \in \mathbb{N}$. For $i \in \{1, 2, \dots, r\}$ let k_i^- and k_i^+ denote the numbers of edges e, that are incident with v_i, and fulfill $g_n(e) = 1 = -g_{n+1}(e)$ and $g_n(e) = -1 = -g_{n+1}(e)$, respectively.
Obviously, $h_{n+1}(v_i) = h_n(v_i) + 2\left(k_i^+ - k_i^-\right)$ for all i, and hence

$$|h_n(v_i)| = \begin{cases} |h_{n+1}(v_i)| - 2\left(k_i^+ - k_i^-\right) & \text{for } v_i \in V_n^+ \cap V_{n+1}^+ \\ |h_{n+1}(v_i)| + 2\left(k_i^+ - k_i^-\right) & \text{for } v_i \in V_n^- \cap V_{n+1}^- \\ -|h_{n+1}(v_i)| + 2\left(k_i^+ - k_i^-\right) & \text{for } v_i \in V_n^- \cap V_{n+1}^+ \\ -|h_{n+1}(v_i)| - 2\left(k_i^+ - k_i^-\right) & \text{for } v_i \in V_n^+ \cap V_{n+1}^-. \end{cases}$$

Summing up these equations yields

$$\begin{aligned} s_n =& s_{n+1} - 2\left(\sum_{v\in V_n^-\cap V_{n+1}^+} |h_{n+1}(v)| + \sum_{v\in V_n^+\cap V_{n+1}^-} |h_{n+1}(v)|\right) \\ &+ 2\left(\sum_{i:v_i\in V_n^-}\left(k_i^+ - k_i^-\right) - \sum_{i:v_i\in V_n^+}\left(k_i^+ - k_i^-\right)\right). \end{aligned} \tag{1}$$

For every $v \in V_n^+$ and every $e = \{v, w\} \in E$ we have $g_n(e) = 1 = -g_{n+1}(e)$ iff $w \in V_n^-$. Therefore $\sum\limits_{i:v_i\in V_n^+} k_i^- = \sum\limits_{i:v_i\in V_n^-} k_i^-$ and analogously $\sum\limits_{i:v_i\in V_n^+} k_i^+ = \sum\limits_{i:v_i\in V_n^-} k_i^+$. So the last two sums in (1) cancel each other and the claim follows. □

Now we have an upper bound for the series (s_n) by

$$s_n \le \sum_{v\in V} \deg(v) + \sum_{v\in V} |1-2c| \deg(v) = 4\alpha|E|$$

with $\alpha = 1-c$ if $c \le \frac{1}{2}$ and $\alpha = c$ if $c > \frac{1}{2}$. For every $v \in V_{n+1}^-$ we have

$$\begin{array}{ll} |h_{n+1}(v)| \ge 1 & \text{if } (1-2c)\deg(v) \in \mathbb{Z}, \\ |h_{n+1}(v)| \ge |1-2c|\deg(v) - \lfloor |1-2c|\deg(v) \rfloor & \text{if } (1-2c)\deg(v) \notin \mathbb{Z}. \end{array}$$

So there is a constant $\beta > 0$ with $|h_{n+1}(v)| \ge \beta$ for all $v \in V_{n+1}^-$ and for all $n \in \mathbb{N}$. It follows

$$\sum_{n\in\mathbb{N}} \left|V_n^+ \cap V_{n+1}^-\right| \le \frac{4\alpha|E| - s_0}{2\beta}. \tag{2}$$

Now we consider any $v \in V$. Let $n_0, n_1, n_2, \ldots$ denote the numbers with $v \in V_{n_i}^- \cap V_{n_i+1}^+$ in increasing order. Then either $v \in V_0^-$ or $v \in V_m^+ \cap V_{m+1}^-$ for an $m < n_0$, and, for all $k \in \mathbb{N}$, $v \in V_m^+ \cap V_{m+1}^-$ for an m with $n_k < m < n_{k+1}$. So with the characteristic functions $\chi_A : V \to \{0,1\}$, $\chi_A(v) = 1$ for $v \in A$ and $\chi_A(v) = 0$ for $v \notin A$ we have

$$\sum_{n\in\mathbb{N}} \chi_{V_n^-\cap V_{n+1}^+}(v) \le \chi_{V_0^-}(v) + \sum_{n\in\mathbb{N}} \chi_{V_n^+\cap V_{n+1}^-}(v).$$

Summing up over V yields

$$\sum_{v\in V}\sum_{n\in\mathbb{N}} \chi_{V_n^-\cap V_{n+1}^+}(v) \le \sum_{v\in V} \chi_{V_0^-}(v) + \sum_{v\in V}\sum_{n\in\mathbb{N}} \chi_{V_n^+\cap V_{n+1}^-}(v),$$

$$\sum_{n\in\mathbb{N}}\sum_{v\in V} \chi_{V_n^-\cap V_{n+1}^+}(v) \le \sum_{v\in V} \chi_{V_0^-}(v) + \sum_{n\in\mathbb{N}}\sum_{v\in V} \chi_{V_n^+\cap V_{n+1}^-}(v).$$

$$\sum_{n\in\mathbb{N}} \left|V_n^- \cap V_{n+1}^+\right| \le \left|V_0^-\right| + \sum_{n\in\mathbb{N}} \left|V_n^+ \cap V_{n+1}^-\right| \le |V_0^-| + \frac{4\alpha|E| - s_0}{2\beta},$$

and hence with (2)

$$\sum_{n\in\mathbb{N}} \left(\left|V_n^- \cap V_{n+1}^+\right| + \left|V_n^+ \cap V_{n+1}^-\right|\right) \leq |V_0^-| + \frac{4\alpha|E| - s_0}{\beta}.$$

Consequently there is an $n_0 \leq |V_0^-| + \frac{4\alpha|E|-s_0}{\beta}$ with $(V_{n_0}^+ \cap V_{n_0+1}^-) \cup (V_{n_0}^- \cap V_{n_0+1}^+) = \emptyset$. From this follows:

Theorem *Let $G = (V, E)$ be a graph with a 2-colouring of V and $0 < c < 1$. In every time step every vertex v changes its colour iff more than $c\deg(v)$ vertices in the neighbourhood of v are coloured in a different way than v. Finally this algorithm runs into a period of length 1 or 2.*

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References

[1] **Poljak, S.**, and **Sůra, M. :** *On periodical behaviour in societies with symmetric influences.* Combinatorica **3** (1) 119-121 (1983)

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