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Existence of Solutions for an Elliptic Equation Involving a Schrödinger Operator with Weight in all of the Space

ABSTRACT. In this paper, we obtain some results about the existence of solutions for the following elliptic semilinear equation $(-\Delta+q)u = \lambda mu + f(x,u)$ in $\mathbb{R}^N$ where q is a positive potential satisfying $\lim_{|x|\to+\infty} q(x) = +\infty$ and m is a bounded positive weight.

1 Introduction

In this paper, we study the existence of solutions for the elliptic semilinear equation:

$$(-\Delta + q)u = \lambda mu + f(x,u) \text{ in } \mathbb{R}^N \tag{1}$$

where the following hypotheses are satisfied:

(h1) $q \in L^2_{loc}(\mathbb{R}^N)$ such that $\lim_{|x|\to+\infty} q(x) = +\infty$ and $q \geq \text{const} > 0$.

(h2) $m \in L^\infty(\mathbb{R}^N)$ such that $\exists m_1 \in \mathbb{R}^{*+}$, $\exists m_2 \in \mathbb{R}^{*+}$, $\forall x \in \mathbb{R}^N$, $0 < m_1 \leq m(x) \leq m_2$.

We will specify later the hypothesis on f. We denote by λ a real parameter.
The variational space is denoted by $V_q(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N),\ (-\Delta+q)u \in L^2(\mathbb{R}^N)\}$ which is the completed of $\mathcal{D}(\mathbb{R}^N)$ for the norm $\|u\|_q = \sqrt{\int_{\mathbb{R}^N} |\nabla u|^2 + qu^2}$.
Recall (see [1] for example) that the embedding of $V_q(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$ is compact.
We denote by $\|u\|_m = \sqrt{\int_{\mathbb{R}^N} mu^2}$ for all $u \in L^2(\mathbb{R}^N)$. According to the hypothesis $(h2)$, $\|.\|_m$ is a norm in $L^2(\mathbb{R}^N)$ equivalent to the usual norm. We denote by M the operator of multiplication by m in $L^2(\mathbb{R}^N)$. The operator $(-\Delta+q)^{-1}M : (L^2(\mathbb{R}^N), \|.\|_m) \to (L^2(\mathbb{R}^N), \|.\|_m)$ is positive self-adjoint and compact. So its spectrum is discrete and consists of a positive sequence $\mu_1 \geq \mu_2 \geq ...\mu_n \to 0$ when $n \to +\infty$. We denote by $\lambda_1 = \frac{1}{\mu_1}$ and u_1 the corresponding eigenfunction which satisfy $(-\Delta+q)u_1 = \lambda_1 mu_1$ in $\mathbb{R}^N$ and $\|u_1\|_m = 1$. (We know that

λ_1 is simple and $u_1 > 0$ (see [2, Th2.2]).) By the Courant-Fischer formulas, λ_1 is given by:

$$\lambda_1 = \inf\{\frac{\int_{\mathbb{R}^N} |\nabla\phi|^2 + q\phi^2}{\int_{\mathbb{R}^N} m\phi^2},\ \phi \in \mathcal{D}(\mathbb{R}^N)\}.$$

We recall now some results already obtained for the existence of solutions in the linear cases or semilinear cases.
Using the Lax-Milgram theorem and the above characterization of λ_1, we obtain the following result:

Theorem 1.1 (see [3],[4]) *We consider the linear case (i.e. $f(x,u) = f(x)$.) Assume that the hypotheses $(h1)$ and $(h2)$ are satisfied and that $f \in L^2(\mathbb{R}^N)$. If $\lambda < \lambda_1$, then the equation (1) has a unique solution $u_\lambda \in V_q(\mathbb{R}^N)$. Moreover, the Maximum Principle is satisfied i.e.: if $f \geq 0$ and $\lambda < \lambda_1$ then $u_\lambda \geq 0$.*
If $\lambda = \lambda_1$ (which is the case of the Fredholm Alternative), then the equation (1) admits a solution iff $\int_{\mathbb{R}^N} fu_1 = 0$.

Using a method of sub- and supersolutions and a Schauder Fixed Point Theorem (see [3]) or an approximation method (see [4]), we get the following results in the semilinear case:

Theorem 1.2

1. *(see [3]). Assume that the hypotheses $(h1)$ and $(h2)$ are satisfied. Assume also that f is Lipschitz in u uniformly in x and that:*
   $\exists\theta \in L^2(\mathbb{R}^N), \theta > 0, \forall u \geq 0,\ 0 \leq f(x,u) \leq su + \theta.$
   If $\lambda < \lambda_1$, the equation (1) has at least a positive solution.

2. *(see [4]). Assume that the hypothesis $(h1)$ is satisfied, $N \geq 3$ and $0 \leq m \in L^{\frac{N}{2}}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$. Assume also that f is Lipschitz in u uniformly in x and that: $\exists\theta \in L^2(\mathbb{R}^N)$, $\forall u \in L^2(\mathbb{R}^N),\ |f(x,u)| \leq \theta$.*
   If $\lambda < \lambda_1$, then the equation (1) has at least a solution.

Finally, for the linear case (i.e. $f(x,u) = f(x)$), assuming $N = 2$, m a radial weight and q a radial potential with some strong properties of growth at infinity (not recalled here) (see [5]), we obtain the following result for the Antimaximum Principle:

Theorem 1.3 (see [5]) *Assume that the hypotheses $(h1)$ and $(h2)$ are satisfied.*
We denote by $X^{1,2} = \{f \in L^2_{loc}(\mathbb{R}^2), \frac{\partial f}{\partial\theta}(r,.) \in L^2(-\pi,\pi)$ for all $r > 0$, and $\exists C \geq 0$, $\|f(r,\theta)\| + (\frac{1}{2\pi}\int_{-\pi}^{\pi} |\frac{\partial f}{\partial\theta}(r,\theta)|^2 d\theta)^{\frac{1}{2}} \leq Cu_1(r)$ for all $r \geq 0$ and $\theta \in]-\pi,\pi].\}$
Assume that $f \geq 0$ in $\mathbb{R}^2$, $f > 0$ in a subset with a non zero Lebesgue measure and $f \in X^{1,2}$.
Let u be a solution of the equation (1).
Then $\exists\delta(f) > 0, \forall\lambda \in (\lambda_1, \lambda_1 + \delta(f)), \exists c(\lambda, f) > 0,\ u \leq -c(\lambda, f)u_1$.

In this paper, we sudy the existence of solutions for the equation (1) in the case $\lambda > \lambda_1$, λ near λ_1.

For the linear case (i.e. $f(x,u) = f(x)$), if $\lambda \in (\lambda_1, \lambda_2)$, $\lambda_2 = \frac{1}{\mu_2}$ where μ_2 is the second eigenvalue of $(-\Delta + q)^{-1}M$, then there are obviously existence and uniqueness of a solution for the equation (1).

In the second section, following a bifurcation method developped in [6], we get the following result:

Theorem 1.4 *Assume that the hypotheses* $(h1)$ *and* $(h2)$ *are satisfied. Assume also that* $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ *(defined by* $f(x,y)$*) satisfies the following hypothesis* $(h3)$*:*

i) $f(x,0) = 0$.

ii) f *is Frechet differentiable with respect to the second variable* y *and its derivative* $f'_y(x,.)$ *is continuous and bounded, uniformly in* x.

iii) $f'_y(x,0) = 0$.

Then there exists for λ *sufficiently near* λ_1 *a nontrivial solution for the equation* (1).

Finally, in the third section, following a method developped in [7] for the p-Laplacian in a bounded domain of $\mathbb{R}^N$, we get results for the case where $f(x,u) = f(x)|u(x)|^{\gamma-2}u(x)$. Before stating the results, we need some notations. We define for $C \in \mathbb{R}^{*+}$ the set $X_{q,C} = \{u \in V_q(\mathbb{R}^N),\ u_1 \le u \le C \quad a.e.\}$.

Let $F(u) := \int_{\mathbb{R}^N} f|u|^\gamma$ for all $u \in V_q(\mathbb{R}^N)$.

Let $\lambda^* = \sup_{u \in V_q(\mathbb{R}^N), u \ge 0}\{\inf_{\phi \in V_q(\mathbb{R}^N)}\{\frac{\int_{\mathbb{R}^N} \nabla u.\nabla\phi + qu\phi}{\int_{\mathbb{R}^N} mu\phi},\ F'(u)(\phi) \ge 0,\ \phi \ge 0\}\}$ and

$\lambda^{**} = \sup_{u \in X_{q,C}}\{\inf_{\phi \in V_q(\mathbb{R}^N)}\{\frac{\int_{\mathbb{R}^N} \nabla u.\nabla\phi + qu\phi}{\int_{\mathbb{R}^N} mu\phi},\ F'(u)(\phi) \ge 0,\ \phi \ge 0\}\}$.

(Note that $\lambda^{**} \le \lambda^*$.)

We consider also hypotheses of the following forms:

(h4) $\lambda_1 < \lambda^{**} \le \lambda^* < +\infty$.

(h5) $f \in L^\infty(\mathbb{R}^N)$.

(h6) The sets $\Omega^+ = \{x \in \mathbb{R}^N, f(x) > 0\}$ and $\Omega^- = \{x \in \mathbb{R}^N, f(x) < 0\}$ have non zero measures.

(h7) $f \ge -\frac{\epsilon u_1 m}{l^{\gamma-2}C^{\gamma-1}}$.

Theorem 1.5 *Assume that the hypotheses* $(h1)$ *and* $(h2)$ *are satisfied,* $N = 3, 4$ *so that* $\gamma = 2^* = \frac{2N}{N-2} \in \mathbb{N}^*$.

1. *If in addition the hypotheses* $(h4)$ *and* $(h5)$ *are satisfied, and if* $\lambda > \lambda^*$, *then the equation* (1) *has no positive solution.*

2. *Assume additionally that the hypotheses* $(h4) - (h7)$ *are satisfied, where the numbers* $l \geq 1$, $\epsilon > 0$, ϵ *involved in (h7) are small enough such that* $\lambda_1 \leq \epsilon\gamma l^{\gamma-2}$ *and* $\epsilon < \frac{\lambda_1}{\gamma}$. *If there holds* $\lambda_1 + \epsilon l^{\gamma-2} < \lambda < \lambda^{**}$ *with the same numbers* ϵ, l *as in (h7), then the equation* (1) *has at least a positive solution.*

2 A bifurcation result

In this section, we follow a method developped in [6].
We obtain some results of the existence of solutions for the semilinear equation

$$(-\Delta + q)u = \lambda m u + f(x, u) \text{ in } \mathbb{R}^N \tag{1}$$

by considering bifurcating solutions from the zero solution. We suppose that the hypotheses $(h1), (h2), (h3)$ are satisfied in all this section. We denote by $< .,. >_q$ the inner product in $V_q(\mathbb{R}^N)$. We define the operator $T : \mathbb{R} \times V_q(\mathbb{R}^N) \to V_q(\mathbb{R}^N)$ by: $\forall \phi \in V_q(\mathbb{R}^N)$

$$< T(\lambda, u), \phi >_q = \int_{\mathbb{R}^N} \nabla u . \nabla \phi + q u \phi - \lambda \int_{\mathbb{R}^N} m u \phi - \int_{\mathbb{R}^N} f(x, u(x))\phi(x) dx.$$

Lemma 2.1 *The operator* T *is well defined.*

Proof: Let $u \in V_q(\mathbb{R}^N)$. We introduce
$F(\phi) = \int_{\mathbb{R}^N} \nabla u . \nabla \phi + q u \phi - \lambda \int_{\mathbb{R}^N} m u \phi - \int_{\mathbb{R}^N} f(x, u(x))\phi(x) dx$ for all $\phi \in V_q(\mathbb{R}^N)$.
Since m is bounded, f is Lipschitz in u uniformly in x and $f(x, 0) = 0$, we deduce that: $\forall \phi \in V_q(\mathbb{R}^N)$, $|F(\phi)| \leq \text{const} \cdot ||u||_q ||\phi||_q$. The operator F is linear and continuous. By the Riesz Theorem, we can well define the operator T. □

Lemma 2.2 *The operator* T *is continuous, Frechet differentiable with continuous derivatives given by:* $\forall \phi \in V_q(\mathbb{R}^N)$, $\forall \psi \in V_q(\mathbb{R}^N)$,

$$< T'_u(\lambda, u)\phi, \psi >_q = \int_{\mathbb{R}^N} \nabla \phi . \nabla \psi + q \phi \psi - \lambda \int_{\mathbb{R}^N} m \phi \psi - \int_{\mathbb{R}^N} f'_y(x, u(x))\phi(x)\psi(x) dx.$$

$$< T'_\lambda(\lambda, u), \phi >_q = - \int_{\mathbb{R}^N} m u \phi \ ; \ < T''_{\lambda u}(\lambda, u)\phi, \psi >_q = - \int_{\mathbb{R}^N} m \phi \psi.$$

Proof: We do not give here the details of the proof which is technical but simple. Since m is bounded and f is Lipschitz in u uniformly in x, we obtain the continuity of T and T'_λ. By using the hypothesis that $f'_y(x, .)$ is bounded uniformly in x and using the Lebesgue Dominated Convergence Theorem, we get the continuity of T'_u. □

Remarks $T'_u(\lambda_1, 0)$ is a continuous self-adjoint operator (by $(h3)$); the kernel $N(T'_u(\lambda_1, 0))$ is generated by u_1. So $dim\, N(T'_u(\lambda_1, 0)) = 1 = dim\, R(T'_u(\lambda_1, 0))$. Moreover $T''_{\lambda u}(\lambda_1, 0)u_1 \notin R(T'_u(\lambda_1, 0))$.
Indeed, denote by $< u_1 >$ the sub-space of $V_q(\mathbb{R}^N)$ generated by u_1. Since $T'_u(\lambda_1, 0)$ is a self-adjoint operator, the range $R(T'_u(\lambda_1, 0))$ of $T'_u(\lambda_1, 0)$ is the orthogonal of $< u_1 >$. But $< T''_{\lambda u}(\lambda_1, 0)u_1, u_1 >_q = -\int_{\mathbb{R}^N} mu_1^2 < 0$.
So $T''_{\lambda u}(\lambda_1, 0)u_1 \notin R(T'_u(\lambda_1, 0))$.
We can now apply the Theorem 1.7 in [8] to obtain a local bifurcation result.

Theorem 2.1 *Assume that the hypotheses* $(h1) - (h3)$ *are satisfied. Then there exist a number* $\epsilon_0 > 0$, *and two continuous functions* $\eta : (-\epsilon_0, \epsilon_0) \to \mathbb{R}$ *and* $\psi : (-\epsilon_0, \epsilon_0) \to < u_1 >^{\perp}$ *such that:* $\eta(0) = \lambda_1$, $\psi(0) = 0$ *and all non trivial solutions of* $T(\lambda, u) = 0$ *in a small neighbourhood of* $(\lambda_1, 0)$ *have the form* $(\lambda_\epsilon, u_\epsilon) = (\eta(\epsilon), \epsilon u_1 + \epsilon\psi(\epsilon))$ *for all* $\epsilon \in (-\epsilon_0, \epsilon_0)$.

Remark $T(\lambda, u) = 0$ iff u is solution of the equation (1). So near λ_1 (including the cases where $\lambda > \lambda_1$), the equation (1) admits non trivial solutions.

Adding another hypothesis on f, we are going to study now the sign of u_ϵ for $\epsilon \in (-\epsilon_0, \epsilon_0)$. First, we study the asymptotic behaviour of each solution of the equation (1).

Lemma 2.3 *Assume that the hypothesis* $(h1) - (h3)$ *are satisfied. Let* u *be a solution of the equation* (1). *Then* $\lim_{|x|\to+\infty} u(x) = 0$.

Proof: We have in a weak sense: $(-\Delta + q)u = \lambda mu + f(x, u) = [\lambda m + \frac{f(x,u)}{u}]u$ in $\mathbb{R}^N$. By $(h3)$, $\exists K > 0,\ |f(x, u)| \leq K|u|$. Using $(h2)$ we obtain that $\lambda m + \frac{f(x,u)}{u} \in L^\infty(\mathbb{R}^N)$. This implies by Theorem 4.1.3 in [3] combining with Theorem 8.17 in [9] that $\lim_{|x|\to+\infty} u(x) = 0$. □

Theorem 2.2 *Assume that the hypotheses* $(h1) - (h3)$ *are satisfied. Assume also that the following hypothesis* $(h'3)$ *is satisfied where:*
$(h'3)$ $\exists R > 0$, $\exists \epsilon^* > 0$, $\forall x \in \mathbb{R}^N$, $\forall y \in \mathbb{R}^{*-}$, $|x| > R$ *and* $|\lambda - \lambda_1| < \epsilon^* \Rightarrow \lambda m(x)y + f(x, y) > 0$.
Then $u_\epsilon \geq 0$ *for* ϵ *small enough.*

Proof:

i) Recall that $\lim_{|x|\to+\infty} u_\epsilon(x) = 0$.

ii) Let $0 < \epsilon < \epsilon_0$. We have: $\forall x,\ u_\epsilon(x) = \epsilon u_1(x) + \epsilon\psi(\epsilon)(x)$. Since $u_1 > 0$ and $\psi(\epsilon) \to 0$ when $\epsilon \to 0$, we deduce that:$\exists \epsilon_1 > 0,\ 0 < \epsilon < \epsilon_1 \Rightarrow \forall x \in B(0, R),\ u_\epsilon(x) > 0$.
We suppose that: $\exists x_0 \in \mathbb{R}^N,\ u_\epsilon(x_0) < 0$. Since $\lim_{|x|\to+\infty} u_\epsilon(x) = 0$, we deduce that

there exists $x_1 \in \mathbb{R}^N$, $|x_1| > R$ such that u_ϵ has a negative minimum in x_1.
If $(-\Delta + q)(u_\epsilon)(x_1) > 0$, then there exists a bounded domain Ω, containing x_1 such that $\forall x \in \Omega$, $(-\Delta + q)(u_\epsilon)(x) \geq 0$.
By the Maximum Principle (see Corollary 3.2 in [9]), we have: $\inf_\Omega u_\epsilon = u_\epsilon(x_1) \geq \inf_{\partial\Omega} u_\epsilon^- \geq 0$ where $u_\epsilon^- = \max\{0, -u_\epsilon\}$. Since $u_\epsilon(x_1) < 0$, we get a contradiction. Therefore $(-\Delta + q)(u_\epsilon)(x_1) \leq 0$. Using $(h'3)$, we have also: $(-\Delta + q)(u_\epsilon)(x_1) = \lambda m(x_1)u_\epsilon(x_1) + f(x_1, u_\epsilon(x_1)) > 0$.
So we get again a contradiction. Therefore $u_\epsilon \leq 0$.

□

We sudy now the global nature of the continuum of solutions obtained by bifurcation from the $(\lambda_1, 0)$ solution. Using Theorems 1.3 and 1.40 in [10], we obtain the following result:

Theorem 2.3 *There exists a continuum $\mathcal{C}$ of non trivial solutions for the equation* (1) *obtained by bifurcation from the $(\lambda_1, 0)$ solution, which is either unbounded or contains a point $(\lambda, 0)$ where $\lambda \neq \lambda_1$ is the inverse of an eigenvalue of the operator L. (L is defined by $< Lu, \phi >_q = \int_{\mathbb{R}^N} mu\phi$.) Since $\lambda_1$is simple, $\mathcal{C}$ has two connected subsets $\mathcal{C}^+$ and $\mathcal{C}^-$ which satisfy also the above alternatives.*

Proof:

i) We define an operator S by setting $S(\lambda, u) = u - T(\lambda, u)$ i.e. $\forall \phi \in V_q(\mathbb{R}^N)$,

$$< S(\lambda, u), \phi >_q = \int_{\mathbb{R}^N} [\lambda mu\phi + f(x, u)\phi].$$

So u is a solution of the equation (1) iff $u = S(\lambda, u)$. We write $S(\lambda, u) = \lambda Lu + H(\lambda, u)$ where $< Lu, \phi >_q = \int_{\mathbb{R}^N} mu\phi$ and $< H(\lambda, u), \phi >_q = \int_{\mathbb{R}^N} f(x, u)\phi$.

ii) For applying the results in [10], we must prove that $S : \mathbb{R} \times V_q(\mathbb{R}^N) \to V_q(\mathbb{R}^N)$ is continuous and compact, that $L : V_q(\mathbb{R}^N) \to V_q(\mathbb{R}^N)$ is linear and compact, that $H(\lambda, u) = O(\|u\|)$ for u near 0 uniformly on bounded intervals of λ and that $\frac{1}{\lambda_1}$ is a simple eigenvalue of L (which is true because it's a simple eigenvalue of $(-\Delta + q)^{-1}M$.)

iii) We show here that S is continuous and compact. S is continuous since T is continuous. Let $((\lambda_n, u_n))_n$ be a bounded sequence in $\mathbb{R} \times V_q(\mathbb{R}^N)$. Since the embedding of $V_q(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$ is compact, there exists a convergent subsequence, denoted also by $((\lambda_n, u_n))_n$ in $\mathbb{R} \times L^2(\mathbb{R}^N)$.
We have: $\forall \phi \in V_q(\mathbb{R}^N)$,

$< S(\lambda_n, u_n) - S(\lambda_p, u_p), \phi >_q = \lambda_n \int_{\mathbb{R}^N} m u_n \phi - \lambda_p \int_{\mathbb{R}^N} m u_p \phi + \int_{\mathbb{R}^N} [f(x, u_n) - f(x, u_p)]\phi$.
So $\|S(\lambda_n, u_n) - S(\lambda_p, u_p)\|_q^2 = (\lambda_n - \lambda_p) \int_{\mathbb{R}^N} m u_n [S(\lambda_n, u_n) - S(\lambda_p, u_p)]$

$$+\lambda_p \int_{\mathbb{R}^N} m(u_n - u_p)[S(\lambda_n, u_n) - S(\lambda_p, u_p)] + \int_{\mathbb{R}^N} [f(x, u_n) - f(x, u_p)][S(\lambda_n, u_n) - S(\lambda_p, u_p)].$$

By $(h2)$ and $(h3)$ we deduce that $(S(\lambda_n, u_n))_n$ is a Cauchy sequence and therefore a convergent sequence. So S is compact.

iv) We show here that L is linear and compact. L is obviously linear and continuous.
Let $(u_n)_n$ be a bounded sequence in $V_q(\mathbb{R}^N)$. Since the embedding of $V_q(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$ is compact, there exists a convergent subsequence, denoted also by $(u_n)_n$ in $L^2(\mathbb{R}^N)$.
We have: $\|Lu_n - Lu_p\|_q^2 = \int_{\mathbb{R}^N} m(u_n - u_p)[Lu_n - Lu_p]$.
By the Cauchy-Schwartz inequality, we get: $\|Lu_n - Lu_p\|_q \leq cst\|u_n - u_p\|_{L^2(\mathbb{R}^N)}$.
Therefore $(Lu_n)_n$ is a Cauchy sequence and so L is compact.

v) Finally note that $H(\lambda, u)$ is independant of λ. We denote it $H(u)$. We have: $\|H(u)\|_q^2 = \int_{\mathbb{R}^N} f(x, u)H(u) \leq cst\|u\|_q\|H(u)\|_q$.
So $H(u) = O(\|u\|)$.

□

3 Existence of positive solutions

We follow here a method developped in [7] for the p-Laplacian in a bounded domain.
Our results are more restrictive than in [7] because of the unboundedness of our domain.
We consider the equation

$$(-\Delta + q)u = \lambda m u + f|u|^{\gamma-2}u \text{ in } \mathbb{R}^N \tag{1}$$

for which the hypotheses $(h1)$ and $(h2)$ are satisfied, and $N = 3, 4$ so that $\gamma = 2^* = \frac{2N}{N-2} \in \mathbb{N}^*$.
Our aim is to study the existence of positive solutions for the equation (1) where $\lambda > \lambda_1$.
We define for $C \in \mathbb{R}^{*+}$, $C \geq u_1$, the set $X_{q,C} = \{u \in V_q(\mathbb{R}^N),\ u_1 \leq u \leq C \quad a.e.\}$
Let $F(u) := \int_{\mathbb{R}^N} f|u|^\gamma$ and $H_\lambda(u) := \int_{\mathbb{R}^N} |\nabla u|^2 + qu^2 - \lambda \int_{\mathbb{R}^N} mu^2$ for all $u \in V_q(\mathbb{R}^N)$.
Let $\lambda^* = \sup_{u \in V_q(\mathbb{R}^N), u \geq 0}\{\inf_{\phi \in V_q(\mathbb{R}^N)}\{\frac{\int_{\mathbb{R}^N} \nabla u.\nabla\phi + qu\phi}{\int_{\mathbb{R}^N} mu\phi},\ F'(u)(\phi) \geq 0,\ \phi \geq 0\}\}$ and
$\lambda^{**} = \sup_{u \in X_{q,C}}\{\inf_{\phi \in V_q(\mathbb{R}^N)}\{\frac{\int_{\mathbb{R}^N} \nabla u.\nabla\phi + qu\phi}{\int_{\mathbb{R}^N} mu\phi},\ F'(u)(\phi) \geq 0,\ \phi \geq 0\}\}$.
(Note that $\lambda^{**} \leq \lambda^*$.)
Let $l \geq 1$, $\epsilon > 0$, ϵ be small enough such that $\lambda_1 \leq \epsilon\gamma l^{\gamma-2}$ and $\epsilon < \frac{\lambda_1}{\gamma}$.

Remark There holds $\lambda_1 \leq \lambda^*$. On the contrary, if $\lambda_1 > \lambda^*$, then by the characterization of λ_1 we have $H_{\lambda_1}(u_1) = 0$. By the definition of λ^*, $\exists \phi \in V_q(\mathbb{R}^N)$, $\phi \geq 0$, $F'(u_1)(\phi) \geq 0$, $\frac{\int_{\mathbb{R}^N} \nabla u_1.\nabla\phi + q u_1 \phi}{\int_{\mathbb{R}^N} m u_1 \phi} \leq \lambda^* < \lambda_1$.
So $H'_{\lambda_1}(u_1)(\phi) < 0$.
We have: $\forall \eta \in \mathbb{R}^{*+}$, $H_{\lambda_1}(u_1 + \eta\phi) = H_{\lambda_1}(u_1) + \eta H'_{\lambda_1}(u_1)(\phi) + \|\eta\phi\| h(\eta\phi)$ with $h(\eta\phi) \to 0$ when $\eta \to 0$. Therefore, for η small enough, we have $H_{\lambda_1}(u_1 + \eta\phi) < 0$ and this contradicts the definition of λ_1.

Theorem 3.1 *Assume that the hypotheses* $(h1) - (h7)$ *are satisfied,* $N = 3, 4$ *and* $\gamma = 2^* = 2N/(N-2)$.

a) *If* $\lambda > \lambda^*$, *then the equation* (1) *has no positive solution.*

b) *If* $\lambda_1 + \epsilon l^{\gamma-2} < \lambda < \lambda^{**}$, *then the equation* (1) *has at least a positive solution.*

Proof:

i) By $(h7)$ we have: $f \geq -\frac{\epsilon u_1 m}{l^{\gamma-2} C^{\gamma-1}} \geq -\frac{\lambda_1 m}{\gamma l^{\gamma-2} C^{\gamma-2}} \geq -\frac{\epsilon m}{u_1^{\gamma-2}}$.

ii) Since $H^1(\mathbb{R}^N) \subset L^{2^*}(\mathbb{R}^N)$ with continuous imbedding, we deduce that $V_q(\mathbb{R}^N) \subset L^{2^*}(\mathbb{R}^N)$ with continuous imbedding.

Note that $\forall \phi \in V_q(\mathbb{R}^N),\ F'(u)(\phi) = \gamma \int_{\mathbb{R}^N} f|u|^{\gamma-2} u\phi$ and
$H'_\lambda(u)(\phi) = 2\int_{\mathbb{R}^N} [\nabla u.\nabla\phi + qu\phi - \lambda m u\phi]$.
Note also that u is a solution of the equation (1) iff $\forall \phi \in V_q(\mathbb{R}^N),\ H'_\lambda(u)(\phi) = \frac{2}{\gamma} F'(u)(\phi)$.
Moreover, if $t \in \mathbb{R}^{*+}$, $F'(tu)(\phi) = t^{\gamma-1} F'(u)(\phi)$ and $H'_\lambda(tu)(\phi) = tH'_\lambda(u)(\phi)$.

Assume here that $\lambda > \lambda^*$.
So: $\forall u \in V_q(\mathbb{R}^N)$, $u \geq 0$, $\exists \phi \geq 0$, $F'(u)(\phi) \geq 0$ and $H'_\lambda(u)(\phi) < 0$. Therefore the equation (1) has no positive solution.

Assume now that $\lambda_1 + \epsilon l^{\gamma-2} < \lambda < \lambda^{**}$.
We are going to prove that the equation (1) admits at least a positive solution by using the sub and supper solutions method and a Schauder Fixed Point Theorem.

a) Note by the definition of λ^{**} that:
$\exists u^* \in X_{q,C}$, $\forall \phi \geq 0$, $F'(u^*)(\phi) \geq 0 \Rightarrow H'_\lambda(u^*)(\phi) > 0$. (e)
We suppose that $\forall 0 < t \leq l$, $\exists \psi_t \geq 0$, $H'_\lambda(tu^*)(\psi_t) < \frac{2}{\gamma} F'(tu^*)(\psi_t)$.

If $\forall t$, $\forall \psi_t$, $F'(tu^*)(\psi_t) \geq 0$, then:

Let $\phi \geq 0$ such that $F'(u^*)(\phi) < 0$.

So $\forall t > 0$, $H'_\lambda(tu^*)(\phi) \geq \frac{2}{\gamma} F'(tu^*)(\phi)$ i.e. $\forall t > 0$, $t^{\gamma-2} \int_{\mathbb{R}^N} f(u^*)^{\gamma-1}\phi \leq \int_{\mathbb{R}^N} [\nabla u^*.\nabla\phi + qu^*\phi - \lambda mu^*\phi]$.

When $t \to 0$, we get: $0 \leq H'_\lambda(u^*)(\phi)$.

So $F'(u^*)(\phi) < 0 \Rightarrow H'_\lambda(u^*)(\phi) \geq 0$.

Using the property (e), we get: $\forall \phi \geq 0$, $H'_\lambda(u^*)(\phi) \geq 0$.

In particular, for $\phi = u_1$, we obtain: $\lambda_1 \int_{\mathbb{R}^N} mu^*u_1 \geq \lambda \int_{\mathbb{R}^N} mu^*u_1 > 0$.

Since $\lambda_1 < \lambda$, we get a contradiction.

If $\forall t$, $\forall \psi_t$, $F'(tu^*)(\psi_t) \leq 0$, then:

Let $\phi \geq 0$ such that $F'(tu^*)(\phi) > 0$. We have $H'_\lambda(tu^*)(\phi) \geq \frac{2}{\gamma} F'(tu^*)(\phi) > 0$.

So $\forall t$, $\int_{\mathbb{R}^N} [\nabla u^*.\nabla\phi + qu^*\phi - \lambda mu^*\phi] \geq t^{\gamma-2} \int_{\mathbb{R}^N} f(u^*)^{\gamma-1}\phi > 0$ and this is impossible for t large enough (because we can take a bigger l.)

Then we have: $\exists \phi \geq 0$, $\exists \psi \geq 0$, $H'_\lambda(u^*)(\phi) < \frac{2}{\gamma} t^{\gamma-2} F'(u^*)(\phi) < 0$ and $0 < H'_\lambda(u^*)(\psi) < \frac{2}{\gamma} t^{\gamma-2} F'(u^*)(\psi)$ (for at least one t).

Since $F'(u^*)$ is a continuous function, $\exists \alpha \in (0,1)$, $F'(u^*)(\alpha\phi + (1-\alpha)\psi) = 0$.

Therefore we deduce that $H'_\lambda(u^*)(\alpha\phi + (1-\alpha)\psi) > 0$.

But: $\frac{\alpha\gamma}{2t^{\gamma-2}} H'_\lambda(u^*)(\phi) < \alpha F'(u^*)(\phi) = -(1-\alpha)F'(u^*)(\psi) < -\frac{(1-\alpha)\gamma}{2t^{\gamma-2}} H'_\lambda(u^*)(\psi)$.

So $\frac{\gamma}{2t^{\gamma-2}} [\alpha H'_\lambda(u^*)(\phi) + (1-\alpha)H'_\lambda(u^*)(\psi)] < 0$ and we get a contradiction.

Therefore $\exists t \in (o, l]$, $\forall \phi \geq 0$, $H'_\lambda(tu^*)(\phi) \geq \frac{2}{\gamma} F'(tu^*)(\phi)$ i.e. tu^* is a supper solution of the equation (1). Note that $tu^* \geq su_1$ if $0 < s \leq t$. Let $s > 0$ such that $\frac{1}{s} \leq l^{\gamma-3}$. This is possible because we can choose l sufficiently big such that $\frac{1}{l^{\gamma-3}} \leq t \leq l$.

b) We show now that su_1 is a sub solution of the equation (1).

We have: $\frac{\lambda_1 - \lambda}{s^{\gamma-2}} < -\epsilon$ (since $l \geq s$) and $f \geq -\frac{\epsilon m}{u_1^{\gamma-2}}$.

So: $fu_1^{\gamma-1} > \frac{\lambda_1-\lambda}{s^{\gamma-2}} mu_1$ and therefore su_1 is a sub solution of the equation (1).

c) Let $\sigma = [su_1, tu^*]$ and the operator T be defined by $T(u) = v$ with v solution of $(-\Delta + q)v = \lambda mu + f|u|^{\gamma-2}u$ in $\mathbb{R}^N$.

We want to prove that $T(\sigma) \subset \sigma$ and that T is a continuous compact operator.

Let $u \in \sigma$ and $T(u) = v$.

We have, in a weak sense: $(-\Delta + q)(v - su_1) = \lambda mu + fu^{\gamma-1} - \lambda_1 msu_1$.

By $(h7)$, $f \geq -\frac{\epsilon u_1 m}{l^{\gamma-2}C^{\gamma-1}}$.

So, since $u > 0$, we have: $\lambda mu + fu^{\gamma-1} - \lambda_1 msu_1 \geq -\frac{\epsilon u_1 m}{l^{\gamma-2}C^{\gamma-1}} u^{\gamma-1} + \lambda mu - \lambda_1 msu_1$.

Moreover $u \in \sigma$ so $u^{\gamma-1} \leq l^{\gamma-1}C^{\gamma-1}$

and $\lambda mu + fu^{\gamma-1} - \lambda_1 msu_1 \geq m[\lambda u - (\lambda_1 + \frac{\epsilon l}{s})su_1] > 0$.

Therefore, since $u \geq su_1$ and $\lambda > \lambda_1 + \epsilon l^{\gamma-2} \geq \lambda_1 + \epsilon\frac{l}{s}$, we obtain that: $(-\Delta + q)(v - su_1) \geq 0$.
By the Maximum Principle, we deduce that $v \geq su_1$.

Moreover we have: $\forall \phi \geq 0$, $< (-\Delta + q)(tu^* - v), \phi >_{L^2(\mathbb{R}^N)} \geq \int_{\mathbb{R}^N} [\lambda m(tu^* - u) + f((tu^*)^{\gamma-1} - u^{\gamma-1})]\phi$.
By $(h7)$, since $t \leq l$ and $\lambda_1 < \lambda$ we have:
$f \geq -\frac{\lambda_1 m}{\gamma C^{\gamma-2} l^{\gamma-2}} \geq -\frac{\lambda_1 m}{\gamma C^{\gamma-2} t^{\gamma-2}} \geq -\frac{\lambda m}{\gamma C^{\gamma-2} t^{\gamma-2}}$.
But $\lambda m(tu^* - u) + f((tu^*)^{\gamma-1} - u^{\gamma-1}) \geq 0$ iff $f \geq -\frac{\lambda m}{\sum_{i=0}^{\gamma-2}(tu^*)^i u^{\gamma-2-i}}$.
Since $\sum_{i=0}^{\gamma-2}(tu^*)^i u^{\gamma-2-i} \leq \gamma C^{\gamma-2} t^{\gamma-2}$, we get $f \geq -\frac{\lambda m}{\sum_{i=0}^{\gamma-2}(tu^*)^i u^{\gamma-2-i}}$.
Therefore, by the Maximum Principle, we obtain $(-\Delta + q)(tu^* - v) \geq 0$ and so $v \leq tu^*$.

d) Let $(u_n)_n$ be a convergent sequence in σ, with limit u for the norm $\|.\|_q$. Let $T(u_n) = v_n$ and $T(u) = v$.
We have: $\forall n$,
$\|v_n - v\|_q^2 \leq cst\|u_n - u\|_q \|v_n - v\|_q + \|f\|_\infty \int_{\mathbb{R}^N} |u_n^{\gamma-1} - u^{\gamma-1}||v_n - v|$.
Since $u_n, u \in \sigma$, $|u_n^{\gamma-1} - u^{\gamma-1}| \leq cst|u_n - u|$ we obtain that:
$\|v_n - v\|_q \leq cst\|u_n - u\|_q$ and so T is a continuous operator. We finish this proof by showing that T is compact. Let now $(u_n)_n$ be a bounded sequence in σ for the norm $\|.\|_q$. Since the embedding of $V_q(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$ is compact, there exists a convergent subsequence, denoted also by $(u_n)_n$, in $L^2(\mathbb{R}^N)$. Let $T(u_n) = v_n$.
We have: $\forall n, p$
$\|v_n - v_p\|_q^2 = \lambda \int_{\mathbb{R}^N} m(u_n - u_p)(v_n - v_p) + \int_{\mathbb{R}^N} f(u_n^{\gamma-1} - u_p^{\gamma-1})(v_n - v_p)$.
Since $|u_n^{\gamma-1} - u_p^{\gamma-1}| \leq cst|u_n - u_p|$ we obtain that:
$\|v_n - v_p\|_q \leq cst\|u_n - u_p\|_{L^2(\mathbb{R}^N)}$.
We can deduce that $(v_n)_n$ is a Cauchy sequence and so T is a compact operator.

□

To finish, we obtain some results assuring the validity of the hypothesis $(h4)$. First, we need the following lemma: (we still follow a method developped in [7]).

Lemma 3.1 $\forall u \in V_q(\mathbb{R}^N)$, $u > 0, \forall \phi \in V_q(\mathbb{R}^N)$, $\phi \geq 0$,
$H'_\lambda(u)((\frac{\phi}{u})^{\gamma-1}\phi) - H'_\lambda(\phi)((\frac{\phi}{u})^{\gamma-1}u) \leq 0$.

Proof: We denote by $A = H'_\lambda(u)((\frac{\phi}{u})^{\gamma-1}\phi) - H'_\lambda(\phi)((\frac{\phi}{u})^{\gamma-1}u)$.
We have: $A = 2\int_{\mathbb{R}^N}[\nabla u.\nabla((\frac{\phi}{u})^{\gamma-1}\phi) - \nabla\phi.\nabla((\frac{\phi}{u})^{\gamma-1}u)]$.
$A = 2\int_{\mathbb{R}^N}[\phi\nabla u.\nabla((\frac{\phi}{u})^{\gamma-1}) - u\nabla\phi.\nabla((\frac{\phi}{u})^{\gamma-1})]$.

Since $\nabla((\frac{\phi}{u})^{\gamma-1}) = (\gamma-1)(\frac{\phi}{u})^{\gamma-2}[\frac{1}{u}\nabla\phi - \frac{\phi}{u^2}\nabla u]$, we get:
$A = 2(\gamma-1)\int_{\mathbb{R}^N}(\frac{\phi}{u})^{\gamma-2}[2\frac{\phi}{u}\nabla u.\nabla\phi - (\frac{\phi}{u})^2|\nabla u|^2 - |\nabla\phi|^2] \leq 0.$ □

So we get the last theorem:

Theorem 3.2 *Assume that the hypotheses* $(h1),(h2),(h5)$ *are satisfied,* $N == 3,4$ *and* $\gamma = 2^*$.

i) *If* $\Omega^+ = \{x \in \mathbb{R}^N,\ f(x) > 0\}$ *is a nonempty, bounded domain of* $\mathbb{R}^N$ *with a smooth frontier* $\partial\Omega^+$*, then* $\lambda^* < +\infty$.

ii) *If* $F(u_1) \geq 0$*, then* $\lambda^* = \lambda_1 < +\infty$.

iii) *Moreover* $\lambda_1 < \lambda^*$ *iff* $F(u_1) < 0$.

Proof:

i) Consider the following equation $(-\Delta + q)u = \lambda mu$ defined in Ω^+ with Dirichlet condition on $\partial\Omega^+$. We denote by λ_{1+} the first eigenvalue (which is simple and positive) and by ϕ_1 the first eigenfunction associated i.e:
$(-\Delta + q)\phi_1 = \lambda_{1+}m\phi_1$ in Ω^+, $\phi_1 > 0$ in Ω^+, $\phi_1 = 0$ on $\partial\Omega^+$.
Since $supp\phi_1 \subset \Omega^+$, by the above lemma, we get:
$\forall u \in \mathcal{D}(\mathbb{R}^{\mathcal{N}}),\ H'_{\lambda_{1+}}(u)((\frac{\phi_1}{u})^{\gamma-1}\phi_1) \leq 0$
i.e. $\forall u \in \mathcal{D}(\mathbb{R}^{\mathcal{N}}),\ u \geq 0$

$$\frac{\int_{\mathbb{R}^N}[\nabla u.\nabla((\frac{\phi_1}{u})^{\gamma-1}\phi_1) + qu(\frac{\phi_1}{u})^{\gamma-1}\phi_1]}{\int_{\mathbb{R}^N} mu(\frac{\phi_1}{u})^{\gamma-1}\phi_1} \leq \lambda_{1+} < +\infty.$$

Moreover, $F'(u)((\frac{\phi_1}{u})^{\gamma-1}\phi_1) = \gamma\int_{\Omega^+} f\phi_1^\gamma \geq 0$.
So $\lambda^* \leq \lambda_{1+} < +\infty$.

ii) As remarked before, there holds always $\lambda^* \geq \lambda_1$. We need to show that $\lambda^* \leq \lambda_1$, under the condition that $F(u_1) \geq 0$. We use again the above lemma.
We have $H'_{\lambda_1}(u_1)((\frac{u_1}{u})^{\gamma-1}u) = 0$ so
$\forall u \in \mathcal{D}(\mathbb{R}^{\mathcal{N}}),\ H'_{\lambda_1}(u)((\frac{u_1}{u})^{\gamma-1}u_1) \leq 0$.
Therefore, $\forall u \in \mathcal{D}(\mathbb{R}^{\mathcal{N}}),\ u \geq 0$

$$\frac{\int_{\mathbb{R}^N}[\nabla u.\nabla((\frac{u_1}{u})^{\gamma-1}u) + qu(\frac{u_1}{u})^{\gamma-1}u_1]}{\int_{\mathbb{R}^N} mu(\frac{u_1}{u})^{\gamma-1}u_1} \leq \lambda_1 < +\infty.$$

Since $F'(u)((\frac{u_1}{u})^{\gamma-1}u_1) = \gamma F(u_1) \geq 0$ we get that $\lambda^* \leq \lambda_1$ and therefore $\lambda^* = \lambda_1$.

iii)

a) Moreover, if $\lambda_1 < \lambda^*$, then, by *ii)* we obtain $F(u_1) < 0$.

b) Assume now that $F(u_1) < 0$.

1. We denote by $\lambda^- = \inf_{\phi \in V_q(\mathbb{R}^N), \phi \geq 0, F(\phi) \geq 0,} \frac{\int_{\mathbb{R}^N} [|\nabla \phi|^2 + q|\phi|^2]}{\int_{\mathbb{R}^N} m|\phi|^2}$.
   We are going to prove that $\lambda_1 < \lambda^-$ then that $\lambda^- \leq \lambda^*$.
   Let $W = \{\phi \in V_q(\mathbb{R}^N), \phi \geq 0, F(\phi) \geq 0\}$. Since $W \subset V_q(\mathbb{R}^N)$, we have $\lambda_1 \leq \lambda^-$. Since $u_1 \notin W$, then $\lambda_1 < \lambda^-$.
   We have to prove now that $\lambda^- \leq \lambda^*$.
2. First we prove that $\exists u^- \in V_q(\mathbb{R}^N)$, $u^- \geq 0$, $F(u^-) \geq 0$,
   $\lambda^- = \frac{\int_{\mathbb{R}^N} [|\nabla u^-|^2 + q|u^-|^2]}{\int_{\mathbb{R}^N} m|u^-|^2}$.
   On the contrary, we suppose that
   $\forall u \in V_q(\mathbb{R}^N)$, $u \geq 0$, $F(u) \geq 0 \Rightarrow \lambda^- < \frac{\int_{\mathbb{R}^N} [|\nabla u|^2 + q|u|^2]}{\int_{\mathbb{R}^N} m|u|^2}$.
   Let $v \geq 0$ such that $F(v) > 0$. Then $H_{\lambda^-}(v) > 0$.
   Since $\lambda_1 < \lambda^-$, we have $H_{\lambda^-}(u_1) < 0$ and so $H_{\lambda^-}(\eta u_1) < 0$ for all $\eta > 0$.
   Since the function H_{λ^-} is continuous, we get:
   $\exists \alpha \in (0,1)$, $H_{\lambda^-}(\alpha \eta u_1 + (1-\alpha)v) = 0$.
   Then $F(\alpha \eta u_1 + (1-\alpha)v) < 0$.
   Since $F((1-\alpha)v) > 0$, there exists $\eta > 0$ small enough such that $F(\alpha \eta u_1 + (1-\alpha)v) > 0$.
   So we get a contradiction and therefore we can deduce the existence of u^-.
3. Finally, we have to prove that $\lambda^- \leq \lambda^*$.
   On the contrary, we suppose that $\lambda^- > \lambda^*$.
   So $\exists \phi \in V_q(\mathbb{R}^N)$, $\phi \geq 0$, $F'(u^-)(\phi) \geq 0$, $\frac{\int_{\mathbb{R}^N} [\nabla u^- . \nabla \phi + q u^- \phi]}{\int_{\mathbb{R}^N} m u^- \phi} < \lambda^-$
   i.e. $H'_{\lambda^-}(u^-)(\phi) < 0$.
   Since $F(u^-) \geq 0$ and $F'(u^-)(\phi) \geq 0$, then $F(u^- + \eta\phi) \geq 0$ for $\eta > 0$ small enough.
   Moreover, since $H'_{\lambda^-}(u^-)(\phi) < 0$ and $H_{\lambda^-}(u^-) = 0$, we can choose $\eta > 0$ small enough such that $H_{\lambda^-}(u^- + \eta\phi) < 0$.
   So we obtain that: $\frac{\int_{\mathbb{R}^N} [|\nabla (u^- + \eta\phi)|^2 + q(u^- + \eta\phi)^2]}{\int_{\mathbb{R}^N} m(u^- + \eta\phi)^2} < \lambda^-$ and this contradicts the definition of λ^-.
   Therefore $\lambda^- \leq \lambda^*$.

□

References

[1] **Abbakhati-Mchachti, A. :** *Systèmes semilinéaires d'équations de Schrödinger.* Thèse numéro 1338, Université de Toulouse III, France 1993

[2] **Agmon, S. :** *Bounds on Exponential Decay of Eigenfunctions of Schrödinger Operators.* Schrödinger Operators, (Como, 1984), Springer-Verlag, pp 1-38, Berlin 1985

[3] **Cardoulis, L. :** *Problèmes elliptiques : applications de la théorie spectrale et étude de systèmes, existence de solutions.* Thèse, Université Toulouse I, 1997

[4] **Cardoulis, L. :** *Existence of Solutions for some Semilinear Elliptic Systems.* Rostock. Math. Kolloq. 56, 29-38, 2002

[5] **Alziary, B.**, and **Cardoulis, L. :** *An Anti-Maximum Principle for a Schrödinger Equation in* $\mathbb{R}^2$ *with an Indefinite Weight Function.* preprint

[6] **Brown, K. J.**, and **Stravakakis, N. :** *Global Bifurcation Results for a Semilinear Elliptic Equation on all of* $\mathbb{R}^n$. Duke Math.J. 85, No.1, 77-94, 1996

[7] **Il'Yasov, Y. :** *On positive solutions of indefinite elliptic equations.* C. R. Acad. Sci. Paris, t.3, Série I, p. 533-538, 2001

[8] **Crandall, M. G.**, and **Rabinowitz, P. H. :** *Bifurcation from Simple Eigenvalues.* Journ. of Funct. Anal. 8, 321-340, 1971

[9] **Gilbarg, D.**, and **Trudinger, N. S. :** *Elliptic Partial Differential Equations of Second Order.* Springer-Verlag, Berlin 1983

[10] **Rabinowitz, P. H. :** *Some Global Results for nonlinear Eigenvalues Problems.* Journ. of Funct. Anal. 7, 487-513, 1971

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