Rostock. Math. Kolloq. **58**, 101–122 (2004)

Subject Classification (AMS)
45D05, 39A10, 26C05, 05C90

Lothar Berg, Manfred Krüppel

Recursions for the Solution of an Integral-Functional Equation

ABSTRACT. In this paper, we continue our considerations in [1, 2, 3] about a homogeneous integral-functional equation with a parameter $a > 1$. Here we assume that $a \geq 2$, disregarding some explicitly mentioned cases where a can be smaller than 2. We derive new recursions which allow to calculate the solution and its derivatives effectively, and which contain formulas of R. Schnabl [8] and W. Volk [10] as special cases for $a = 2$.

KEY WORDS. Integral-functional equation, generating functions, Cantor sets, relations containing polynomials, recursions, directed graphs.

1 Introduction

There exists a long history concerning compactly supported C^∞-functions, which are solutions of differential-functional equations, cf. [7], [3] and the literature quoted there. By integration these equations can be transformed into integral-functional equations. Here, we deal with the special equation

$$\phi(t) = b \int_{at-a+1}^{at} \phi(\tau) d\tau \qquad \left(b = \frac{a}{a-1}\right) \tag{1.1}$$

with the real variable t and a parameter $a > 1$. Applications of (1.1) to probability problems were given by G.J. Wirsching for $a = 3$ in [11], and for $a \geq \frac{3}{2}$ in [12].

In this paper, we continue our considerations in [1, 2, 3] concerning the solutions of (1.1) under the assumption $a \geq 2$ disregarding some explicitly mentioned cases where a can be smaller than 2, in particular, in Section 8. We derive new recursions which allow to calculate the solution and its derivatives effectively, and which contain formulas of R. Schnabl [8] and W. Volk [10] as special cases for $a = 2$. For convenience of the reader we first list those results from [1, 2, 3] which are needed later on.

For $a > 1$ equation (1.1) has a C^∞-solution with the support $[0, 1]$, which is uniquely determined by means of the normalization

$$\int_0^1 \phi(t)dt = 1 \ . \tag{1.2}$$

In particular, it is $\phi(0) = \phi(1) = 0$. The solution of (1.1)-(1.2) is symmetric with respect to the point $\frac{1}{2}$, monotone at both sides of $\frac{1}{2}$ and it is strictly positive for $t \in (0, 1)$. The Laplace transform Φ of the solution ϕ of (1.1)-(1.2) is an entire function satisfying $\Phi(0) = 1$ and the functional equation

$$\Phi(z) = \frac{1 - e^{-z/b}}{z/b} \Phi\left(\frac{z}{a}\right) . \tag{1.3}$$

It has the Taylor series

$$\Phi(z) = \sum_{n=0}^{\infty} \frac{\rho_n(a)}{n!} z^n \qquad (z \in \mathbb{C})$$

where the coefficients are rational functions with respect to a and, starting with $\rho_0(a) = 1$ for $n \geq 1$, they can be determined by means of the recursion

$$\rho_n(a) = \frac{1}{(n+1)(a^n - 1)} \sum_{\nu=0}^{n-1} \binom{n+1}{\nu} \rho_\nu(a)(1-a)^{n-\nu} \ . \tag{1.4}$$

Moreover, for fixed n, the functions $(-1)^n \rho_n(a)$ are increasing for $a \geq 1$ and it holds

$$\frac{1}{2^n} \leq (-1)^n \rho_n(a) \leq \frac{1}{n+1}, \tag{1.5}$$

cf. [1, (2.14)].

For $a > 2$, the solution ϕ of (1.1)-(1.2) is a polynomial on each component of an open Cantor set with Lebesgue measure 1. These polynomials can be expressed by means of the polynomials

$$\psi_n(t) = \sum_{\nu=0}^{n} \binom{n}{\nu} \rho_{n-\nu}(a) t^\nu, \tag{1.6}$$

which have the special values

$$\psi_n(0) = \rho_n(a), \qquad \psi_n(1) = (-1)^n \rho_n(a), \tag{1.7}$$

and which have the generating function

$$e^{tz}\Phi(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \psi_n(t) z^n \tag{1.8}$$

so that they are Appell polynomials, cf. [8], [1]. Note that in [1] we have used the abbreviation ψ_n for the polynomials (1.6) with $\frac{1}{a}$ instead of a. In [3] we have modified the polynomials (1.6) by

$$f_n(t) = c_n \psi_n(t) \tag{1.9}$$

where c_n is given by

$$c_n = \frac{b^{n+1}}{n!\, a^{\frac{n(n+1)}{2}}} = \frac{1}{n!\, a^{\frac{(n+1)(n-2)}{2}} (a-1)^{n+1}} \; . \tag{1.10}$$

These polynomials can be calculated recursively by

$$f_n(t) = \frac{b}{na^n} \left(t - \frac{1}{2} \right) f_{n-1}(t) + \frac{1}{n} \sum_{\nu=2}^{n} \frac{1}{\nu!} B_\nu \frac{a^{\frac{1}{2}\nu(\nu+1-2n)}}{a^\nu - 1} f_{n-\nu}(t) \quad (n \geq 1), \tag{1.11}$$

starting with $f_0(t) = b$ and using the Bernoulli numbers

$$B_0 = 1 \; , \quad B_1 = -\frac{1}{2} \; , \quad B_2 = \frac{1}{6} \; , \quad B_3 = 0 \; , \quad B_4 = -\frac{1}{30} \; , \quad \ldots \; .$$

They satisfy the relations

$$f_n(t) - f_n(t - a + 1) = f_{n-1}\left(\frac{t}{a} \right) \tag{1.12}$$

and

$$f_n(t) = (-1)^n f_n(1-t) \tag{1.13}$$

with $n \in \mathbb{N}_0$ and $f_{-1} = 0$. The simplest connection between the solution ϕ of (1.1)-(1.2) and polynomials f_n $(n \geq -1)$ is valid for $a \geq 2$, and reads

$$\phi\left(\frac{\tau}{a^{n+1}} \right) = f_n(\tau) \qquad (1 \leq \tau \leq a-1). \tag{1.14}$$

In particular for $n = 0$, ϕ attains its maximum $\phi(t) = b$ for $\frac{1}{a} \leq t \leq 1 - \frac{1}{a}$. In order to state more complicated connections between ϕ and f_n we need an auxiliary sequence $\gamma_k = \gamma_k(a)$ defined as follows: If $k \in \mathbb{N}$ has the dyadic representation $k = d_p \ldots d_1 d_0$ with $d_p = 1$ and $d_\nu \in \{0, 1\}$ then

$$\gamma_k = (a-1) \sum_{\nu=0}^{p} d_\nu a^\nu. \tag{1.15}$$

The sequence γ_k $(k \in \mathbb{N}_0)$ can also be defined by

$$\gamma_{2k} = a\gamma_k \; , \quad \gamma_{2k+1} = a\gamma_k + a - 1 \; , \quad k = 0, 1, 2, \ldots \; , \tag{1.16}$$

so that in particular $\gamma_0 = 0$ and $\gamma_1 = a - 1$. For $p \in \mathbb{N}_0$ these numbers satisfy the relations

$$\gamma_{2^p} = (a-1)a^p, \qquad \gamma_{2^p - 1} = a^p - 1, \qquad \gamma_{2^p - 2} = a^p - a \quad (p \neq 0), \tag{1.17}$$

$$\gamma_{2k+1} = \gamma_{2k} + \gamma_1, \qquad \gamma_{2^\sigma\kappa} = a^\sigma \gamma_\kappa \qquad (\sigma, \kappa \in \mathbb{N}_0), \tag{1.18}$$

$$\gamma_k + \gamma_u + 1 = a^{p+1} \qquad \text{if} \qquad k + u + 1 = 2^{p+1} \tag{1.19}$$

and the inequality

$$\gamma_{k+1} \geq \gamma_k + \gamma_1 \qquad (k \in \mathbb{N}_0,\, a \geq 2). \tag{1.20}$$

For integers a also the numbers γ_k are integers. In particular, for $a = 2$, we have $\gamma_k = k$. Moreover, we need the sign sequence $\varepsilon_k = (-1)^{\nu(k)}$, where $\nu(k)$ denotes the number of "1s" in the dyadic representation of k, i.e. $\nu(k)$ is the binary sum-of-digits function (cf. [4]).

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
ε_k	1	-1	-1	1	-1	1	1	-1	-1	1	1	-1	1	-1	-1	1

Table 1: The first numbers ε_k

In the case $a \geq 2$ we define the following closed intervals

$$\overline{G}_{kn} = \left[\frac{\gamma_{2k}+1}{a^{n+1}}, \frac{\gamma_{2k+1}}{a^{n+1}}\right], \qquad F_{kn} = \left[\frac{\gamma_k}{a^n}, \frac{\gamma_k + 1}{a^n}\right] \tag{1.21}$$

with $\overline{G}_{kn} \subset F_{kn}$, since

$$F_{kn} = F_{2k,n+1} \cup \overline{G}_{kn} \cup F_{2k+1,n+1} \tag{1.22}$$

$k = 0, 1, \ldots, 2^n - 1$, $n \in \mathbb{N}_0$ (for $a = 2$ the intervals $\overline{G}_{kn}$ degenerate to a single point). In the intervals $\overline{G}_{kn}$, the solution ϕ of (1.1)-(1.2) has the representation

$$\phi(t) = \sum_{\nu=0}^{2k} \varepsilon_\nu f_n(a^{n+1}t - \gamma_\nu) \quad (t \in \overline{G}_{kn}) \tag{1.23}$$

for $k = 0, 1, \ldots, 2^n - 1$, $n \in \mathbb{N}_0$. Moreover, for $t \in F_{kn}$, i.e. $t = \frac{\gamma_k + \tau}{a^n}$ with $0 \leq \tau \leq 1$ we have the *main formula*

$$\phi\left(\frac{\gamma_k + \tau}{a^n}\right) - \varepsilon_k \phi\left(\frac{\tau}{a^n}\right) = \sum_{\nu=0}^{k-1} \varepsilon_\nu f_{n-1}(\gamma_k + \tau - \gamma_\nu). \tag{1.24}$$

Another relation is

$$\phi\left(\frac{\gamma_k + \tau}{a^{n+1}}\right) + \phi\left(\frac{\gamma_\ell + \tau}{a^{n+1}}\right) = f_{n-p}\left(\frac{\gamma_k + \tau}{a^p}\right) \quad (0 \leq \tau \leq a,\, a \geq 2,\, n \geq p) \tag{1.25}$$

where k is even and $k = 2^p + \ell$ $(0 \leq \ell < 2^p, p \in \mathbb{N})$.

Remark 1.1 Formula (1.25) is also valid for arbitrary $k \in \mathbb{N}$ ($p \in \mathbb{N}_0$) when $0 \leq \tau \leq 1$. If $k = 2^\sigma \kappa$ and $\ell = 2^\sigma \lambda$ with $\sigma \in \mathbb{N}$ and integers κ, λ then $\kappa = 2^{p-\sigma} + \lambda$ and, according to (1.18), it holds

$$\frac{\gamma_k + \tau}{a^{n+1}} = \frac{\gamma_\kappa + \frac{\tau}{a^\sigma}}{a^{n-\sigma+1}}$$

and an analogous formula with ℓ and λ instead of k and τ, respectively. Writing (1.25) for $0 \leq \tau \leq 1$ with κ, λ, $n - \sigma$, $p - \sigma$ and $\frac{\tau}{a^\sigma}$ instead of k, ℓ, n, p and τ, respectively, we see that (1.25) is even valid for $0 \leq \tau \leq a^\sigma$ (i.e. at least for $0 \leq \tau \leq a$ when k is even). This assertion is already contained in [2, Proposition 6.1], however without proof.

Finally, we quote a relation which is valid even for $a \geq \frac{3}{2}$, namely

$$\phi\left(\frac{\tau}{a^{n+1}}\right) + (-1)^n \phi\left(\frac{1-\tau}{a^{n+1}}\right) = f_n(\tau) \qquad (2 - a \leq \tau \leq a - 1,\ n \geq -1), \tag{1.26}$$

and the relation valid for $a > 1$

$$\sum_{\nu=-\infty}^{+\infty} \phi\left(t - \frac{\nu}{b}\right) = b \qquad (t \in \mathbb{R}). \tag{1.27}$$

2 Polynomial relations

First we state two sets of new formulas for the polynomials f_n.

Proposition 2.1 *The polynomials f_n ($n \in \mathbb{N}_0$) satisfy the addition theorem*

$$f_n(at + (1-a)s) = a^n \sum_{\nu=0}^{n} \frac{(-1)^\nu}{\nu!} B_\nu(s) a^{\frac{\nu}{2}(\nu-1-2n)} f_{n-\nu}(t) \qquad (s, t \in \mathbb{R}) \tag{2.1}$$

where $B_\nu(s)$ are the Bernoulli polynomials, and the multiplication theorem

$$f_n(t) = a^{-\frac{n(n+1)}{2}} \sum_{\nu=0}^{n} \frac{(-1)^{n-\nu}}{(n+1-\nu)!} a^{\frac{\nu(\nu-1)}{2}} f_\nu(at) \qquad (t \in \mathbb{R}). \tag{2.2}$$

Proof: Equation (1.3) can be written in the form

$$e^{(at+(1-a)s)\frac{z}{a}} \Phi\left(\frac{z}{a}\right) = \frac{-\frac{z}{b} e^{\frac{-sz}{b}}}{e^{-\frac{z}{b}} - 1} e^{tz} \Phi(z) \qquad (s, t \in \mathbb{R}),$$

since $b = \frac{a}{a-1}$. Expanding both sides into power series with respect to z, using (1.8) and the generating function of the Bernoulli polynomials, and comparing the coefficients we obtain the formula

$$\psi_n(at + (1-a)s) = \sum_{\nu=0}^{n} \binom{n}{\nu} (1-a)^\nu B_\nu(s) a^{n-\nu} \psi_{n-\nu}(t) \qquad (n \in \mathbb{N}_0).$$

In view of (1.9) and

$$\frac{c_n}{c_{n-\nu}} = \frac{(n-\nu)!}{n!}\frac{a^{\frac{\nu}{2}(\nu+1-2n)}}{(a-1)^\nu}$$

the last equation turns over into (2.1).

From (1.3) and (1.8) we obtain analogously

$$\psi_n(t) = n!\sum_{\nu=0}^{n}\frac{\psi_\nu(at)}{a^\nu \nu!}\frac{1}{(n+1-\nu)!}\left(-\frac{1}{b}\right)^{n-\nu},$$

and in view of (1.9) and

$$\frac{c_n}{c_\nu} = \frac{\nu!}{n!}b^{n-\nu}\frac{a^{\frac{\nu(\nu+1)}{2}}}{a^{\frac{n(n+1)}{2}}} \tag{2.3}$$

we obtain (2.2) ■

Formula (2.1) is equivalent for $a = \frac{1}{2}$ and $s = t$ to [8, (C)], and for $s = 0$ to a formula in [2, p.1012].

Relation (2.2) is a generalization of (1.4), because for $t = 0$ it can be transferred into (1.4), using (1.9) and the first relation of (1.7). Though formula (2.2) is not a usual recursion, it is possible to calculate $f_n(t)$ recursively by means of it if we additionally use from (1.6) and (1.9) that the polynomial must have the main term $c_n t^n$. Relation (2.2) can be considered as the inversion of (2.1) with $s = 0$ and vice versa.

3 Special recursions for the solutions

Formula [10, (1.14)] from W. Volk can be generalized to the case $a \geq 2$, which shall be the general assumption in the Sections 3 − 7.

Proposition 3.1 *For $n \geq 2$ we have the recursion*

$$\phi\left(\frac{1}{a^n}\right) = \frac{1}{1-a^{1-n}}\sum_{\nu=2}^{n}\frac{1}{\nu!}a^{\frac{1}{2}\nu(\nu+1-2n)}\phi\left(\frac{1}{a^{n+1-\nu}}\right) \tag{3.1}$$

with the initial value $\phi(\frac{1}{a}) = b$.

Proof: According to (1.14) with $\tau = 1$ we have $\phi(\frac{1}{a^{n+1}}) = f_n(1)$ so that (1.9) and (1.7) yield

$$\phi\left(\frac{1}{a^{n+1}}\right) = (-1)^n c_n \rho_n(a) \tag{3.2}$$

and in particular $\phi(\frac{1}{a}) = b$. Substituting (3.2) into (1.4) we get

$$\phi\left(\frac{1}{a^{n+1}}\right) = \frac{1}{(n+1)(a^n-1)} \sum_{\nu=0}^{n-1} \binom{n+1}{\nu} \frac{c_n}{c_\nu} (a-1)^{n-\nu} \phi\left(\frac{1}{a^{\nu+1}}\right),$$

and in view of (2.3) we obtain the equation

$$\phi\left(\frac{1}{a^{n+1}}\right) = \frac{a^n}{a^n-1} \sum_{\mu=2}^{n+1} \frac{1}{\mu!} a^{\frac{1}{2}\mu(\mu-1-2n)} \phi\left(\frac{1}{a^{n+2-\mu}}\right),$$

which turns over into (3.1) replacing n by $n-1$ ■

Formula [10, (1.14)] is the special case $a = 2$. Besides of (3.1) we also can state recursions for $\phi(\frac{\tau}{a^n})$. Inserting (1.14) into (1.11) with $t = \tau$ and into (2.1) with $t = s = \tau$, respectively, we immediately obtain

Corollary 3.2 *For $1 \le \tau \le a-1$ and $n \ge 1$ we have the recursion formulas*

$$\phi\left(\frac{\tau}{a^{n+1}}\right) = \frac{(\tau - \frac{1}{2})}{n(a-1)a^{n-1}} \phi\left(\frac{\tau}{a^n}\right) + \frac{1}{n} \sum_{\nu=2}^{n} \frac{1}{\nu!} B_\nu \frac{a^{\frac{\nu}{2}(\nu+1-2n)}}{a^\nu - 1} \phi\left(\frac{\tau}{a^{n-\nu}}\right) \tag{3.3}$$

and

$$\phi\left(\frac{\tau}{a^{n+1}}\right) = \frac{a^n}{1-a^n} \sum_{\nu=1}^{n} \frac{(-1)^\nu}{\nu!} B_\nu(\tau) a^{\frac{\nu}{2}(\nu-1-2n)} \phi\left(\frac{\tau}{a^{n+1-\nu}}\right), \tag{3.4}$$

both with the initial value $\phi(\frac{\tau}{a}) = b$.

Equations (3.3) and (3.4), both for $\tau = 1$, lead to new recursions for $\phi(\frac{1}{a^n})$ which are different from (3.1). Moreover, for $\tau = a-1$ both equations yield recursions for $\phi(\frac{\gamma_1}{a^n})$ which are the initial values for more general recursions yielding $\phi(\frac{\gamma_k}{a^n})$. In order to state such recursions we apply Taylor's formula and hence we need the derivatives of higher order of the solutions. Moreover, we have to extend the interval of validity of the main formula (1.24).

4 The domain of validity of the main formula

We preserve the assumption $a \ge 2$ and show that formula (1.24) with $n \in \mathbb{N}_0$ and $k \in \{0, 1, \ldots, 2^n - 1\}$ has in fact a greater interval of validity when $a > 2$.

Proposition 4.1 *The main formula* (1.24) *for the solution ϕ of* (1.1)-(1.2) *is valid even for $2 - a \le \tau \le a - 1$.*

Proof: Since (1.24) is trivial for $k = 0$ we assume that $k \geq 1$ and therefore also $n \geq 1$. For convenience we introduce the notation $\overline{G}_{2^m,m} = [1, 1 + \frac{a-2}{a^{m+1}}]$ where $m \in \mathbb{N}_0$. Then according to (1.21) every F_{kn} has two $\overline{G}_{\ell m}$ with $m \leq n-1$ as neighbouring intervals. Since $t = \frac{\gamma_k+\tau}{a^n} \in F_{kn}$ for $0 \leq \tau \leq 1$ and $|\overline{G}_{\ell,n-1}| = \frac{a-2}{a^n}$, we see that t lies in intervals $\overline{G}_{\ell m}$ both for $2-a \leq \tau \leq 0$ and for $1 \leq \tau \leq a-1$. Hence, in both cases $\phi(t)$ is a polynomial. But in both cases also $\phi(\frac{\tau}{a^n})$ is a polynomial, namely 0 and $f_{n-1}(\tau)$, respectively, cf. (1.14). This implies that the left-hand side of (1.24) is a polynomial spline for $2-a \leq \tau \leq a-1$. But it is also a C^∞-function, i.e. it must be a unique polynomial ■

The interval $2-a \leq \tau \leq a-1$ is optimal if k is odd, cf. (1.22). The case that $k = 2^\sigma\kappa$ is even can be reduced to the odd case as in Remark 1.1 using $\gamma_k = a^\sigma\gamma_\kappa$.

As consequence of Proposition 4.1, formula (1.26) can be generalized in the case $a \geq 2$ as follows:

Proposition 4.2 *For $n \in \mathbb{N}_0$, $k \in \{0, 1, \ldots, 2^n - 1\}$ and $2-a \leq \tau \leq a-1$, the solution ϕ of (1.1)-(1.2) has the property*

$$\phi\left(\frac{\gamma_k+\tau}{a^n}\right) + (-1)^{n-1}\phi\left(\frac{\gamma_k+1-\tau}{a^n}\right) = P(\tau) \tag{4.1}$$

where P is the polynomial

$$P(\tau) = \varepsilon_k f_{n-1}(\tau) + \sum_{\nu=0}^{k-1} \varepsilon_\nu \left[f_{n-1}(\gamma_k+\tau-\gamma_\nu) + f_{n-1}(\gamma_\nu+\tau-\gamma_k)\right]. \tag{4.2}$$

Proof: The inequality $2-a \leq \tau \leq a-1$ implies $2-a \leq 1-\tau \leq a-1$. Hence, according to Proposition 4.1 besides of (1.24) we also have

$$\phi\left(\frac{\gamma_k+1-\tau}{a^n}\right) - \varepsilon_k\phi\left(\frac{1-\tau}{a^n}\right) = \sum_{\nu=0}^{k-1} \varepsilon_\nu f_{n-1}(\gamma_k+1-\tau-\gamma_\nu). \tag{4.3}$$

Multiplying the last equation with $(-1)^{n-1}$, using (1.13) and (1.26) with $n-1$ instead of n, we obtain the assertion by adding (1.24) ■

5 On the derivatives of higher order

As before it shall be $a \geq 2$. Besides of the intervals (1.21) we need the open intervals

$$\overset{\circ}{F}_{kn} = \left(\frac{\gamma_k}{a^n}, \frac{\gamma_k+1}{a^n}\right) \qquad (k = 0, 1, \ldots, 2^n-1,\ n \in \mathbb{N}_0), \tag{5.1}$$

with the decomposition

$$\overset{\circ}{F}_{kn} = \overset{\circ}{F}_{2k,n+1} \cup \overline{G}_{kn} \cup \overset{\circ}{F}_{2k+1,n+1} \tag{5.2}$$

where the three sets on the right-hand side are disjoint. As in [3] we introduce the set

$$M = \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{2^n-1} \overline{G}_{kn}$$

and its complement $CM = (0,1) \setminus M$ which can also be represented as

$$CM = \bigcap_{n=0}^{\infty} \bigcup_{k=0}^{2^n-1} \overset{\circ}{F}_{kn}\ . \tag{5.3}$$

For t in one of the intervals $\overset{\circ}{F}_{\ell m}$ it holds

$$\phi^{(m)}(t) = \varepsilon_\ell a^{\frac{m(m+1)}{2}} b^m \phi(a^m t - \gamma_\ell) \quad (t \in \overset{\circ}{F}_{\ell m}) \tag{5.4}$$

and otherwise we have $\phi^{(m)}(t) = 0$, cf. [2]. This means that for fixed $t \in (0,1)$ and $m \in \mathbb{N}_0$ we have $\phi^{(m)}(t) \neq 0$ if and only if there is an index ℓ satisfying

$$0 < a^m t - \gamma_\ell < 1, \tag{5.5}$$

i.e. $t \in \overset{\circ}{F}_{\ell m}$. Note that there exists at most one number $\ell = \ell_m$ with (5.5), since the intervals $F_{\ell m}$ $(\ell = 0, 1, \ldots, 2^m - 1)$ are pairwise disjoint. Next, we modify [3, Definition 2.2]:

Definition 5.1 *For given $t \in (0,1)$ we define a sequence $\delta_m = \delta_m(a,t)$ $(m \in \mathbb{N}_0)$ by $\delta_m = 1$ if (5.5) is satisfied for a certain index $\ell = \ell_m$, and by $\delta_m = 0$ elsewhere.*

Lemma 5.2 *If for given $t \in (0,1)$ it holds $\delta_m = 1$ $(m \in \mathbb{N})$ then $\delta_{m-1} = 1$, too, with the corresponding index $\ell_{m-1} = [\frac{\ell_m}{2}]$.*

Proof: We have $\delta_m = 1$ if and only if $t \in \overset{\circ}{F}_{\ell m}$ with $\ell = \ell_m$. But, according to (5.2), $t \in \overset{\circ}{F}_{\ell m}$ implies that $t \in \overset{\circ}{F}_{k,m-1}$ with $k = [\frac{\ell}{2}]$. This yields the assertion ■

Proposition 5.3 *The derivatives of the solution ϕ of (1.1)-(1.2) have the following property:*

1. *For $t \in M$, i.e. $t \in \overline{G}_{kn}$ with fixed k, n, it holds $\phi^{(m)}(t) \neq 0$ when $0 \leq m \leq n$ and $\phi^{(m)}(t) = 0$ when $m \geq n+1$.*

2. *For $t \in CM$ it holds $\phi^{(m)}(t) \neq 0$ for all $m \in \mathbb{N}_0$.*

Proof: **1.** In the case $a = 2$, where the interval $\overline{G}_{kn}$ degenerates to the point $t = \frac{2k+1}{2^{n+1}}$, it is known that $\phi^{(m)}(t) = 0$ for $m > n$, cf. [2, (4.8)] or [9, p.575]. For $a > 2$ and $t \in \overline{G}_{kn}$ the function ϕ is a polynomial of degree n according to (1.23) and hence $\phi^{(m)}(t) = 0$ for $m > n$, too. But (5.2) shows that it is also $t \in \overset{\circ}{F}_{kn}$ and hence $\phi^{(n)}(t) \neq 0$ in view of (5.4). Lemma 5.2 implies $\phi^{(m)}(t) \neq 0$ for $m \leq n$.

2. In view of (5.3) the supposition $t \in CM$ implies that for each m we have $t \in \overset{\circ}{F}_{\ell m}$ with $\ell = \ell_m$ defined above, and hence (5.4) implies $\phi^{(m)}(t) \neq 0$ ■

In order to determine the sequences δ_m and ℓ_m from Definition 5.1 explicitly for a given $t = \frac{\gamma_k}{a^n} \in (0,1)$, which is necessary for a later application, we introduce the dyadic representation

$$k = d_p d_{p-1} \dots d_1 d_0 \tag{5.6}$$

with $d_j \in \{0,1\}$, i.e. $k = d_p 2^p + \dots + d_1 2 + d_0$, where $p < n$ since $k < 2^n$. For convenience we extend the coefficients by $d_j = 0$ for $p+1 \leq j \leq n$. In the next lemma we shall show that in Definition 5.1 it holds $\ell_m = [\frac{k}{2^{n-m}}]$, i.e.

$$\ell_m = d_{n-m} + d_{n-m+1} 2 + \dots + d_n 2^m \tag{5.7}$$

when $m \in \{0, \dots, n-1\}$.

Lemma 5.4 *Assume that $t = \frac{\gamma_k}{a^n} \in (0,1)$ with k from* (5.6) *and $n \in \mathbb{N}$. If k has the form $k = 2^\sigma(2\kappa+1)$ with $\sigma, \kappa \in \mathbb{N}_0$ then it holds $\delta_m = 1$ for $m \in \{0, \dots, n-\sigma-1\}$ with the corresponding index* (5.7)*, and $\delta_m = 0$ for $m \geq n - \sigma$.*

Proof: With (1.15) and the above notations we have

$$a^m t - \gamma_{\ell_m} = \gamma_1 \left(\frac{d_0}{a^{n-m}} + \dots + \frac{d_{n-m-1}}{a} \right). \tag{5.8}$$

The assumption $k = 2^\sigma(2\kappa+1)$ means $d_\sigma = 1$ and in the case $\sigma > 0$ additionally $d_j = 0$ for $0 \leq j < \sigma$, so that (5.8) reduces to

$$a^m t - \gamma_{\ell_m} = \gamma_1 \left(\frac{1}{a^{n-\sigma-m}} + \frac{d_{\sigma+1}}{a^{n-\sigma-m-1}} + \dots + \frac{d_{n-m-1}}{a} \right). \tag{5.9}$$

Choosing $m = n - \sigma$ we obtain $a^{n-\sigma} t - \gamma_{\ell_m} = 0$. In view of (1.20) this implies that $a^{n-\sigma} t - \gamma_\nu \geq 1$ for $\nu < \ell_m$ and that $a^{n-\sigma} t - \gamma_\nu \leq 0$ for $\nu \geq \ell_m$, i.e. $\delta_{n-\sigma} = 0$. Lemma 5.2 yields $\delta_m = 0$ for all $m \geq n - \sigma$. For $m \in \{0, \dots, n-\sigma-1\}$ equation (5.9) implies that

$$0 < \frac{a-1}{a^{n-\sigma-m}} \leq a^m t - \gamma_{\ell_m} \leq 1 - \frac{1}{a^{n-\sigma-m}} < 1.$$

Hence, for these m it holds $\delta_m = 1$ and the corresponding index reads (5.7) ■

6 More general recursions

The announced recursion formula for $\phi(\frac{\gamma_k}{a^n})$ in the case $a \geq 2$ is a consequence of the following

Theorem 6.1 *Assume that $n \in \mathbb{N}$ and that $k = 2^\sigma(2\kappa+1)$, $0 < k < 2^n$, has the dyadic representation* (5.6)*. Then for $2 - a \leq \tau \leq a - 1$ it holds*

$$\phi\left(\frac{\gamma_k + \tau}{a^n}\right) = \varepsilon_k \phi\left(\frac{\tau}{a^n}\right) + \sum_{m=0}^{n-\sigma-1} \varepsilon_{\ell_m} \frac{\tau^m}{m!} b^m a^{\frac{m(m+1-2n)}{2}} \phi\left(\frac{\gamma_{r_m}}{a^{n-m}}\right) \tag{6.1}$$

where ℓ_m is given by (5.7) *and*

$$r_m = d_0 + d_1 2 + \ldots + d_{n-m-1} 2^{n-m-1}, \tag{6.2}$$

i.e. $k = 2^{n-m}\ell_m + r_m$.

Proof: Owing to Proposition 4.1, the function

$$f(\tau) = \phi\left(\frac{\gamma_k + \tau}{a^n}\right) - \varepsilon_k \phi\left(\frac{\tau}{a^n}\right)$$

is a polynomial of degree at most n when $2 - a \leq \tau \leq a - 1$. According to Taylor's formula and $\phi^{(m)}(0) = 0$ for all m, we get

$$f(\tau) = \sum_{m=0}^{n} \frac{1}{m!} \phi^{(m)}(t) \left(\frac{\tau}{a^n}\right)^m$$

where $t = \frac{\gamma_k}{a^n}$. Using Definition 5.1 and (5.4) we get equation

$$f(\tau) = \sum_{m=0}^{n} \delta_m \varepsilon_{\ell_m} \frac{\tau^m}{m!} b^m a^{\frac{m}{2}(m+1-2n)} \phi(a^m t - \gamma_{\ell_m}).$$

With (5.7) and (6.2), equation (5.8) can be written as

$$a^m t - \gamma_{\ell_m} = \frac{\gamma_{r_m}}{a^{n-m}}, \tag{6.3}$$

and the assertion follows from Lemma 5.4 ■

Applying formula (6.1) with even k and $\tau = a - 1$, then in view of $\gamma_k + a - 1 = \gamma_{k+1}$, cf. (1.18), and $b = \frac{a}{a-1}$ we obtain

Corollary 6.2 *Assume that $n \in \mathbb{N}$ and that $k = 2^\sigma(2\kappa+1)$ is even, $0 < k < 2^n$, with the dyadic representation* (5.6)*. Then it holds*

$$\phi\left(\frac{\gamma_{k+1}}{a^n}\right) = \varepsilon_k \phi\left(\frac{\gamma_1}{a^n}\right) + \sum_{m=0}^{n-\sigma-1} \varepsilon_{\ell_m} \frac{1}{m!} a^{\frac{m}{2}(m+3-2n)} \phi\left(\frac{\gamma_{r_m}}{a^{n-m}}\right) \tag{6.4}$$

with the notations (5.7) *and* (6.2).

Note that the values $\phi(\frac{\gamma_1}{a^n})$ can be determined recursively by both formulas of Corollary 3.2 with $\tau = \gamma_1$. Using $\phi(\frac{\gamma_1}{a^n})$ as initial values, all further $\phi(\frac{\gamma_k}{a^n})$ with $1 < k < 2^n$ can be computed recursively by means of (6.4) in view of $\frac{\gamma_{2\ell}}{a^n} = \frac{\gamma_\ell}{a^{n-1}}$.

The formulas (6.4) are recursions for the right end points of the intervals $\overline{G}$ from (1.21). Left end points can be reduced to right ones by means of the symmetry of ϕ with respect to $\frac{1}{2}$.

According to (6.3), Proposition 6.1 with $a = 2$ and $\tau = 1$ yields the

Corollary 6.3 *For $a = 2$, $n \in \mathbb{N}$ and $k = 1, 2, \ldots, 2^n - 1$, we have the equation*

$$\phi\left(\frac{k+1}{2^n}\right) = \varepsilon_k \phi\left(\frac{1}{2^n}\right) + \sum_{m=0}^{n-\sigma-1} \frac{1}{m!} \varepsilon_{\ell_m} 2^{\frac{m}{2}(m+3-2n)} \phi\left(\frac{k}{2^{n-m}} - \ell_m\right), \tag{6.5}$$

where $\ell_m = [\frac{k}{2^{n-m}}]$ and $k = 2^\sigma(2\kappa + 1)$.

Note that after a simple calculation, (6.5) for $k = 1$ and $n + 1$ instead of n yields

$$\phi\left(\frac{1}{2^n}\right) = \frac{1}{1 - 2^{1-n}} \sum_{m=2}^{n} \frac{1}{m!} 2^{\frac{m}{2}(m+1-2n)} \phi\left(\frac{1}{2^{n+1-m}}\right), \tag{6.6}$$

i.e. (3.1) with $a = 2$, cf. [10, (1.14)]. Therefore, (6.5) with the initial value $\phi(\frac{1}{2}) = 2$ is a recursion for all $\phi(\frac{k}{2^n})$ without additional knowledge where it suffices to use it only for even k with $k < 2^{n-2}$, considering the symmetry of ϕ and the relation

$$\phi(t) + \phi\left(\frac{1}{2} - t\right) = 2 \qquad \left(0 \le t \le \frac{1}{2}\right),$$

cf. (1.26) for $n = 0$ and $a = 2$. The first $\phi(\frac{k}{2^n})$ are calculated in [3, p.216].

7 Reduced polynomial representations

The polynomial representation (1.23) for ϕ is rather redundant, since the terms can be reduced by means of (1.12). One reduced formula was already set up in [2, (6.3)], to which we shall come back later on after some preliminaries. Though the following results are valid also for $a = 2$, they are only interesting in the case $a > 2$.

Let k, ℓ, m be even and u, v odd numbers from $\mathbb{N}_0$, such that, for some numbers p, q from $\mathbb{N}$, we have

$$k = 2^p + \ell \quad (0 \le \ell \le 2^p - 2), \qquad k + u = 2^{p+1} - 1 \tag{7.1}$$

and for the same or another odd u

$$u = 2^q + v \quad (1 \le v \le 2^q - 1), \qquad u + m = 2^{q+1} - 1. \tag{7.2}$$

In (7.1) it is always $k \geq 2$, and in (7.2) it is $u \geq 3$. For fixed $n \in \mathbb{N}$ we introduce the notations

$$\varphi_k = \phi\left(\frac{\gamma_k + \tau}{a^{n+1}}\right), \qquad \varphi_u = (-1)^n \phi\left(\frac{\gamma_u + 1 - \tau}{a^{n+1}}\right). \tag{7.3}$$

Proposition 7.1 *In the case* (7.1) *it holds*

$$\varphi_k = -\varphi_\ell + f_{n-p}\left(\frac{\gamma_k + \tau}{a^p}\right), \tag{7.4}$$

$$\varphi_k = (-1)^p \varphi_u + f_{n-p-1}\left(\frac{\gamma_k + \tau}{a^{p+1}}\right), \tag{7.5}$$

when $n \geq p$, *and in the case* (7.2)

$$\varphi_u = (-1)^q \varphi_m + (-1)^{q+1} f_{n-q-1}\left(\frac{\gamma_m + \tau}{a^{q+1}}\right), \tag{7.6}$$

$$\varphi_u = -\varphi_v + (-1)^q f_{n-q}\left(\frac{\gamma_m + \tau}{a^q} - \gamma_1\right), \tag{7.7}$$

when $n \geq q$, *all formulas are valid for*

$$0 \leq \tau \leq a. \tag{7.8}$$

Proof: Relation (7.4) is only another notation for (1.25). Replacing in (1.26) n by $n-p-1$ as well as τ by $\frac{\gamma_k+\tau}{a^{p+1}}$ and considering (1.19) we obtain (7.5). The condition concerning τ is equivalent to

$$a^{p+1}(2-a) - \gamma_k \leq \tau \leq a^{p+1}(a-1) - \gamma_k,$$

and these inequalities are satisfied in view of (7.8), $2 \leq a$, $k \leq 2^{p+1} - 2$, (1.20) and (1.17). Solving (7.5) with respect to φ_u and putting $k = m$ as well as $p = q$ we obtain (7.6). Given (7.2), we can write $u - 1 = k$, $v - 1 = \ell$, and we obtain the first relation of (7.1) with q instead of p. Replacing τ in (7.4) by $a - \tau$, whereby the condition (7.8) remains invariant, and considering

$$\gamma_k + a - \tau = \gamma_u + 1 - \tau$$

(cf.(1.18)) as well as (1.13) with $n - q$ instead of n and (1.19) concerning the second relation of (7.2), we obtain (7.7) ■

In the following we restrict (7.8) to the inequality

$$1 \leq \tau \leq a - 1$$

which includes the condition $a \geq 2$. In view of (1.14), (7.6) and (7.3) it holds for these τ

$$\varphi_0 = f_n(\tau), \qquad \varphi_1 = f_n(\tau - \gamma_1). \tag{7.9}$$

By means of the formulas of Proposition 7.1 we can reduce the index k of φ_k successively down to 1 or 0, where we have the representations (7.9). In this way it is possible to arrive at formulas of the type

$$\varphi_k = \sum_{j=0}^{p+1} \sigma_j f_{n-j}(\cdot) \tag{7.10}$$

with $\sigma_j \in \{-1, 0, 1\}$ and suitable arguments by the polynomials f. It would be sufficient to carry out this reduction only by means of (7.4). Then (7.10) is the already mentioned formula [2, (6.3)] and the signs of the non-vanishing terms in (7.10) alternate. But there are further possibilities.

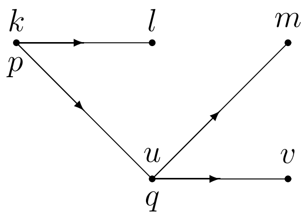

Figure 1: Graph of (7.1) and (7.2)

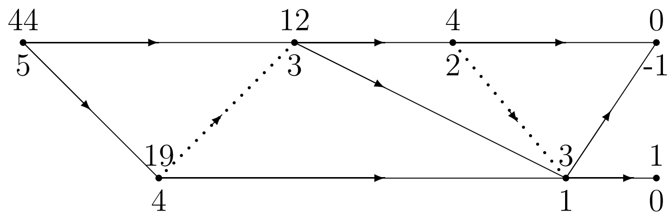

Figure 2: Graph in the case $k = 44$

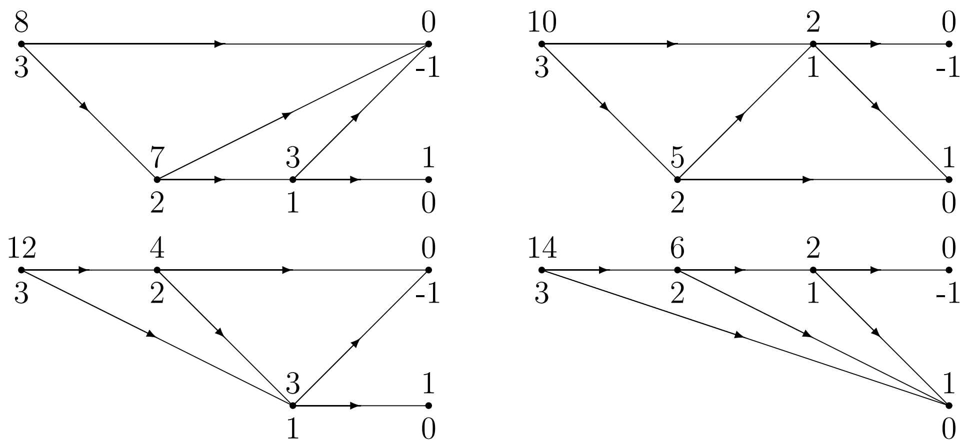

Figure 3: The cases $8 \leq k < 16$

In order to describe them in detail we identify u in (7.1) and (7.2), visualize these relations by the directed graph in Figure 1, and proceed with the nodes ℓ, m, v analogously down to the endpoints 1 and 0, respectively. There are two possibilities to label the nodes, namely, either by means of the numbers k and u (written over the node), or by means of the exponents p and q (written down the node) corresponding to them in (7.1-2). For the end points 1 and 0 we define $q = 0$ in the first and $p = -1$ in the second case. For example, Figure 2 shows the graph in the case $k = 44$ ($p = 5$), and Figure 3 the graphs in the cases $8 \le k < 16$ ($p = 3$). In the following we mainly characterize the nodes by means of the exponential labels p, q.

In general for $p \in \mathbb{N}$, $d_j \in \{0, 1\}$, assume that (5.6) is the dyadic representation of a given even number k with $d_p = 1$ and $d_0 = 0$. For $0 \le j \le p$ we introduce the extended notations

$$k_j = d_j d_{j-1} \dots d_0, \qquad u_j = \overline{d}_j \overline{d}_{j-1} \dots \overline{d}_0 \tag{7.11}$$

with d_j from (5.6) and $\overline{d}_j = 1 - d_j$. The directed graph belonging to $k \ge 2$ has the following structure. It has $p + 2$ nodes $p, \dots, 1, 0, -1$ with the root p and two end points 0, -1. For convenience the nodes j are placed on a first line with the end point -1 when $d_j = 1$, whereas they are placed on a second line with the endpoint 0 when $\overline{d}_j = 1$. The corresponding number (7.11) belonging to a fixed node j is k_j on the first line ($k_{-1} = 0$) and u_j on the second one. Every node, which is no end point, is the start point of exactly two arcs, one to the next smallest j on the same line, and one to the next smallest j on the other line. In particular, for every $j \ge 1$ there is an arc from j to $j - 1$.

Let ℓ be the length of a fixed path from p to one of the end points, obviously $1 \le \ell \le p$, where there always exist two paths of maximal length $\ell = p$. But we are interested in shortest paths.

Proposition 7.2 *(i) For $j = p, p-1, \dots, 1$ let $\widehat{jx_j}$, $\widehat{jy_j}$ be the arcs of the graph belonging to a given even integer k. We get a shortest path, if we choose successively the arcs $\widehat{jz_j}$ with $z_j = \min(x_j, y_j)$.*

(ii) Suppose that in the representation (5.6) of k there are $\ell - 1 \ge 0$ disjunct pairs (d_{j+1}, d_j) of the form $(1, 0)$ or $(0, 1)$ for $j = p - 2, \dots, 1$. Then ℓ is the length of the shortest path.

Proof: (i) Let $\ell(j)$ be the length of a shortest path from p to j, so that $\ell(p) = 0$. Let J be the set of the nodes j belonging to the path with the arcs $\widehat{jz_j}$. This path is a shortest path if Bellman's equation

$$\ell(z_j) = \min_{\widehat{iz_j}} \ell(i) + 1 \tag{7.12}$$

is satisfied for all $j \in J$ with $j \ge 1$, cf. [5, p. 101]. For all these j it is $\max(x_j, y_j) = j - 1$ and therefore $z_j \le j - 2$. This means $z_j + 1 \le j - 1$, where z_j and $z_j + 1$ lie on different lines. Hence, for all $j \in J$ with $j < p$ the nodes j and $j + 1$ lie on different lines. For the first arc $\widehat{pz_p}$ it is $\ell(z_p) = 1 \leq \ell(z_p + 1)$. If $\ell(j) \leq \ell(j+1)$ for a fixed $j \in J$ with $j < p$, then $\ell(z_j) = \ell(j) + 1 = \ell(j-1) \leq \ell(z_j + 1)$, which implies that $\ell(j) \leq \ell(j+1)$ for all $j \in J$ with $j < p$. Moreover, we see that the possible i in (7.12) are either $j, j-1, \ldots, z_j + 1$ or $j+1, j, \ldots, z_j + 1$, cf. Figure 4 or an analogous figure with interchanged lines, and Bellman's equation (7.12) is satisfied indeed.

(*ii*) To every arc $\widehat{jz_j}$ of the just constructed shortest path with $p - 2 \geq z_j \geq 1$ we consider the nodes i with $j > i \geq z_j$. These nodes contain the pair $(z_j + 1, z_j)$ with nodes on different lines, but no other such pairs which are disjoint to $(z_j + 1, z_j)$, cf. Figure 4. These pairs correspond to the pairs (d_{j+1}, d_j) of the proposition. Since we have to consider also the arc with the end point -1 or 0 the number ℓ of all arcs exceeds the number of the just mentioned pairs by 1 ■

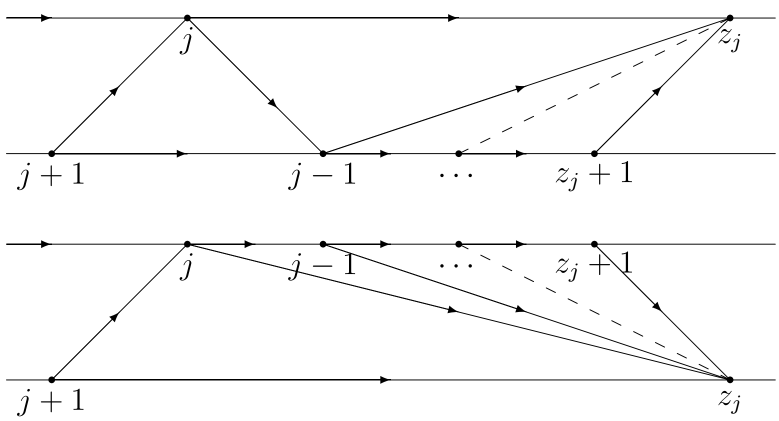

Figure 4: The neighbourhood of $\widehat{jz_j}$

As a simple consequence of Proposition 7.2/(ii) we get

Corollary 7.3 *The length ℓ_k of the shortest path belonging to the even k from* (7.1) *satisfies the estimate*

$$\ell_k \leq \left[\frac{p+1}{2}\right]. \tag{7.13}$$

The smallest numbers k such that $\ell_k = n \in \mathbb{N}$ are $\pi_n = \frac{2}{3}(4^n - 1)$ since these are the numbers $k = 2^p + 2^{p-2} + \ldots + 2 = \frac{2}{3}(2^{p+1} - 1)$ with odd p and $n = \frac{p+1}{2}$.

For a given $k \geq 2$ formula (7.10) or a more complicated formula arises, if we construct the corresponding graph, choose a path from p to -1 or 0 and apply the formulas of Proposition 7.1 as well as (7.9). If we take the path along the first line, then we only have to apply formula (7.4). This possibility is preferable if many of the d_j in (5.6) vanish. In the case that many $\overline{d}_j$ vanish it is preferable to use the path from p down to and then along the second line, i.e. to apply first formula (7.5) and then always (7.7). Another possibility yields the zigzag path, where the formulas (7.5), (7.6) are applied alternately.

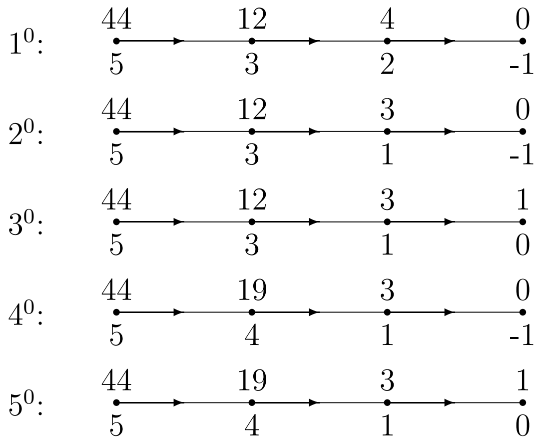

Figure 5: Shortest paths of Figure 2

We call (7.10) a *minimal formula* if we have used a shortest path for the construction. The graph in Figure 2 for $k = 44 = 2^5 + 2^3 + 2^2$ and $u = 19 = 2^4 + 2^1 + 2^0$ has five shortest paths which we obtain if we disregard the dotted arcs, and which are shown in Figure 5. To these shortest paths belong the formulas

$$1^0: \quad \phi\left(\tfrac{\gamma_{44}+\tau}{a^{n+1}}\right) = f_{n-5}\left(\tfrac{\gamma_{44}+\tau}{a^5}\right) - f_{n-3}\left(\tfrac{\gamma_{12}+\tau}{a^3}\right) + f_{n-2}\left(\tfrac{\gamma_4+\tau}{a^2}\right) - f_n(\tau),$$

$$2^0: \quad \phi\left(\tfrac{\gamma_{44}+\tau}{a^{n+1}}\right) = f_{n-5}\left(\tfrac{\gamma_{44}+\tau}{a^5}\right) - f_{n-4}\left(\tfrac{\gamma_{12}+\tau}{a^4}\right) + f_{n-2}\left(\tfrac{\tau}{a^2}\right) - f_n(\tau),$$

$$3^0: \quad \phi\left(\tfrac{\gamma_{44}+\tau}{a^{n+1}}\right) = f_{n-5}\left(\tfrac{\gamma_{44}+\tau}{a^5}\right) - f_{n-4}\left(\tfrac{\gamma_{12}+\tau}{a^4}\right) + f_{n-1}\left(\tfrac{\tau}{a} - \gamma_1\right) - f_n(\tau - \gamma_1),$$

$$4^0: \quad \phi\left(\tfrac{\gamma_{44}+\tau}{a^{n+1}}\right) = f_{n-6}\left(\tfrac{\gamma_{44}+\tau}{a^6}\right) - f_{n-4}\left(\tfrac{\gamma_{12}+\tau}{a^4} - \gamma_1\right) + f_{n-2}\left(\tfrac{\tau}{a^2}\right) - f_n(\tau),$$

$$5^0: \quad \phi\left(\tfrac{\gamma_{44}+\tau}{a^{n+1}}\right) = f_{n-6}\left(\tfrac{\gamma_{44}+\tau}{a^6}\right) - f_{n-4}\left(\tfrac{\gamma_{12}+\tau}{a^4} - \gamma_1\right) + f_{n-1}\left(\tfrac{\tau}{a} - \gamma_1\right) - f_n(\tau - \gamma_1)$$

with $n \geq 5$. The equivalence of these formulas can be checked by means of (1.12). The first formula is that one where only (7.4) is applied. The second one is the formula corresponding to the path of Proposition 7.2 and the last one is that where after the first step only (7.7) is applied. It is remarkable that all these minimal formulas are alternating. The zigzag case $44 - 19 - 12 - 3 - 0$ does not yield a minimal formula.

A minimal formula (7.10) is called *optimal formula*, if the indices j with $\sigma_j \neq 0$ are maximal, i.e. if the degrees of the polynomials are minimal. In the foregoing examples formula 4^0 is optimal. However, since the practical advantage of optimal formulas is small, we do not investigate existence and uniquiness of them.

8 Formulas for a greater domain of a

Finally, we give up the general assumption $a \geq 2$.

8.1. Recursions in the case $a \geq \frac{3}{2}$. First we remark that for $\tau = \frac{1}{2}$ equation (1.26), which is valid for $a \geq \frac{3}{2}$, implies

$$\phi\left(\frac{1}{2a^{2n+1}}\right) = \frac{1}{2} f_{2n}\left(\frac{1}{2}\right) \qquad (n \in \mathbb{N}_0) \tag{8.1}$$

and $n = 0$ yields

$$\phi\left(\frac{1}{2a}\right) = \frac{b}{2} \tag{8.2}$$

in view of $f_0(t) = b$.

Proposition 8.1 *For $a \geq \frac{3}{2}$ and $n \geq 1$ we have the recursions*

$$\phi\left(\frac{1}{2a^{2n+1}}\right) = \frac{1}{2n} \sum_{\nu=1}^{n} \frac{1}{(2\nu)!} B_{2\nu} \frac{a^{\nu(2\nu+1-4n)}}{a^{2\nu} - 1} \phi\left(\frac{1}{2a^{2n-2\nu+1}}\right) \tag{8.3}$$

and

$$\phi\left(\frac{1}{2a^{2n+1}}\right) = \frac{a^{2n}}{a^{2n} - 1} \sum_{\nu=1}^{n} \frac{1 - 2^{1-2\nu}}{(2\nu)!} B_{2\nu} a^{\nu(2\nu-1-4n)} \phi\left(\frac{1}{2a^{2n-2\nu+1}}\right) \tag{8.4}$$

both with the initial value (8.2).

Proof: Substituting (8.1) into (1.11) with $t = \frac{1}{2}$ and $2n$ instead of n, we get (8.3). From (2.1) with $t = s = \frac{1}{2}$ and $2n$ instead of n, we obtain analogously

$$\phi\left(\frac{1}{2a^{2n+1}}\right) = a^{2n} \sum_{\nu=0}^{n} \frac{1}{(2\nu)!} B_{2\nu}\left(\frac{1}{2}\right) a^{\nu(2\nu-1-4n)} \phi\left(\frac{1}{2a^{2n-2\nu+1}}\right),$$

and by means of the well-known relation $B_\nu\left(\frac{1}{2}\right) = -\left(1 - 2^{1-\nu}\right) B_\nu$, cf. [6, p.22], it follows (8.4) ∎

8.2. The maximum value. Equation (1.27) yields for $a \geq \frac{4}{3}$ the relation

$$\phi\left(t - \frac{1}{b}\right) + \phi(t) + \phi\left(t + \frac{1}{b}\right) = b \qquad \left(\frac{2}{a} - 1 \leq t \leq 2 - \frac{2}{a}\right). \tag{8.5}$$

Putting $t = \frac{1}{2}$ in (8.5), we obtain for the maximum value of the solution ϕ of (1.1)-(1.2) that

$$\phi\left(\frac{1}{2}\right) = b - 2\phi\left(\frac{1}{a} - \frac{1}{2}\right) \qquad \left(a \geq \frac{4}{3}\right), \tag{8.6}$$

since $\frac{1}{a} + \frac{1}{b} = 1$ and ϕ is symmetric. In order to give an application for (8.1), we define α_n ($n \in \mathbb{N}$) as the real solution of $a^{2n}(2-a) = 1$ which is different from 1 where $\alpha_1 = \frac{1}{2}(1+\sqrt{5}) = 1.618\ldots$, $\alpha_n < \alpha_{n+1} < 2$ and

$$\alpha_n = 2 - \frac{1}{4^n} + \mathcal{O}\left(\frac{n}{16^n}\right) \qquad (n \to \infty).$$

Hence, by (8.1) with $a = \alpha_n > \frac{3}{2}$, formula (8.6) turns over into the explicit formula

$$\phi\left(\frac{1}{2}\right) = b - f_{2n}\left(\frac{1}{2}\right) \qquad (a = \alpha_n,\ n \in \mathbb{N}).$$

For arbitrary $a > 1$ it follows from (1.27) with $t = \frac{1}{2}$ that the maximum value $\phi(\frac{1}{2})$ has the form

$$\phi\left(\frac{1}{2}\right) = c(a)b$$

where $c(a) = 1$ for $a \geq 2$, and where $0 < c(a) < 1$ for $1 < a < 2$. Moreover, $c(a) \to 0$ as $a \to 1$ in view of

$$c(a) = \frac{1}{b}\phi\left(\frac{1}{2}\right) = 1 - \frac{2}{b}\sum_{\nu=1}^{\infty}\phi\left(\frac{1}{2} - \frac{\nu}{b}\right) \to 1 - 2\int_0^{1/2}\phi(t)dt = 0,$$

where we have used (1.27), $\frac{1}{b} \to 0$, the symmetry of ϕ and (1.2).

On the other side, $\phi(\frac{1}{2}) \to \infty$ as $a \to 1$, since otherwise we would get a contradiction to the solution $\phi(t) = \delta(t - \frac{1}{2})$ of (1.1)-(1.2) for $a = 1$, cf. [1, p.164].

8.3. Special series. We denote by a_p ($p \in \mathbb{N}_0$) the positive solution of $a^p(2-a) = a - 1$. Then $a_0 = \frac{3}{2}$, $a_1 = \alpha_1$ and $a_p < a_{p+1} < 2$. Moreover, it is

$$a_p = 2 - \frac{1}{2^p} + \mathcal{O}\left(\frac{p}{4^p}\right) \qquad (p \to \infty)$$

and $a_{2n} > \alpha_n$ for $n \in \mathbb{N}$.

Lemma 8.2 *For $a \geq a_p$ and $n \in \mathbb{N}_0$ we have*

$$\phi\left(\frac{1}{a^n(a^p+1)}\right) = (-1)^n\phi\left(\frac{a^p}{a^n(a^p+1)}\right) + f_{n-1}\left(\frac{1}{a^p+1}\right). \tag{8.7}$$

Proof: Applying (1.26) with $\tau = \frac{1}{a^p+1}$ and $n-1$ instead of n yields (8.7) in view of $1 - \tau = \frac{a^p}{a^p+1}$, when

$$2 - a \leq \frac{1}{a^p+1} \leq a - 1.$$

The first inequality is equivalent to

$$a^p(2-a) \leq a - 1 \tag{8.8}$$

and therefore valid for $a \geq a_p$. The second inequality is equivalent to $a^p(1-a) \leq a-2$ which follows from (8.8) in view of $a^p > 1$ ■

In the case $a \geq 2$ equation (8.7) is valid for all $p \in \mathbb{N}_0$ since $a_p < 2$. Owing to $\phi(0) = 0$ and (1.13), $p \to \infty$ yields the known formula $\phi\left(\frac{1}{a^n}\right) = f_{n-1}(1)$, cf. (1.14).

Proposition 8.2 *Assume that $a \geq a_p$ with $p \in \mathbb{N}$, $q \in \mathbb{Z}$ and $q \leq p$. Then the solution ϕ of* (1.1)-(1.2) *has the expansion*

$$\phi\left(\frac{a^q}{a^p+1}\right) = -\sum_{\nu=1}^{\infty} \eta_\nu f_{\nu p-q-1}\left(\frac{1}{a^p+1}\right) \tag{8.9}$$

where

$$\eta_\nu = (-1)^{\frac{\nu(\nu+1)p}{2}+\nu q}. \tag{8.10}$$

Proof: For $\nu \in \mathbb{N}$ we have $n = \nu p - q \in \mathbb{N}_0$ and equation (8.7) reads

$$\phi\left(\frac{a^q}{a^{\nu p}(a^p+1)}\right) = (-1)^{\nu p-q}\phi\left(\frac{a^q}{a^{(\nu-1)p}(a^p+1)}\right) + f_{\nu p-q-1}\left(\frac{1}{a^p+1}\right).$$

Multiplication with η_ν from (8.10) yields the relation

$$\eta_\nu\phi_\nu = \eta_{\nu-1}\phi_{\nu-1} + \eta_\nu f_{\nu p-q-1}\left(\frac{1}{a^p+1}\right) \tag{8.11}$$

where $\phi_\nu = \phi(\frac{a^q}{a^{\nu p}(a^p+1)})$. In view of $\eta_0 = 1$ and $\phi_\nu \to 0$ as $\nu \to \infty$ we obtain by summation over $\nu \geq 1$ that

$$0 = \phi_0 + \sum_{\nu=1}^{\infty} \eta_\nu f_{\nu p-q-1}\left(\frac{1}{a^p+1}\right)$$

and this implies the assertion ■

Remark 8.4 **1.** The coefficients η_ν, given by (8.10), are 4-periodic with $\eta_1 = (-1)^{p+q}$, $\eta_2 = (-1)^p$, $\eta_3 = (-1)^q$ and $\eta_4 = 1$. By means of (1.5) and (1.7) it can be shown that, for $0 \leq t \leq 1$, the polynomial f_n satisfies the inequality $|f_n(t)| \leq \frac{1}{n+1}c_n$ with c_n from (1.10). This means that the series (8.9) are rapidly convergent.

2. In the case $a \geq a_1$ equation (8.9) for $p = 1$ and $q = 0$ yields

$$\phi\left(\frac{1}{a+1}\right) = \sum_{\nu=0}^{\infty}(-1)^{\frac{\nu(\nu+3)}{2}} f_\nu\left(\frac{1}{a+1}\right).$$

In view of (1.12) with $t = \frac{a^2}{a+1}$ and (1.13) with $t = \frac{a}{a+1}$ it is easy to see that the foregoing equation is equivalent to

$$\phi\left(\frac{1}{a+1}\right) = \sum_{\nu=0}^{\infty}(-1)^\nu f_{2\nu+1}\left(\frac{a^2}{a+1}\right),$$

i.e. [3, (5.9)] is not only true for $a \geq 2$ but even for $a \geq a_1$.

3. Since the number $x = \frac{1}{a+1}$ has the expansion

$$\frac{1}{a+1} = \gamma_1 \sum_{\nu=1}^{\infty} \frac{1}{a^{2\nu}}$$

it follows by [3, Proposition 4.4] that x belongs to CM. This means for $a \geq 2$ that $\frac{1}{a+1}$ never lies in one of the intervals $\overline{G}_{kn}$, so that $\phi(\frac{1}{a+1})$ cannot be calculated by means of the formulas in [1] or [2]. Analogously, this comes true for the more general left-hand side of (8.9).

References

[1] **Berg, L.**, and **Krüppel, M.** : *On the solution of an integral-functional equation with a parameter.* Z. Anal. Anw. 17 (1998), 159-181

[2] **Berg, L.**, and **Krüppel, M.** : *Cantor sets and integral-functional equations.* Z. Anal. Anw. 17 (1998), 997-1020

[3] **Berg, L.**, and **Krüppel, M.** : *Series expansions for the solution of an integral-functional equation with a parameter.* Results Math. 41 (2001), 213-228

[4] **Flajolet, F., Grabner, P., Prodinger, H.**, and **Tichy, R. F.** : *Mellin transform and asymptotics: digital sums.* Theoret. Comput. Sci. 123 (1994), 291-314

[5] **Jungnickel, D.** : *Graphen, Netzwerke und Algorithmen.* BI Wissenschaftsverlag, Mannheim et. al., 1994

[6] **Nörlund, N. E.** : *Vorlesungen über Differenzenrechnung.* Berlin, 1924

[7] **Rvachev, V. A.** : *Compactly supported solutions of functional-differential equations and their applications.* Russian Math. Surveys 45:1 (1990), 87-120

[8] **Schnabl, R.** : *Über eine C^∞-Funktion.* In: E. Hlawka (Ed.): *Zahlentheoretische Analysis.* Lecture Notes in Mathematics 1114 (1985), 134-142

[9] **Volk, W.** : *Properties of subspaces generated by an infinitely often differentiable function and its translates.* Z. Ang. Math. Mech. (ZAMM) 76 Suppl.1 (1996), 575-576.

[10] **Volk, W.** : *Evaluation of the solution of an integral-functional-equation.* J. Comp. Appl. Math. (to appear)

[11] **Wirsching, G. J. :** *The Dynamical System Generated by the* $3n+1$ *Function.* Lecture Notes in Mathematics 1681 (1998)

[12] **Wirsching, G. J. :** *Balls in constrained urns and Cantor-like sets.* Z. Anal. Anw. 17 (1998), 979-996

received: February 3, 2004

Authors:

Lothar Berg
Universität Rostock
Fachbereich Mathematik
18051 Rostock
Germany

e-mail:
lothar.berg@mathematik.uni-rostock.de

Manfred Krüppel
Universität Rostock
Fachbereich Mathematik
18051 Rostock
Germany

e-mail:
manfred.krueppel@mathematik.uni-rostock.de