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Oscillating Solutions of Rational Difference Equations

ABSTRACT. For a special rational difference equation of order two oscillating series solution are constructed. An example is given where Bessel functions arise as coefficients.

KEY WORDS. Rational difference equation, oscillating solutions, periodic solutions, Bessel functions

A detailed investigation of the rational difference equation

$$x_{n+2} = \frac{\alpha + \beta x_{n+1} + \gamma x_n}{A + B x_{n+1} + C x_n} \quad (n \in \mathbb{N}_0) \tag{1}$$

with non-negative parameters ($A+B+C > 0$) is contained in the book Kulenović and Ladas [3]. Under the conditions

$$A + B + C = \alpha + \beta + \gamma = 1 \tag{2}$$

it has the positive equilibrium $\tilde{x} = 1$, and the corresponding linearized equation has the characteristic equation $D(s) = 0$ with

$$D(s) = s^2 + (B - \beta)s + C - \gamma\,. \tag{3}$$

In the case that the zeros of (3) are real, series solutions of (1) were constructed in [2]. Here, we deal with the case

$$C > \gamma + \frac{1}{4}(\beta - B)^2 \tag{4}$$

where the zeros

$$z = \frac{1}{2}\left(\beta - B + i\sqrt{4(C-\gamma) - (\beta - B)^2}\right) \tag{5}$$

and $\overline{z}$ of (3) are complex, and we construct series solutions which are oscillating. In the following we use the notation

$$g_{jk}(r) = \frac{1 - r^{j+1}}{1 - r}\frac{1 - r^{k+1}}{1 - r} - 1 - r^{j+k} \tag{6}$$

with $j, k \in \mathbb{N}_0$. Moreover, we put $r = |z|$ where (2), (4) and (5) imply that $0 < r = \sqrt{C-\gamma} \le 1$. Some calculations were carried out by means of the DERIVE system.

Proposition 1 *Under the assumptions* (2), (4) *and* $r < 1$ *the difference equation* (1) *has the solution*

$$x_n = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} c_{jk} a^j z^{nj} \overline{a}^k \overline{z}^{nk} \tag{7}$$

with $c_{00} = c_{10} = c_{01} = 1$*, an arbitrary complex* a*, and*

$$c_{jk} = -\frac{1}{D(z^j \overline{z}^k)} {\sum_{\mu=0}^{j}}' {\sum_{\nu=0}^{k}}' c_{\mu\nu} c_{j-\mu,k-\nu} z^\mu \overline{z}^\nu (B z^j \overline{z}^k + C z^\mu \overline{z}^\nu) \tag{8}$$

for $j+k \ge 2$*, where the primes at the sums shall indicate that the pairs* $(0,0)$ *and* (j,k) *are excluded for* (μ,ν)*. The series* (7) *converges for*

$$\lambda |a| r^n < 1 \tag{9}$$

where

$$\lambda = \sup_{j+k\ge 2} \frac{1}{|D(z^j \overline{z}^k)|} \left(B r^{j+k} g_{jk}(r) + C g_{jk}(r^2) \right) . \tag{10}$$

Proof: Writing (1) in the form

$$x_{n+2}(A + Bx_{n+1} + Cx_n) = \alpha + \beta x_{n+1} + \gamma x_n$$

and replacing x_n by means of (7) with $c_{00} = c_{10} = c_{01} = 1$, we obtain by comparing coefficients that the coefficients c_{jk} can be determined recursively by (8), whereas a remains arbitrary.

In order to prove the convergence condition (9) we show that

$$|c_{jk}| \le \lambda^{j+k-1} \tag{11}$$

for $j+k \ge 1$. This estimate is valid in the case $j+k = 1$. Assuming that $|c_{\mu\nu}| \le \lambda^{\mu+\nu-1}$ is valid for $0 \le \mu \le j$, $0 \le \nu \le k$ but $1 \le \mu+\nu < j+k$, then (8) implies the estimate

$$|c_{jk}| \le \frac{1}{|D(z^j \overline{z}^k)|} \left(B r^{j+k} g_{jk}(r) + C g_{jk}(r^2) \right) \lambda^{j+k-2} ,$$

and (11) is proved by induction in view of (10) $\square$

The coefficients of (7) satisfy $c_{jk} = \overline{c}_{jk}$. Writing $z = re^{i\varphi}$, $c_{jk} a^j \overline{a}^k = \varrho_{jk} e^{i\vartheta_{jk}}$ and using $c_{jk} a^j z^{nj} \overline{a}^k \overline{z}^{nk} + c_{kj} a^k z^{nk} \overline{a}^j \overline{z}^{nj} = 2\varrho_{jk} r^{(j+k)n} \cos(n\varphi(j-k) + \vartheta_{jk})$, we see that the solution (7) oscillates around the equilibrium 1 when $a \ne 0$.

The estimate (9) implies that (7) converges at least for

$$n > \frac{\ln(\lambda|a|)}{\ln \frac{1}{r}} .$$

Proposition 2 *The supremum* (10) *allows the estimate*

$$\lambda \leq \frac{2(Br+C)}{(1-r)^2}\,. \tag{12}$$

Proof: We use the abbreviations $x = r^j$, $y = r^k$. In view of $D(s) = (z-s)(\overline{z}-s)$ we have

$$\left|D\left(z^j\overline{z}^k\right)\right| \geq (r-xy)^2\,,$$

so that (12) is valid if we show that both

$$0 \leq \frac{2r}{(1-r)^2} - \frac{xy}{(r-xy)^2}\left(\frac{(1-rx)(1-ry)}{(1-r)^2} - 1 - xy\right) \tag{13}$$

and

$$0 \leq \frac{2}{(1-r)^2} - \frac{1}{(r-xy)^2}\left(\frac{(1-r^2x^2)(1-r^2y^2)}{(1-r^2)^2} - 1 - x^2y^2\right)\,. \tag{14}$$

The right-hand side of (13) can be written as

$$ry(1-x)(r-x) + rx(1-y)(r-y) + r(r-y)(r-xy) + (r-x)(r^2-xy^2) \tag{15}$$

divided by the positive denominator $(1-r)^2(r-xy)^2$, and the right-hand side of (14) as

$$r^2(x-y)^2 + 3(r^2-xy)^2 + 4r(r^2-xy)(1-xy) \tag{16}$$

divided by the positive denominator $(1-r^2)^2(r-xy)^2$. In view of $xy \leq r^2 < 1$ the expression (16) is always non-negative. For both $x \leq r$ and $y \leq r$ also the expression (15) is non-negative. In the case $x = 1$ and $y \leq r^2$ the expression (15) can be written as

$$r(r-y)^2 + (r-y)(r^2-y)$$

so that it is also non-negative and, in view of the symmetry of (14), also the case $x \leq r^2$ and $y = 1$ is settled □

The right-hand sides of (13) and (14) vanish for $x = y = r$.

Example 3 Pielou's equation

$$x_{n+2} = \frac{2x_{n+1}}{1+x_n}\,,$$

cf. [3, Theorem 4.4.1 (b)], is a special case of (1), (2) with the non-vanishing coefficients $A = C = \frac{1}{2}$, $\beta = 1$. Hence, $z = \frac{1}{2}(1+i)$ with $r = \frac{1}{\sqrt{2}}$, and

$$c_{20} = \frac{1}{5}(2-i)\,, \quad c_{11} = 0\,, \quad c_{30} = \frac{1}{15}(1-2i)\,, \quad c_{21} = \frac{1}{5}(1+2i)\,.$$

The estimate (12) seems to be rather bad because it only yields $\lambda \leq 6 + 4\sqrt{2}$.

The case $r > 1$ is impossible in view of (2). If $r = 1$ without z being a root of unity, then the coefficients (8) exist, but the convergence of (7) is an open problem. If z is a root of unity, then the general solution of (1) is periodic and we do not need the expansion (7). A special example is Lyness' equation with $C = 1$ and $\alpha = \beta^2$ having 5-periodic solutions, cf. [3, p. 71].

A further one is

Example 4 with $C = \beta = 1$, i.e.

$$x_{n+2} = \frac{x_{n+1}}{x_n} \tag{17}$$

and 6-periodic solutions, cf. [3, p. 48]. The general positive solution of (17) reads

$$x_n = \exp(az^n + \overline{a}\,\overline{z}^n) \tag{18}$$

with $z = e^{\frac{i\pi}{3}}$ and an arbitrary complex constant a. In this case the corresponding expansion (7) has the coefficients $c_{jk} = \frac{1}{j!k!}$ and it can be written in a finite form. In order to show this we introduce the notation $a = \varrho e^{i\vartheta}$ and write it first as

$$x_n = \sum_{\ell=-\infty}^{+\infty} \mathrm{I}_\ell(2\varrho) \exp\left[i\left(\frac{\pi n}{3} + \vartheta\right)\ell\right] \tag{19}$$

with the Bessel functions

$$\mathrm{I}_\ell(2\varrho) = \sum_{k=0}^{\infty} \frac{1}{k!(k+\ell)!}\varrho^{\ell+2k}.$$

Setting $\ell = 6\mu + \nu$, expression (19) turns over into the finite Fourier sum

$$x_n = \sum_{\nu=0}^{5} C_\nu(\varrho, \vartheta) \exp\left[i\left(\frac{\pi n}{3} + \vartheta\right)\nu\right] \tag{20}$$

with

$$C_\nu(\varrho, \vartheta) = \sum_{\mu=-\infty}^{+\infty} \mathrm{I}_{6\mu+\nu}(2\varrho) \exp(6i\vartheta\mu). \tag{21}$$

The series (21) converges in view of

$$\mathrm{I}_\ell(2\varrho) = \mathrm{I}_{-\ell}(2\varrho) \sim \frac{\varrho^\ell}{\ell!}$$

as $\ell \to \infty$. The coefficients in (20) can be simplified using the discrete Fourier transform as in [1, p. 1073].

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