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Somashekhar Naimpally

Proximity and Hyperspace Topologies

Dedicated to my friend Professor Dr. Harry Poppe on his 70th birthday.

ABSTRACT. In this paper we give a survey of the use of proximities in hyperspace topologies. A proximal hypertopology corresponding to a LO-proximity is a g eneralization of the well known Vietoris topology. In case we start with an EF-proximity, the proximal hypertopology equals the Hausdorff uniform topology corresponding to the totally bounded uniformity and, being contained in both the Vietoris and Hausdorff uniform topologies, serves as a bridge between the two. Wattenberg and Beer-Himmelberg-Prickry-Van Vleck showed that the locally finite hypertopology induced by a metrizable space is the sup of the Hausdorff metric topologies induced by all compatible metrics. Naimpally-Sharma showed that this follows from the fact that a Tychonoff space is normal iff its fine uniformity induces the locally finite hypertopology. Di Concilio-Naimpally-Sharma showed that in a Tychonoff space the fine uniformity induces the proximal locally finite hypertopology.

We study DELTA topologies introduced by Poppe, and their proximal variations. We show that a short proof can be given of the Beer-Tamaki result concerning the uniformizability of (proximal) DELTA hypertopologies via the Attouch-Wets approach used by Beer in dealing with the Fell topology. Finally we present a result concerning (Proximal) DELTA-U-hypertopolgies. Several new hypertopologies are introduced.

KEY WORDS AND PHRASES. proximity, hyperspace, Δ-topology, proximal Δ-topology, U-topology, ΔU-topology, proximal ΔU-topology, Function space, Vietoris topology, Fell topology, Hausdorff uniformity.

1 Introduction

Suppose (X, T) (respectively (X, V)) is a T_1 topological space (respectively a uniform space). Then it is well known that on $\mathrm{CL}(X)$, the hyperspace of all non-empty closed subsets of X, one can define Vietoris topology $\tau(\mathrm{V})$ (respectively a Hausdorff uniformity V_{H}) such that X

is topologically (respectively, uniformly) embedded in $\mathrm{CL}(X)$. But it is not known how one can define directly a proximity on the hyperspace of a given proximity space (X, δ). Nachman ([21]) tackled this problem in the case of an EF-proximity δ on X via Hausdorff uniformities associated with compatible uniformities on X. An attempt was made to use proximity in hyperspaces in [16] and a little later in [4]. Since the paper [16] remains unpublished and the paper [4] dealt with proximities in the context of metric spaces, an impression continues in the literature that proximal hypertopologies exist only in metric spaces. The aim of this paper is to correct this impression and show that proximal topologies can be defined using LO-proximities in any T_1 space. Recently there has been some work done in the general case. (See e. g. [9], [10]) See [1] for the latest results on compactness in function spaces via hyperspaces and (uniform) convergence structures.

(X, T) denotes a T_1-topological space and δ denotes any compatible LO-proximity on X. The symbol δ_0 denotes the fine LO-proximity and it is well known (Urysohn Theorem) that it is EF iff X is normal. If (X, T) is Tychonoff, then we generally choose δ to be EF. $\mathrm{CL}(X)$ denotes the family of all non-empty closed subsets of X and $\mathrm{K}(X)$ denotes the family of all non-empty compact subsets. We use the symbol Δ to denote a subfamily of $\mathrm{CL}(X)$ and we assume, without any loss of generality, that it is a cover of X and is closed under finite unions and contains all singletons.

For any set $E \subset X$ and $\mathsf{E} \subset \mathsf{T}$ we use the following notation:

$E^- = \{A \in \mathrm{CL}(X) : A \cap E \neq \emptyset\}$

$\mathsf{E}^- = \{A \in \mathrm{CL}(X) : A \cap E \neq \emptyset\}$ for each $E \in \mathsf{E}\}$

$E^{++} = \{A \in \mathrm{CL}(X) : A \ll E$ w. r. t. δ i. e. $A \underline{\delta} E^c\}$

$E^+ = \{A \in \mathrm{CL}(X) : A \subset E$ i. e. $A \ll E$ w. r. t. $\delta_0\}$

The **Δ-topology** $\tau(\Delta)$ is generated by a basis of the form $E^+ \vee \mathsf{E}^-$, where $E^c \in \Delta$ and $\mathsf{E} \subset \mathsf{T}$ is finite. ([26], [27])

The **proximal Δ-topology** (w. r. t. δ) $\sigma(\delta\Delta)$ is generated by a basis of the form $E^{++} \vee \mathsf{E}^-$, where $E^c \in \Delta$ and $\mathsf{E} \subset \mathsf{T}$ is finite. We omit δ if it is obvious from the context.

If in the above, the family E is locally finite, then we have the **locally finite Δ-topology** $\tau(\mathrm{LF}\Delta)$ and the **proximal locally finite Δ-topology** (w. r. t. δ) $\sigma(\mathrm{LF}\delta\Delta)$.

The **ΔU-topology** $\tau(\Delta\mathrm{U})$ is generated by a basis of the form $E^+ \vee \mathsf{E}^-$, where $E^c \in \Delta$ or $\mathrm{cl}E \in \Delta$ and $\mathsf{E} \subset \mathsf{T}$ is finite.

The **proximal ΔU-topology** (w. r. t. δ) $\sigma(\delta\Delta\mathrm{U})$ is generated by a basis of the form $E^{++} \vee \mathsf{E}^-$, where $E^c \in \Delta$ or $\mathrm{cl}E \in \Delta$ and $\mathsf{E} \subset \mathsf{T}$ is finite.

If in the above, the family E is locally finite, then we have the **locally finite ΔU-topology** $\tau(\mathrm{LF}\Delta\mathrm{U})$ and the **proximal locally finite ΔU-topology** (w. r. t. δ) σ $(\mathrm{LF}\delta\Delta\mathrm{U})$.

Well known special cases are:

(a) when $\Delta = \mathrm{CL}(X)$, $\tau(\Delta) = \tau(\mathrm{V})$ the **Vietoris or finite topology** ([20])
$\sigma(\delta\Delta) = \sigma(\delta)$ the **proximal topology** ([16])
$\tau(\mathrm{LF}\Delta) = \tau(\mathrm{LF})$ the **locally finite topology** ([19])
$\sigma(\mathrm{LF}\delta\Delta) = \sigma(\mathrm{LF}\delta)$ the **proximal locally finite topology** ([16])

To make the notation simpler, we'll omit δ from all proximal topologies whenever it is understood from the context: thus we'll use σ for $\sigma(\delta)$, $\sigma(\mathrm{LF})$ for $\sigma(\mathrm{LF}\delta)$ etc.

(b) When $\Delta = \mathrm{K}(X)$, $\tau(\Delta) = \tau(\mathrm{F})$ the **Fell topology** ([17])
and we define three new ones
$\sigma(\delta\Delta) = \sigma(\delta\mathrm{F})$ the **proximal Fell topology**
$\tau(\mathrm{LF}\Delta) = \tau(\mathrm{LFF})$ the **locally finite Fell topology**
$\sigma(\mathrm{LF}\delta\Delta) = \sigma(\mathrm{LF}\delta\mathrm{F})$ the **proximal locally finite Fell topology**
$\tau(\Delta\mathrm{U}) = \tau(\mathrm{U})$ the **U-topology** ([8])
and we define three new ones
$\sigma(\delta\Delta\mathrm{U}) = \sigma(\delta\mathrm{U})$ the **proximal U-topology**
$\tau(\mathrm{LF}\Delta\mathrm{U}) = \tau(\mathrm{LFU})$ the **locally finite U-topology**
$\sigma(\mathrm{LF}\delta\Delta\mathrm{U}) = \sigma(\mathrm{LF}\delta\mathrm{U})$ the **proximal locally finite U-topology**

Of course, if the proximity δ is EF or R (and so X is Tychonoff or regular respectively) then $\tau(\mathrm{F}) = \sigma(\mathrm{F})$, $\tau(\mathrm{LFF}) = \sigma(\mathrm{LFF})$, $\tau(\mathrm{U}) = \sigma(\mathrm{U})$ and $\tau(\mathrm{LFU}) = \sigma(\mathrm{LFU})$.

Many interesting properties of the Fell topology stem from the fact that it is also a proximal topology!. In generalizing results from the Fell topology to Δ-topologies, we find that some hold for $\tau(\Delta)$ and others for $\sigma(\Delta)$!!

(c) If (X,d) is a metric space, δ is the metric proximity induced by d and Δ denotes the ring generated by closed balls of non-negative radii, then

$\tau(\Delta) = \tau(\mathrm{B})$ the **Ball topology** ([2])
$\sigma(\Delta) = \sigma(\mathrm{B})$ the **proximal Ball topology** ([14])
and we introduce two new ones
$\tau(\mathrm{LF}\Delta) = \tau(\mathrm{LFB})$ the **locally finite Ball topology**,
$\sigma(\mathrm{LF}\Delta) = \sigma(\mathrm{LFB})$ the **proximal locally finite Ball topology**

In addition we have the well known **Hausdorff metric** d_{H} and the **Hausdorff metric topology** $\tau(d_{\mathrm{H}})$.

If (X,V) is a uniform space, then we have the **Hausdorff uniformity** V_{H} and the **Hausdorff uniform topology** $\tau(\mathsf{V}_{\mathrm{H}})$.

[2] is a standard reference on hyperspace topologies and we give below other relevant bibliography for the interested reader.

2 VIETORIS, PROXIMAL AND (PROXIMAL) LOCALLY FINITE TOPOLOGIES

Suppose (X, T) is a T_1 topological space, δ any compatible LO-proximity on X and δ_0 the fine LO-proximity. If (X, T) is Tychonoff, the fine EF-proximity is denoted by $\delta^{\#}$ (the functionally indistinguishable EF-proximity), the fine uniformity is denoted by $\mathsf{V}^{\#}$, and the finest totally bounded uniformity is denoted by V^{*}. If δ is a compatible EF-proximity on X, then $\Pi(\delta)$ is the family of all uniformities which induce δ and V_{ω} denotes the coarsest totally bounded member of $\Pi(\delta)$ ([25]). We note that since, in general, there are many proximities compatible with (X, T), proximal hypertopologies provide us with a large number of hypertopologies. For further details see [16].

Theorem 2.1 ([16]) (a) $\tau(\mathrm{V}) = \sigma(\delta_0)$

(b) $\tau(\mathrm{LF}) = \sigma(\mathrm{LF}\delta_0)$

(c) $\sigma \subset \sigma(\mathrm{LF})$ *and* $\tau(\mathrm{V}) \subset \tau(\mathrm{LF})$.

In each case $\subset$ *is replaced by* $=$ *if and only if* X *is feebly compact (i. e. every locally finite family of open sets in* X *is finite).*

(d) *In general* $\tau(\mathrm{V})$ *and* σ *are independent.*

(e) *If* $\delta < \delta'$ *and* δ *is* EF, *then* $\sigma(\delta) \subset \sigma(\delta')$ *and* $\sigma(\mathrm{LF}\delta) \subset \sigma.(\mathrm{LF}\delta')$.

Consequently, $\sigma \subset \tau(\mathrm{V})$.

(f) *If* δ *is EF and* $\delta \neq \delta_0$, *then* $\sigma \neq \tau(\mathrm{V})$ *and* $\sigma(\mathrm{LF}) \neq \tau(\mathrm{LF})$.

Corollary 2.2 ([16]) *If* δ *is* EF, *then* (a), (b) *and* (c) *are mutually equivalent and each implies* (d)*:*

(a) $\tau(\mathrm{V}) = \sigma$

(b) $\tau(\mathrm{LF}) = \sigma(\mathrm{LF})$

(c) $\delta = \delta_0$

(d) (X, T) *is normal.*

Corollary 2.3 ([4]) *If (X, d) is a metric space and δ is the metric proximity, then the following are equivalent:*

(a) *(X, d) is Atsuji or* UC

(i. e. every real valued continuous function on X is uniformly continuous.)

(b) $\tau(\mathrm{V}) \subset \tau(d_{\mathrm{H}})$

(c) $\delta = \delta_0$

(d) $\tau(\mathrm{V}) = \sigma$

Theorem 2.4 ([16]) *Suppose (X, δ) is an* EF*-proximity space and V and V_ω are in $\Pi(\delta)$. Then*

(a) $\sigma = \tau(\mathsf{V}_{\omega\mathrm{H}}) \subset \tau(\mathsf{V}_{\mathrm{H}}) \subset \sigma(\mathrm{LF}\delta^{\#}) \subset \tau(\mathrm{LF})$ *and* $\sigma = \tau(\mathsf{V}_{\mathrm{H}})$ *implies* $\mathsf{V} = \mathsf{V}_\omega$.

It follows that if (X, T) is normal, then $\tau(\mathrm{V}) = \tau(\mathsf{V}^*_{\mathrm{H}}) \subset \tau(\mathrm{LF})$.

(b) $\sigma = \tau(\mathsf{V}_{\omega\mathrm{H}}) \subset \tau(\mathrm{V}) \subset \tau(\mathrm{LF})$

(c) $\sigma(\mathrm{LF}\delta^{\#}) = \tau(\mathsf{V}^*_{\mathrm{H}})$

(d) $\sigma(\delta^{\#}) = \tau(\mathsf{V}^*_{\mathrm{H}})$

Corollary 2.5 ([24]) *The following are equivalent:*

(a) *(X, T) is normal.*

(b) *δ_0 is* EF.

(c) $\tau(\mathsf{V}^{\#}_{\mathrm{H}}) = \tau(\mathrm{LF})$.

(d) $\sigma(\delta^{\#}) = \tau(\mathrm{LF})$.

The following Hesse diagram shows the various relationships:

$$\begin{tikzcd}
& \tau(\mathrm{LF}) = \sigma(\mathrm{LF}\delta_0)\,. \arrow[dl, dash] \arrow[dr, dash] & \\
\sigma(\mathrm{LF}\delta^{\#}) = \tau(\mathsf{V}^{\#}_{\mathrm{H}}) \arrow[dr, dash] & & \tau(\mathrm{V}) = \sigma(\delta_0) \arrow[dl, dash] \\
& \sigma(\delta^{\#}) = \tau(\mathsf{V}^*_{\mathrm{H}}) \arrow[d, dash] & \\
& \sigma = \tau(\mathsf{V}_{\omega\mathrm{H}}) &
\end{tikzcd}$$

Remark 2.6 In [5] it was shown that if X is a metrizable space, the locally finite topology $\tau(\mathrm{LF})$ on $\mathrm{CL}(X)$ is the sup of the Hausdorff metric topologies $\{\tau(d_{\mathrm{H}})\}$ corresponding to equivalent compatible metrics $\{d\}$ on X. This result was generalized in ([24]) to : a Tychonoff space X is normal if and only if the locally finite topology $\tau(\mathrm{LF})$ equals $\tau(\mathsf{V}^{\#}_{\mathrm{H}})$, the topology induced by the Hausdorff uniformity corresponding to the fine uniformity $\mathsf{V}^{\#}_{\mathrm{H}}$. The question then arises: in a non-normal Tychonoff space what is $\tau(\mathsf{V}^{\#}_{\mathrm{H}})$? The answer was provided in ([16]) that it is precisely the proximal locally finite topology $\sigma(\mathrm{LF}\delta^{\#})$ induced by the fine proximity $\delta^{\#}$ on X. This shows the importance of proximal topologies in this problem.

3 (PROXIMAL) DELTA TOPOLOGIES

Poppe ([26], [27]) first studied Δ-topologies as generalizations of the Fell topology. On the other hand, many workers in this area have used, in the context of metric spaces, *bounded* sets to study new hyperspace topologies e.g. the bounded Vietoris (proximal) topology ([12]), Attouch-Wets topology ([3]) etc. It is not widely known that boundedness can also be defined in general topological spaces in an abstract way ([18]) and this provides a technique to give simple proofs and generalizations of several results in this area. In this section we give a glimpse of this approach and refer the interested readers to ([15]) for further information.

A boundedness $\mathbf{H}$ in a T_1-space (X, T) is a non-empty family of subsets of X which contains finite unions and subsets of its members. Well known examples of $\mathbf{H}$ include

(a) metrically bounded subsets of a metric space

(b) the family of subsets of compact sets in a topological space

(c) the family of all totally bounded subsets of a uniform space etc.

In what follows we'll usually take $\Delta\mathbf{H} \cap \mathrm{CL}(X)$. Then $\tau(\Delta)$ is a generalization of the **bounded Vietoris topology**. We note here that the **upper Δ-topology** $\tau^+(\Delta)$ is generated by $\{E^+ : E^c \in \Delta\}$ and we have similar definitions of other "upper" topologies.

Theorem 3.1 (cf. [12]) *Suppose* (X, T) *is a* T_1*-topological space and* Δ, Δ' *are two subrings of* $\mathrm{CL}(X)$.

Then the following are equivalent on $\mathrm{CL}(X)$:

(a) $\tau(\Delta) \subset \tau(\Delta')$

(b) $\tau^+(\Delta) \subset \tau^+(\Delta')$

(c) *For each* $B \in \Delta - \{X\}$, $B \subset U \in \mathsf{T}$ *implies the existence of* $B' \in \Delta'$ *such that* $B \subset B' \subset U$.

Corollary 3.2 *Suppose* $\mathbf{H}$ *is a boundedness in a is a* T_1*-topological space* (X, T) *such that* $\mathrm{K}(X) \subset \mathbf{H}$ *and* $\Delta \mathbf{H} \cap \mathrm{CL}(X)$. *Then the following are equivalent:*

(a) $\tau(\mathrm{F}) = \tau(\Delta)$ *on* $\mathrm{CL}(X)$,

(b) $\Delta - \{X\} \subset \mathrm{K}(X)$,

(c) X *is boundedly compact.*

We have the well known results:

Corollary 3.3 (a) *If* (X, d) *is a metric space, then on* $\mathrm{CL}(X)$ *the Fell topology equals the bounded Vietoris topology if and only if* X *is boundedly compact.*

Replacing the metric d *by the equivalent bounded metric* $d' = \min\{d, 1\}$, *we get the result:*

(b) $\tau(\mathrm{F}) = \tau(\mathrm{V})$ *on* $\mathrm{CL}(X)$ *if and only if* X *is compact.*

There are analogous results using abstract boundedness in proximity and uniform spaces and we refer to ([15]).

We now prove some simple results that will generalize relations between $\tau(\mathrm{F})$, $\tau(\mathrm{V})$ *as well as between* $\tau(\mathrm{F})$, $\tau(\mathsf{V}_H)$. *Proofs in the following are similar to those in* (3.1).

Theorem 3.4 *Suppose* (X, T) *is a* T_1*-space and suppose* $\Delta \subset \mathrm{CL}(X)$ *is a ring containing* $\mathrm{K}(X)$. *Then the following are equivalent:*

(a) $\tau(\Delta) = \tau(\Delta\mathrm{U})$

(b) $\tau^+(\Delta) = \tau^+(\Delta\mathrm{U})$

(c) $X \in \Delta$.

(d) $\tau(\Delta) = \tau(\mathrm{V})$.

Theorem 3.5 *Suppose* (X, V) *is a Hausdorff uniform space with a compatible* EF*-proximity* δ. *Suppose* $\Delta \subset \mathrm{CL}(X)$ *is a ring containing* $\mathrm{K}(X)$. *Then the following are equivalent:*

(a) $\sigma(\Delta) = \sigma(\Delta\mathrm{U})$

(b) $\sigma^+(\Delta) = \sigma^+(\Delta\mathrm{U})$

(c) $X \in \Delta$.

(d) $\sigma(\Delta) = \tau(\mathsf{V}_{\mathrm{H}})$

Remark 3.6 As we noted above, $\tau(\mathrm{F}) = \sigma(\mathrm{F})$ and so the two above results generalize the well known result that the following are equivalent:

(a) $\tau(\mathrm{F}) = \tau(\mathrm{V})$

(b) $\tau(\mathrm{F}) = \tau(\mathsf{V}_{\mathrm{H}})$

(c) X is compact.

4 UNIFORMIZATION OF (PROXIMAL) DELTA TOPOLOGIES

Beer and Tamaki ([6], [7]) investigated necessary and sufficient conditions for the unformizability of (proximal) Δ-topologies. Their proof involves construction of special Urysohn functions. In ([22]) we study these and (proximal) ΔU-topologies and provide an alternate approach.

Suppose (X, T) is a Tychonoff space with a compatible EF-proximity δ and suppose $\Delta \subset \mathrm{CL}(X)$ is a ring.

(a) Δ is called **proximally Urysohn** iff whenever $D \in \Delta$ and $A \in \mathrm{CL}(X)$ are far w. r. t. δ then there exists an $S \in \Delta$ such that $D \ll S \subset A^c$.

It is easy to see that the above definition is equivalent to one where the last relation is replaced by $D \ll S \ll A^c$.

Δ is called **Urysohn** if Δ is **proximally Urysohn** w. r. t. the LO-proximity δ_0.

(b) Δ is called a **local family** iff for each $x \in X$ and $V \in \mathsf{T}$ with $x \in V$, implies the existence of a $D \in \Delta$ such that $x \in D^o \subset D \subset V$.

(c) For each $K \in \Delta$ and $W \in \mathsf{V}$, we set

$$[K, W] = \{(A, B) \in \mathrm{CL}(X) \times \mathrm{CL}(X) : A \cap K \subset W(B) \text{ and } B \cap K \subset W(A)\}.$$

The family $\{[K, W] : K \in \Delta \text{ and } W \in \mathsf{V}\}$ is a base for a uniformity on $\mathrm{CL}(X)$ called the **Attouch-Wets uniformity** V_Δ.

The proof of the following result due to Beer and Tamaki uses the Attouch-Wets technique developed by Beer in ([7]) for studying the Fell topology.

Theorem 4.1 *Suppose (X, T) is a Tychonoff space with a compatible EF-proximity δ, V_ω the unique totally bounded uniformity compatible with δ. Suppose Δ is a local proximally Urysohn family. Then the proximal Δ-topology $\sigma(\Delta)$ on $\mathrm{CL}(X)$ is the topology $\tau(\mathsf{V}_{\omega\Delta})$ induced by the Δ-Attouch-Wets uniformity $\mathsf{V}_{\omega\Delta}$ and hence is Tychonoff.*

Conversely, if $\sigma(\Delta)$ is Tychonoff then Δ is a local proximally Urysohn family.

Corollary 4.2 *Suppose (X, T) is a Tychonoff space. Then Δ is a local Urysohn family if and only if $(\mathrm{CL}(X), \tau(\Delta))$ is Tychonoff.*

We conclude with a characterization of completely regular proximal $\Delta\mathrm{U}$-topology $\sigma(\Delta\mathrm{U})$.

Theorem 4.3 *Suppose (X, T) is a Tychonoff space with a compatible LO-proximity δ and Δ is a proximally Urysohn family. Then δ' defined by*

$$A\underline{\delta}'B \quad \textit{iff} \quad \mathrm{cl}\,A \in \Delta \quad \textit{or} \quad \mathrm{cl}\,B \in \Delta \quad \textit{and} \quad A\underline{\delta}B \tag{$*$}$$

is a compatible EF-proximity on X. Further $\delta' \leq \delta$ and $\sigma(\Delta\mathrm{U}) = \sigma(\delta')$.

Corollary 4.4 *Suppose (X, T) is a Tychonoff space and Δ is a Urysohn family. Then δ' defined by*

$$A\underline{\delta}'B \quad \textit{iff} \quad \mathrm{cl}\,A \in \Delta \quad \textit{or} \quad \mathrm{cl}\,B \in \Delta \quad \textit{and} \quad A\underline{\delta_0}B \tag{$**$}$$

is a compatible EF-proximity on X . Further $\delta' \leq \delta_0$ and $\tau(\Delta\mathrm{U}) = \sigma(\delta')$.

Thus $\tau(\Delta\mathrm{U})$ is completely regular.

Corollary 4.5 *Suppose (X, T) is a locally compact Hausdorff space. Then the U-topology $\tau(\mathrm{U})$ is the proximal topology corresponding to the EF-proximity δ_1 induced by the one-point-compactification of X viz:*

$$A\underline{\delta_1}B \quad \textit{iff} \quad \mathrm{cl}\,A \quad \textit{or} \quad \mathrm{cl}\,B \quad \textit{is compact and} \quad A\underline{\delta_0}B\,. \tag{$***$}$$

Remark 4.6 (a) It is interesting to note that in (4.5) if (X, T) is not normal, then δ_0 is not EF but the proximity δ_1 induced by $\Delta = \mathrm{K}(X)$ is indeed EF!

(b) If (X, T) is not locally compact, then the proximity δ_1 as defined by $(***)$ is not EF but is a compatible LO-proximity on X. In this case for any compatible EF-proximity δ on X, $\delta_1 < \delta$ but $\sigma(\delta_1) \not\subset \sigma(\delta)$. (Cf. (2.1)(e)).

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Author:

Somashekhar Naimpally
96 Dewson Street
Toronto, Ontario
M6H 1H3 CANADA

e-mail: sudha@accglobal.net