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Katharina Habermann, Andreas Klein

Lie derivative of symplectic spinor fields, metaplectic representation, and quantization

ABSTRACT. In the context of Riemannian spin geometry it requires skilful handling to define a Lie derivative of (Riemannian) spinor fields.

A Lie derivative of symplectic spinor fields in the direction of Hamiltonian vector fields can be defined in a very natural way. It is the aim of this note to present this construction. Furthermore, an immediate interpretation of this Lie derivative in the language of natural ordering quantization is given.

Introduction

In the context of Riemannian spin geometry, the general question of constructing a Lie derivative for spinor fields has been studied by several authors. Yvette Kosmann, for instance, gave a geometric construction of a so-called metric Lie derivative of spinor fields in [12]. This approach was extended by Jean–Pierre Bourguignon and Paul Gauduchon in [2]. The problem with it is to compare spinor fields for different metrics, since a diffeomorphism ϕ transforms the metric tensor g to ϕ^*g and the (Riemannian) spinor fields over (M, g) will be transformed into spinor fields over (M, ϕ^*g). Other studies focussed on relations between Killing vector fields and Killing spinors such as [14] by Andrei Moroianu and [1] Dmitri Alekseevsky *et al.* A further result in this direction was the finding of Katharina Habermann that conformal vector fields act by a certain kind of conformal Lie derivative on the space of solutions of the twistor equation. In [7] she discussed the relevant $\mathbb{Z}_2$-graded algebra.

Studing the problem in the symplectic setting, one deals with symplectic spinor fields over (M, ω) and $(M, \phi^*\omega)$, respectively. In the case of a Hamiltonian vector field all spinor fields live over the same symplectic manifold and a definition of a Lie derivative for symplectic spinor fields in the direction of a Hamiltonian vector field in the classical way of defining a Lie derivative for geometrical objects is possible. It is the aim of this note to present this construction.

Furthermore, an immediate interpretation of this Lie derivative in the language of natural ordering quantization is given. This interpretation was inspired by Theorem 1 in the book [6] of Maurice de Gosson. The observation is that there is a one-parameter group of metaplectic operators, which is associated to a quadratic Hamiltonian and gives solutions of a Schrödinger equation. A similar Schrödinger equation but without any spinorial context was established in the book [5] of Victor Guillemin and Shlomo Sternberg. Moreover, a detailed discussion of this Schrödinger equation can be found in the mentioned book of Maurice de Gosson. In this paper, we put the Schrödinger equation in the context of symplectic spin geometry and give a new and completely self-contained proof. Finally, the Schrödinger equation gives the Lie derivative of constant symplectic spinor fields on $\mathbb{R}^{2n}$ in the direction of the Hamiltonian vector field associated to the quadratic Hamiltonian.

Altogether, our computations also illustrate a remark of Bertram Kostant in his paper on symplectic spinors. There, symplectic spinor fields were introduced in order to give the construction of the half-form bundle and the half-form pairings in the context of geometric quantization. These half–densities are related to a certain line subbundle of the symplectic spinor bundle, which sometimes is also known as metaplectic correction. And Bertram Kostant notices that Hamiltonian vector fields clearly operate as Lie differentiation on smooth symplectic spinor fields ([13] 5.5).

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1 Preparations

1.1 Some Notations

We consider the standard space $\mathbb{R}^{2n}$ with the Euklidean product $\langle\ ,\ \rangle$. Further, let J be the $2n \times 2n$-matrix given by

$$J = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} ,$$

where $\mathbf{1}$ denotes the $n \times n$-matrix $\mathbf{1} = diag(1, \dots, 1)$. Then the standard symplectic structure ω_0 on $\mathbb{R}^{2n}$ is defined to be

$$\omega_0(\ ,\ ) = \langle J\ ,\ \rangle .$$

We remark that for local coordinates $(p, q) = (p_1, \dots, p_n, q_1, \dots, q_n)$ on $\mathbb{R}^{2n}$ the standard symplectic structure ω_0 writes as

$$\omega_0 = \sum_{j=1}^{n} dp_j \wedge dq_j \ .$$

For the canonical standard basis $\{a_1, \dots, a_n, b_1, \dots, b_n\}$ of $\mathbb{R}^{2n}$ one computes readily

$$\omega_0(a_j, a_k) = 0 \ , \quad \omega_0(b_j, b_k) = 0 \ , \quad \text{and} \quad \omega_0(a_j, b_k) = \delta_{jk} \qquad \text{for} \quad j, k = 1, \dots, n \ .$$

This says that $\{a_1, \dots, a_n, b_1, \dots, b_n\}$ is a symplectic basis of the symplectic vector space $(\mathbb{R}^{2n}, \omega_0)$.

The symplectic group $Sp(2n, \mathbb{R})$ is the group of real $2n \times 2n$-matrices leaving the standard symplectic structure ω_0 on $\mathbb{R}^{2n}$ invariant, i.e. the group $Sp(2n, \mathbb{R})$ consists of those real $2n \times 2n$-matrices A satisfying the relation

$$A^\top J A = J \ . \tag{1.1}$$

Thus, the Lie algebra $\mathfrak{sp}(2n, \mathbb{R})$ of the symplectic group is given by the space of all real $2n \times 2n$-matrices B with

$$B^\top J + J B = 0 \ . \tag{1.2}$$

Moreover, let

$$B_{jk} = \begin{array}{cl} \begin{pmatrix} 0 & \vdots & 0 \\ \dots & 1 & \dots \\ 0 & \vdots & 0 \end{pmatrix} & \leftarrow j\text{-th row} \\ \uparrow k\text{-th column} & \end{array}$$

be the $n \times n$-matrix with a 1 as the only nonvanishing entry at the j-th row and the k-th column for $j, k = 1, \dots, n$. Using these $n \times n$-matrices, we introduce the following $2n \times 2n$-matrices

$$X_{jk} = \begin{pmatrix} B_{jk} & 0 \\ 0 & -B_{kj} \end{pmatrix} \ , \quad Y_{jk} = \begin{pmatrix} 0 & B_{jk} + B_{kj} \\ 0 & 0 \end{pmatrix} \ , \quad \text{and} \quad Z_{jk} = \begin{pmatrix} 0 & 0 \\ B_{jk} + B_{kj} & 0 \end{pmatrix}$$

for $j, k = 1, \dots, n$. Now, it is a well known fact that the set

$$\{Y_{jk} \text{ and } Z_{jk} \text{ for } 1 \leq j \leq k \leq n \ , \ X_{jk} \text{ for } 1 \leq j, k \leq n\}$$

of $2n \times 2n$-matrices is a basis of the symplectic Lie algebra $\mathfrak{sp}(2n, \mathbb{R})$.

1.2 The Metaplectic Representation and symplectic Clifford multiplication

This section recalls well known basics on the metaplectic group and its representation. See also [10, 15].

For the symplectic group, the subgroup $Sp(2n,\mathbb{R}) \cap O(2n,\mathbb{R}) \cong U(n)$ is maximal compact. This implies $\pi_1(Sp(2n,\mathbb{R})) \cong \mathbb{Z}$ for the fundamental group of $Sp(2n,\mathbb{R})$. Consequently, the symplectic group has a – up to isomorphism – uniquely determined covering group of order 2. The metaplectic group $Mp(2n,\mathbb{R})$ is defined to be this two-fold covering group of $Sp(2n,\mathbb{R})$, giving the exact sequence

$$1 \quad \to \quad \mathbb{Z}_2 \quad \to \quad Mp(2n,\mathbb{R}) \quad \overset{\rho}{\to} \quad Sp(2n,\mathbb{R}) \quad \to \quad 1$$

with double covering map ρ. For our computations, it is sufficient to know the differential $\rho_* : \mathfrak{mp}(2n,\mathbb{R}) \to \mathfrak{sp}(2n,\mathbb{R})$ of this double covering. Due to Crumeyrolle [3], the Lie algebra of the metaplectic group is given by the set of all symmetric homogeneous polynomials of degree 2 in the elements of $\mathbb{R}^{2n}$. Thus, the set

$$\{a_j \cdot a_k \text{ and } b_j \cdot b_k \text{ for } 1 \le j \le k \le n \ , \ a_j \cdot b_k + b_k \cdot a_j \text{ for } 1 \le j,k \le n\}$$

is a basis of the metaplectic Lie algebra $\mathfrak{mp}(2n,\mathbb{R})$. This Lie algebra may be represented as a Lie subalgebra of the symplectic Clifford algebra. So we write formally $v \cdot w$ for the polynomial given by the two vectors v and w. Later, this notation will be consistent with the Clifford multiplication of vectors and functions.

Then one proves (cf. [9] Proposition 1.2)

Lemma 1.1 *The differential $\rho_* : \mathfrak{mp}(2n,\mathbb{R}) \to \mathfrak{sp}(2n,\mathbb{R})$ is given by $\rho_*(a_j \cdot a_k) = -Y_{jk}$, $\rho_*(b_j \cdot b_k) = Z_{jk}$, and $\rho_*(a_j \cdot b_k + b_k \cdot a_j) = 2\, X_{jk}$ for $j,k = 1,\dots,n$.* □

The Schrödinger quantization prescription

$$\begin{aligned}
1 \in \mathbb{R} \quad &\mapsto \quad \sigma(1) := \text{multiplication by } i, \\
a_j \in \mathbb{R}^{2n} \quad &\mapsto \quad \sigma(a_j) := \text{multiplication by } ix_j \ , \qquad \text{and} \\
b_j \in \mathbb{R}^{2n} \quad &\mapsto \quad \sigma(b_j) := \frac{\partial}{\partial x_j} \qquad\qquad \text{for} \quad j = 1,\dots,n \ ,
\end{aligned}$$

where the operators $\sigma(1)$, $\sigma(a_j)$, and $\sigma(b_j)$ for $j = 1,\dots,n$ are continuous operators acting on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing smooth functions on $\mathbb{R}^n$, gives the symplectic

Clifford multiplication

$$\begin{array}{rcl}\mu : \mathbb{R}^{2n} \times \mathcal{S}(\mathbb{R}^n) & \to & \mathcal{S}(\mathbb{R}^n) \\ (v, f) & \mapsto & \mu(v, f) = v \cdot f := \sigma(v)f \ .\end{array}$$

It is an elementary computation to prove the relation

$$v \cdot w \cdot f - w \cdot v \cdot f = -i\omega_0(v, w)f$$

for vectors $v, w \in \mathbb{R}^{2n}$ and functions $f \in \mathcal{S}(\mathbb{R}^n)$.

The metaplectic group has a natural representation acting on the Hilbert space $L^2(\mathbb{R}^n)$. A concrete realization of this representation is given by the following specification (cf. [10]).

Consider $g(a) = \left(\sqrt{\det(a)}, \begin{pmatrix} a & 0 \\ 0 & (a^\top)^{-1} \end{pmatrix}\right)$ where $a \in GL(n, \mathbb{R})$. Choosing a square root of $\det(a)$, one has $g(a) \in Mp(2n, \mathbb{R})$ and

$$(L(g(a))f)(x) = \sqrt{\det(a)} f(a^\top x) \ , \quad x \in \mathbb{R}^n \ . \tag{1.3}$$

The set of all matrices $\tau(b) = \begin{pmatrix} \mathbf{1} & b \\ 0 & \mathbf{1} \end{pmatrix}$, where $b^\top = b$ and $\mathbf{1}$ denotes the $n \times n$-matrix $\mathbf{1} = diag(1, \ldots, 1)$ is simply connected. Thus, $\tau(b)$ can be understood as an element of $Mp(2n, \mathbb{R})$, such that $t(0)$ is the unit element in $Mp(2n, \mathbb{R})$. For $\tau(b)$ it is

$$(L(\tau(b))f)(x) = \mathrm{e}^{-\frac{i}{2}\langle bx, x\rangle} f(x) \ , \quad x \in \mathbb{R}^n \ . \tag{1.4}$$

Choosing a square root $i^{1/2}$, the element $\sigma = (i^{1/2}, J)$ can be considered as an element $\sigma \in Mp(2n, \mathbb{R})$. Here, one obtains

$$(L(\sigma)f)(x) = \left(\frac{i}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \mathrm{e}^{i\langle x, y\rangle} f(y) dy \ , \quad x \in \mathbb{R}^n \ . \tag{1.5}$$

That gives $L(\sigma) = i^{\frac{n}{2}} \mathcal{F}^{-1}$, where $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ denotes the usual Fourier transform. Finally, we remark that the metaplectic group is generated by all these types of elements, since the corresponding matrices in $Sp(2n, \mathbb{R})$ already give the whole symplectic group.

With respect to this representation, the symplectic Clifford multiplication is $Mp(2n, \mathbb{R})$-equivariant, i.e. we have the relation

$$\mu(\rho(g)v, L(g)f) = L(g)\mu(v, f)$$

for all $g \in Mp(2n, \mathbb{R})$, $v \in \mathbb{R}^{2n}$, and $f \in \mathcal{S}(\mathbb{R}^n)$.

The differential of the metaplectic representation is more interesting for our computations. In order to be able to give precise calculations, we are going to deduce the differential detailly.

Proposition 1.2 *The differential $L_* : \mathfrak{mp}(2n,\mathbb{R}) \to \mathfrak{u}(\mathcal{S}(\mathbb{R}^n))$ of the metaplectic representation L is given by*

$$\begin{aligned} L_*(a_j \cdot a_k)(f) &= -ia_j \cdot a_k \cdot f \\ L_*(b_j \cdot b_k)(f) &= -ib_j \cdot b_k \cdot f \\ L_*(a_j \cdot b_k + b_k \cdot a_j)(f) &= -i(a_j \cdot b_k + b_k \cdot a_j) \cdot f \end{aligned}$$

for $j, k = 1, \ldots, n$.

Proof: Generally, the differential $L_* : \mathfrak{mp}(2n,\mathbb{R}) \to \mathfrak{u}(\mathcal{S}(\mathbb{R}^n))$ may be computed via

$$L_*(X)f = \frac{d}{dt}(L(\exp(tX))f_{|t=0} \; .$$

We will make use of this formula in the progress of this proof.

First, the relation $\rho(\exp(tX)) = \exp(t\rho_*(X))$ gives

$$\begin{aligned} \rho(\exp(t\, a_j \cdot a_k)) &= \exp(t\rho_*(a_j \cdot a_k)) \quad = \quad \exp(-tY_{jk}) \\ &= \exp\begin{pmatrix} 0 & -t(B_{jk}+B_{kj}) \\ 0 & 0 \end{pmatrix} \quad = \quad \begin{pmatrix} 1 & -t(B_{jk}+B_{kj}) \\ 0 & 1 \end{pmatrix} , \end{aligned}$$

$$\begin{aligned} \rho(\exp(t\, b_j \cdot b_k)) &= \exp(t\rho_*(b_j \cdot b_k)) \quad = \quad \exp(tZ_{jk}) \\ &= \exp\begin{pmatrix} 0 & 0 \\ t(B_{jk}+B_{kj}) & 0 \end{pmatrix} \quad = \quad \begin{pmatrix} 1 & 0 \\ t(B_{jk}+B_{kj}) & 1 \end{pmatrix} \\ &= J\begin{pmatrix} 1 & -t(B_{jk}+B_{kj}) \\ 0 & 1 \end{pmatrix} J^{-1} , \end{aligned}$$

and

$$\begin{aligned} \rho(\exp(t(a_j \cdot b_k + b_k \cdot a_j))) &= \exp(t\rho_*(a_j \cdot b_k + b_k \cdot a_j)) \quad = \quad \exp(2t\, X_{jk}) \\ &= \exp\begin{pmatrix} 2tB_{jk} & 0 \\ 0 & -2tB_{kj} \end{pmatrix} \\ &= \begin{pmatrix} \exp(2tB_{jk}) & 0 \\ 0 & \exp(-2tB_{jk})^\top \end{pmatrix} . \end{aligned}$$

Thus,

$$\exp(t\, a_j \cdot a_k) = \tau(-t(b_{jk} + B_{kj})) \; ,$$

$$\exp(t\, b_j \cdot b_k) = \sigma\, \tau(-t(B_{jk} + B_{kj}))\, \sigma^{-1},$$

and

$$\exp(t(a_j \cdot b_k + b_k \cdot a_j)) = g(\exp(2tB_{jk})) \; .$$

Finally, this gives

$$\begin{aligned}
(L_*(a_j\cdot a_k)f)(x) &= \frac{d}{dt}(L(\exp(t\,a_j\cdot a_k))f)(x)_{|t=0}\\
&= \frac{d}{dt}e^{\frac{i}{2}t\langle(B_{jk}+B_{kj})x,x\rangle}f(x)_{|t=0}\\
&= \frac{i}{2}\langle(B_{jk}+B_{kj})x,x\rangle f(x)\\
&= ix_jx_k\,f(x) \quad = \quad -ia_j\cdot a_k\cdot f(x)\ ,
\end{aligned}$$

$$\begin{aligned}
(L_*(b_j\cdot b_k)f)(x) &= \frac{d}{dt}(L(\exp(t\,b_j\cdot b_k))f)(x)_{|t=0}\\
&= \frac{d}{dt}(L(\sigma)\circ L(\tau(-t(B_{jk}+B_{kj})))\circ L(\sigma)^{-1}(f))(x)_{|t=0}\\
&= i\mathcal{F}^{-1}(x_jx_k\mathcal{F}(f))(x)\\
&= -i\frac{\partial^2 f}{\partial x_j\partial x_k}(x) \quad = \quad -ib_j\cdot b_k\cdot f(x)\ ,
\end{aligned}$$

and

$$\begin{aligned}
(L_*(a_j\cdot b_k+b_k\cdot a_j)f)(x) &= \frac{d}{dt}(L(\exp(t(a_j\cdot b_k+b_k\cdot a_j)))f)(x)_{|t=0}\\
&= \frac{d}{dt}\sqrt{\det(\exp(2tB_{jk}))}\,f(\exp(2tB_{jk})^\top x)_{|t=0}\\
&= \frac{1}{2}\frac{d}{dt}\det(\exp(2tB_{jk}))_{|t=0}\ f(x)+\frac{d}{dt}f(\exp(2tB_{jk})^\top x)_{|t=0}\\
&= \frac{1}{2}Tr\left(\frac{d}{dt}\exp(2tB_{jk})_{|t=0}\right)\ f(x)+df\left(\frac{d}{dt}\exp(2tB_{jk})^\top_{|t=0}x\right)\\
&= \frac{1}{2}Tr(2B_{jk})f(x)+df(2B_{kj}x)\\
&= \delta_{jk}f(x)+2x_j\frac{\partial f}{\partial x_k}(x)\\
&= x_j\frac{\partial f}{\partial x_k}(x)+\frac{\partial}{\partial x_k}x_jf(x) \quad = \quad -i(a_j\cdot b_k+b_k\cdot a_j)\cdot f(x)\ ,
\end{aligned}$$

which are the asserted relations. □

1.3 Symplectic Spinor Fields

Let (M,ω) be a $2n$-dimensional symplectic manifold and R the $Sp(2n,\mathbb{R})$-principal fibre bundle of all symplectic frames over M. A metaplectic structure on (M,ω) is a principal fibre bundle P over M having $Mp(2n,\mathbb{R})$ as structure group together with a bundle morphism $f:P\to R$ which is equivariant with respect to the homomorphism $\rho: Mp(2n,\mathbb{R})\to Sp(2n,\mathbb{R})$.

That is, we have the following commutative diagram

$$\begin{tikzcd} P \times Mp(2n,\mathbb{R}) \arrow[r] \arrow[d, "f \times \rho"] & P \arrow[d, "f"] \arrow[dr] & \\ & & M \\ R \times Sp(2n,\mathbb{R}) \arrow[r] & R \arrow[ur] & \end{tikzcd}$$

such that a metaplectic structure can be understood as a lift of the symplectic frame bundle R with respect to the double covering ρ.

Generally, one has a cohomological obstruction to lifting the structure group of a principal fibre bundle. The topological condition to the existence of a metaplectic structure is given by $c_1(M) \equiv 0 \bmod 2$.

If (M, ω) is a $2n$-dimensional symplectic manifold with fixed metaplectic structure P then the symplectic spinor bundle is defined to be the associated Hilbert bundle

$$\mathcal{Q} = P \times_L L^2(\mathbb{R}^n).$$

Furthermore, we need the subbundle

$$\mathcal{S} = P \times_L \mathcal{S}(\mathbb{R}^n).$$

Observing that the symplectic Clifford multiplication is $Mp(2n, \mathbb{R})$-equivariant, it lifts to the bundle level to a symplectic Clifford multiplication

$$\begin{aligned} \mu : TM \otimes \mathcal{S} &\rightarrow \mathcal{S} \\ X \otimes \varphi &\mapsto \mu(X, \varphi) = X \cdot \varphi \end{aligned}$$

on the symplectic spinor bundle $\mathcal{S}$. Obviously, we have the relation

$$X \cdot Y \cdot \varphi - Y \cdot X \cdot \varphi = -i\omega(X, Y)\varphi$$

for vector fields X, Y and spinor fields φ.

Furthermore, the $L^2(\mathbb{R}^n)$-scalar product on the fibres gives a canonical Hermitian scalar product $\langle\ ,\ \rangle$ on $\mathcal{Q}$. $\Gamma(\mathcal{Q}) = \Gamma(\mathcal{S})$ denotes the space of all smooth symplectic spinor fields.

Moreover, any symplectic covariant derivative on the tangent bundle TM of (M, ω) induces a covariant derivative on the symplectic spinor bundle $\mathcal{Q}$, the spinor derivative

$$\nabla : \Gamma(\mathcal{Q}) \rightarrow \Gamma(T^*M \otimes \mathcal{Q}),$$

which in the following will also be denoted by ∇. If $e_1, \ldots, e_n, f_1, \ldots, f_n$ denotes any local symplectic frame on (M, ω) then the spinor derivative writes as

$$\nabla_X \varphi = X(\varphi) + \frac{i}{2} \sum_{j=1}^{n} \{e_j \cdot \nabla_X f_j - f_j \cdot \nabla_X e_j\} \cdot \varphi\ . \tag{1.6}$$

Here a covariant derivative $\nabla : \Gamma(TM) \to \Gamma(T^*M \otimes TM)$ on a symplectic manifold (M, ω) is called symplectic if and only if $\nabla\omega = 0$. The torsion of such a connection is defined to be

$$T^\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

Then the connection is said to be torsionfree, if and only if $T^\nabla \equiv 0$.

Finally, for the Clifford multiplication, the spinor derivative, and the Hermitian scalar product we have the following relations

$$\begin{array}{rcl} (X \cdot Y - Y \cdot X) \cdot \varphi &=& -i\omega(X,Y)\varphi \\ \langle X \cdot \varphi, \psi \rangle &=& -\langle \varphi, X \cdot \psi \rangle \\ \nabla_X (Y \cdot \varphi) &=& (\nabla_X Y) \cdot \varphi + Y \cdot \nabla_X \varphi \\ X\langle \varphi, \psi \rangle &=& \langle \nabla_X \varphi, \psi \rangle + \langle \varphi, \nabla_X \psi \rangle \\ \langle \varphi, \psi \rangle &=& \overline{\langle \psi, \varphi \rangle}. \end{array}$$

1.4 Symplectic Spinor Fields and Diffeomorphisms

In order to define the Lie derivative of symplectic spinor fields we first illustrate how symplectic spinor fields behave under diffeomorphisms. In Riemannian spin geometry, the problem of transforming a spinor field under diffeomorphisms of the manifold is studied in detail in the paper [4] of Dabrowski and Percacci. This method can be carried over to our situation of symplectic spinor fields.

Let (M, ω) be a $2n$-dimensional symplectic manifold and let ϕ be any orientation preserving diffeomorphism of M. Then ϕ induces an isomorphism ϕ_* of the $Sp(2n, \mathbb{R})$-principal frame bundles R^ϕ and R according to the symplectic structures $\phi^*\omega$ and ω

$$\begin{array}{rcl} \phi_* &:& R^\phi \to R \\ && (e_1, \dots, e_n, f_1, \dots, f_n) \mapsto (\phi_* e_1, \dots, \phi_* e_n, \phi_* f_1, \dots, \phi_* f_n)\ . \end{array}$$

This isomorphism maps symplectic frames with respect to $\phi^*\omega$ to symplectic frames for the symplectic structure ω.

Let (P, f) be a fixed metaplectic structure for (M, ω). Moreover, (P^ϕ, f^ϕ) denotes the metaplectic structure for $(M, \phi^*\omega)$ such that ϕ_* lifts to an isomorphism $\tilde{\phi}_* : P^\phi \to P$, i.e. such that the following diagramm commutes

$$\begin{array}{ccccc} & & \tilde{\phi}_* & & \\ & P^\phi & \to & P & \\ f^\phi & \downarrow & & \downarrow & f\ . \\ & R^\phi & \to & R & \\ & & \phi_* & & \end{array}$$

Let $\mathcal{Q} = P \times_L L^2(\mathbb{R}^n)$ and $\mathcal{Q}^\phi = P^\phi \times_L L^2(\mathbb{R}^n)$ denote the corresponding symplectic spinor bundles. A symplectic spinor field over (M, ω) is a section of the symplectic spinor bundle $\mathcal{Q}$, or, equivalently, an L-equivariant map $\varphi : P \to L^2(\mathbb{R}^n)$. Now, we define the transformed symplectic spinor field $(\phi^{-1})_*\varphi$ by the equation

$$(\phi^{-1})_*\varphi = \varphi \circ \tilde{\phi_*} : P^\phi \to L^2(\mathbb{R}^n) \, ,$$

where this spinor field also is regarded as an L-equivariant map. Then $(\phi^{-1})_*\varphi$ is a symplectic spinor field over $(M, \phi^*\omega)$ with respect to the metaplectic structure (P^ϕ, f^ϕ).

Obviously, ϕ is a symplectomorphism between the symplectic manifolds (M, ω) and $(M, \phi^*\omega)$. Thus, if ∇ is any symplectic covariant derivative on (M, ω) then ∇^ϕ defined by

$$\nabla^\phi_{(\phi^{-1})_*X}(\phi^{-1})_*Y = (\phi^{-1})_*(\nabla_X Y)$$

for vector fields X and Y gives a symplectic covariant derivative for $(M, \phi^*\omega)$. This implies that the induced spinor derivative in $\mathcal{Q}^\phi$ which we also denote by ∇^ϕ satisfies

$$\nabla^\phi_{(\phi^{-1})_*X}(\phi^{-1})_*\varphi = (\phi^{-1})_*(\nabla_X \varphi).$$

Furthermore,

$$((\phi^{-1})_*X) \cdot ((\phi^{-1})_*\varphi) = (\phi^{-1})_*(X \cdot \varphi)$$

holds true for the symplectic Clifford multiplication.

2 The Lie Derivative of Symplectic Spinor Fields

In this section, we will define the Lie derivative of symplectic spinor fields in the direction of Hamiltonian vector fields. This can be done in a very natural way.

Let (M, ω) be a symplectic manifold. A vector field X over M is called Hamiltonian vector field if there is a smooth function $h : M \to \mathbb{R}$ such that

$$\omega(X, \quad) = dh \, .$$

The Hamiltonian vector field given by a function h often is denoted also by X_h. Further, let $\mathcal{L}_X$ denote the Lie derivative in the direction of X. Then, the well known relation

$$\mathcal{L}_X = d \circ i_X + i_X \circ d$$

gives

$$\mathcal{L}_X\omega = d \circ i_X\omega + i_X \circ d\omega = d(\omega(X, \quad)) = ddh = 0.$$

For the sake of simplicity, we assume M to be a closed manifold. For the Hamiltonian vector field X, let $\{\phi_t : M \to M\}_{t\in\mathbb{R}}$ be the one-parameter transformation group of diffeomorphisms induced by X, i.e. we have

$$X(x) = \frac{d}{dt}\phi_t(x)_{|t=0} \qquad \text{for } x \in M\ .$$

Then, $\mathcal{L}_X\omega = 0$ gives

$$\phi_t^*\omega = \omega \qquad \text{for } t \in \mathbb{R}.$$

Let $\mathcal{Q}$ and $\mathcal{S}$ denote the symplectic spinor bundles with respect to a fixed metaplectic structure P over (M, ω).

In section 1.4 we gave a description how a diffeomorphism $\phi : M \to M$ for a given symplectic spinor field $\varphi \in \Gamma(\mathcal{Q})$ over (M, ω) induces a symplectic spinor field $(\phi^{-1})_*\varphi$ over $(M, \phi^*\omega)$. Since $\phi_t^*\omega = \omega$, in our situation each $({\phi_t}^{-1})_*\varphi$ is a symplectic spinor field over (M, ω), i.e. lies in $\Gamma(\mathcal{Q})$. This allows the following definition.

Definition 2.1 *The Lie derivative of the symplectic spinor field $\varphi \in \Gamma(\mathcal{Q})$ in the direction of the Hamiltonian vector field X is defined to be*

$$\mathcal{L}_X\varphi \ = \ \frac{d}{dt}(\phi_t^{-1})_*\varphi_{|t=0},$$

where $\{\phi_t\}_{t\in\mathbb{R}}$ denotes the one-parameter transformation group induced by X.

Recalling the construction of $(\phi^{-1})_* : \Gamma(\mathcal{Q}) \to \Gamma(\mathcal{Q}^\phi)$ in section 1.4, one sees that $(\phi^{-1})_*$ is determined only up to sign. For this reason we additionally require $(\phi_0^{-1})_* = \mathrm{id}_{\Gamma(\mathcal{Q})}$ for the smooth family of mappings $(\phi_t^{-1})_* : \Gamma(\mathcal{Q}) \to \Gamma(\mathcal{Q})$.

Proposition 2.2 *Let ∇ be any torsionfree symplectic connection on (M, ω) and let X be any fixed Hamiltonian vector field. Then the Lie derivative of symplectic spinor fields in the direction of X can be expressed in the following form*

$$\mathcal{L}_X\varphi = \nabla_X\varphi + \frac{i}{2}\sum_{j=1}^{n}\{\nabla_{e_j}X \cdot f_j - \nabla_{f_j}X \cdot e_j\} \cdot \varphi \qquad \textit{for } \varphi \in \Gamma(\mathcal{Q}),$$

where $e_1, \ldots, e_n, f_1, \ldots, f_n$ denotes any local symplectic frame on (M, ω).

Proof: First, one has the relation

$$(\mathcal{L}_X\omega)(Y, Z) = X(\omega(Y, Z)) - \omega([X, Y], Z) - \omega(Y, [X, Z])$$

for vector fields X, Y, Z. Let $e_1, \ldots, e_n, f_1, \ldots, f_n$ be a local symplectic frame on (M, ω) and $\varphi \in \Gamma(\mathcal{Q})$ a symplectic spinor field. Then one obtains for the torsionfree symplectic connection ∇

$$\begin{aligned}\sum_{j=1}^{n}\{\omega(\nabla_{e_j}X, f_j) + \omega(e_j, \nabla_{f_j}X)\} &= \sum_{j=1}^{n}\{\omega(\nabla_X e_j, f_j) - \omega([X, e_j], f_j) \\ &\qquad +\omega(e_j, \nabla_X f_j) - \omega(e_j, [X, f_j])\} \\ &= \sum_{j=1}^{n}\{X(\omega(e_j, f_j)) - \omega([X, e_j], f_j) - \omega(e_j, [X, f_j])\} \\ &= \sum_{j=1}^{n}(\mathcal{L}_X\omega)(e_j, f_j) \\ &= 0, \end{aligned} \tag{2.7}$$

by $\mathcal{L}_X\omega = 0$.

Let $\overline{s} : U \to P$ be a lift of the local symplectic frame $s = (e_1, \ldots, e_n, f_1, \ldots, f_n) : U \to R$ into the metaplectic structure. We consider the sections

$$s_t = ((\phi_t^{-1})_* e_1, \ldots, (\phi_t^{-1})_* e_n, (\phi_t^{-1})_* f_1, \ldots, (\phi_t^{-1})_* f_n) : \phi_t^{-1}(U) \to R \quad \text{for } t \in \mathbb{R}$$

and lifts $\overline{s}_t : \phi_t^{-1}(U) \to P$ of s_t, such that $\overline{s}_t$ gives a smooth family satisfying $\overline{s}_0 = \overline{s}$. If φ is locally given by $\varphi_{|U} = [\overline{s}, u]$ then

$$(\phi_t^{-1})_* \varphi_{|\phi_t^{-1}(U)} = [\overline{s}_t, u \circ \phi_t].$$

Furthermore, we have mappings $g_t : U \cap \phi_t^{-1}(U) \to Mp(2n, \mathbb{R})$ given by

$$\overline{s}_t = \overline{s} g_t \ .$$

With

$$\begin{aligned}&((\phi_t^{-1})_* e_1, \ldots, (\phi_t^{-1})_* e_n, (\phi_t^{-1})_* f_1, \ldots, (\phi_t^{-1})_* f_n) = \\ &\qquad = (e_1, \ldots, e_n, f_1, \ldots, f_n)\begin{pmatrix} \omega((\phi_t^{-1})_* e_l, f_k) & \omega((\phi_t^{-1})_* f_l, f_k) \\ \omega(e_k, (\phi_t^{-1})_* e_l) & \omega(e_k, (\phi_t^{-1})_* f_l)\end{pmatrix}_{k,l=1,\ldots,n}\end{aligned}$$

one derives

$$\rho(g_t) = \begin{pmatrix} \omega((\phi_t^{-1})_* e_l, f_k) & \omega((\phi_t^{-1})_* f_l, f_k) \\ \omega(e_k, (\phi_t^{-1})_* e_l) & \omega(e_k, (\phi_t^{-1})_* f_l)\end{pmatrix}_{k,l=1,\ldots,n},$$

where $\rho : Mp(2n, \mathbb{R}) \to Sp(2n, \mathbb{R})$ denotes the double covering. With

$$\mathcal{L}_X Y = \frac{d}{dt}(\phi_t^{-1})_* Y = [X, Y]$$

for all vector fields Y on M, one sees

$$
\begin{aligned}
\frac{d}{dt}\rho(g_t)_{|t=0} &= \begin{pmatrix} \omega(\mathcal{L}_X e_l, f_k) & \omega(\mathcal{L}_X f_l, f_k) \\ \omega(e_k, \mathcal{L}_X e_l) & \omega(e_k, \mathcal{L}_X f_l) \end{pmatrix}_{k,l=1,\ldots,n} \\
&= \begin{pmatrix} \omega([X, e_l], f_k) & 0 \\ 0 & \omega(e_k, [X, f_l]) \end{pmatrix}_{k,l=1,\ldots,n} \\
&\qquad + \begin{pmatrix} 0 & \omega([X, f_l], f_k) \\ 0 & 0 \end{pmatrix}_{k,l=1,\ldots,n} \\
&\qquad + \begin{pmatrix} 0 & 0 \\ \omega(e_k, [X, e_l]) & 0 \end{pmatrix}_{k,l=1,\ldots,n} .
\end{aligned}
$$

Having $\mathcal{L}_X\omega = 0$, we conclude

$$\omega(e_k, [X, f_l]) = X(\omega(e_k, f_l)) - \omega([X, e_k], f_l) = -\omega([X, e_k], f_l)$$

as well as

$$\omega([X, f_l], f_k) = \omega([X, f_k], f_l) \qquad \text{and} \qquad \omega([X, e_k], e_l) = \omega([X, e_l], e_k)\ .$$

We obtain

$$
\begin{aligned}
\frac{d}{dt}\rho(g_t)_{|t=0} &= \sum_{k,l=1}^{n} \Bigg\{ \omega([X, e_l], f_k) \begin{pmatrix} B_{kl} & 0 \\ 0 & -B_{lk} \end{pmatrix} \\
&\qquad + \frac{1}{2}\omega([X, f_l], f_k) \begin{pmatrix} 0 & B_{kl} + B_{lk} \\ 0 & 0 \end{pmatrix} \\
&\qquad + \frac{1}{2}\omega(e_k, [X, e_l]) \begin{pmatrix} 0 & 0 \\ B_{kl} + B_{lk} & 0 \end{pmatrix} \Bigg\} \\
&= \sum_{k,l=1}^{n} \left\{ \omega([X, e_l], f_k) X_{kl} + \frac{1}{2}\omega([X, f_l], f_k) Y_{kl} + \frac{1}{2}\omega(e_k, [X, e_l]) Z_{kl} \right\} \\
&= \frac{1}{2}\sum_{k,l=1}^{n} \rho_*\big(\omega([X, e_l], f_k)(a_k \cdot b_l + b_l \cdot a_k) + \omega(f_k, [X, f_l]) a_k \cdot a_l \\
&\qquad + \omega(e_k, [X, e_l]) b_k \cdot b_l\big).
\end{aligned}
$$

With

$$\frac{d}{dt}\rho(g_t)_{|t=0} = \rho_*\left(\frac{d}{dt} g_{t|t=0}\right),$$

the definition of the Clifford multiplication, Proposition 1.2, equation (2.7), and relation

(1.6), we compute on U

$$\begin{aligned}
\mathcal{L}_X\varphi &= \frac{d}{dt}[s_t, u\circ\phi_t]_{|t=0}\\
&= \frac{d}{dt}[sg_t, u\circ\phi_t]_{|t=0}\\
&= \frac{d}{dt}[s, L(g_t)(u\circ\phi_t)]_{|t=0}\\
&= [s, L_*(\frac{d}{dt}g_{t|t=0})u+\frac{d}{dt}u\circ\phi_{t|t=0}]\\
&= X(\varphi)-\frac{i}{2}\sum_{k,l=1}^{n}\{\omega([X,e_l],f_k)(e_k\cdot f_l+f_l\cdot e_k)\\
&\qquad +\omega(f_k,[X,f_l])e_k\cdot e_l+\omega(e_k,[X,e_l])f_k\cdot f_l\}\cdot\varphi\\
&= X(\varphi)-\frac{i}{4}\sum_{k=1}^{n}\{[X,e_k]\cdot f_k+f_k\cdot[X,e_k]-[X,f_k]\cdot e_k-e_k\cdot[X,f_k]\}\cdot\varphi\\
&= X(\varphi)-\frac{i}{4}\sum_{k=1}^{n}\{\nabla_X e_k\cdot f_k-\nabla_{e_k}X\cdot f_k+f_k\cdot\nabla_X e_k-f_k\cdot\nabla_{e_k}X\\
&\qquad -\nabla_X f_k\cdot e_k+\nabla_{f_k}X\cdot e_k-e_k\cdot\nabla_X f_k+e_k\cdot\nabla_{f_k}X\}\cdot\varphi\\
&= X(\varphi)+\frac{i}{2}\sum_{k=1}^{n}\{e_k\cdot\nabla_X f_k-f_k\cdot\nabla_X e_k\}\cdot\varphi\\
&\qquad +\frac{i}{4}\sum_{k=1}^{n}\{i\omega(e_k,\nabla_X f_k)-i\omega(f_k,\nabla_X e_k)\}\varphi\\
&\qquad +\frac{i}{2}\sum_{k=1}^{n}\{\nabla_{e_k}X\cdot f_k-\nabla_{f_k}X\cdot e_k\}\cdot\varphi\\
&\qquad +\frac{i}{4}\sum_{k=1}^{n}\{i\omega(\nabla_{e_k}X,f_k)-i\omega(\nabla_{f_k}X,e_k)\}\varphi\\
&= \nabla_X\varphi-\frac{1}{4}\sum_{k=1}^{n}X(\omega(e_k,f_k))\varphi\\
&\qquad +\frac{i}{2}\sum_{k=1}^{n}\{\nabla_{e_k}X\cdot f_k-\nabla_{f_k}X\cdot e_k\}\cdot\varphi\\
&= \nabla_X\varphi+\frac{i}{2}\sum_{k=1}^{n}\{\nabla_{e_k}X\cdot f_k-\nabla_{f_k}X\cdot e_k\}\cdot\varphi\,,
\end{aligned}$$

which proves the proposition. □

As it is well known, the commutator of two Hamiltonian vector fields is a Hamiltonian vector field, too. Ideed, if $X=X_h$ is given by the function h and $Y=X_g$ by a function g then the commutator is the Hamiltonian vector field defined by the Poisson bracket of g and h, i.e.

$$[X_h, X_g] = -X_{\{h,g\}} \ . \tag{2.8}$$

For the Lie derivative in the direction of the commutator one has the following relation.

Corollary 2.3 *Let $\varphi \in \Gamma(\mathcal{Q})$ a symplectic spinor field and let X, Y are Hamiltonian vector fields on (M, ω), then*

$$\mathcal{L}_{[X,Y]}\varphi = [\mathcal{L}_X, \mathcal{L}_Y]\varphi.$$

Proof: Using (2.8) and Proposition 2.2, this proof is immediate. □

In case that M is not closed, all considerations hold true locally.

3 The Lie Derivative as Schrödinger Equation

This section illustrates how the Schrödinger equation for a quadratic Hamiltonian function relates to the Lie derivative of a constant symplectic spinor field over $\mathbb{R}^{2n}$.

3.1 The Schrödinger equation for quadratic Hamiltonians

We consider quadratic Hamiltonians H of the form $H(z) = z^\top Qz$ for $z \in \mathbb{R}^{2n}$, where Q is any real $2n \times 2n$-matrix. In general, one could add an additional absolut real term. But, this is completely inessential, because it does not play any role for the dynamics of the system. Or, physically speaking, the choice of the zero-energy-level is arbitrary.

Lemma 3.1 *Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a quadratic Hamiltonian on $\mathbb{R}^{2n}$, which is given by $H(z) = z^\top Qz$, where Q is any $2n \times 2n$-matrix. Then, there exists a $2n \times 2n$-matrix $A \in \mathfrak{sp}(2n, \mathbb{R})$ such that the Hamiltonian vector field X_H of H is given by $X_H(z) = Az$ for $z \in \mathbb{R}^{2n}$.*

Proof: Let $\gamma(t)$ be a curve in $\mathbb{R}^{2n}$ with $\gamma(0) = z$ and $\dot\gamma(0) = w$. Then

$$dH(w)_z = \frac{d}{dt}H(\gamma(t))_{|t=0} = \frac{d}{dt}(\gamma(t))^\top Q(\gamma(t))_{|t=0} = w^\top Qz + z^\top Qw = w^\top (Q + Q^\top)z \ .$$

On the other hand, the Hamiltonian vector field X_H of H is given by

$$dH(w) = \omega_0(X_H, w) = -\langle Jw, X_H\rangle = -(Jw)^\top X_H = -w^\top J^\top X_H = w^\top JX_H \ .$$

Thus, at any point $z \in \mathbb{R}^{2n}$ we have $JX_H(z) = (Q + Q^\top)z$, and consequently

$$X_H(z) = -J(Q + Q^\top)z = J^\top(Q + Q^\top)z = ((Q + Q^\top)J)^\top z \ .$$

Taking $A = -J(Q + Q^\top)$, we have $X_H(z) = Az$ for $z \in \mathbb{R}^{2n}$. Furthermore, this A satisfies

$$A^\top J = -(Q + Q^\top)J^\top J = -(Q^\top + Q)$$

as well as

$$-JA = -(Q + Q^\top) ,$$

which gives $A^\top J + JA = 0$, or equivalently, $A \in \mathfrak{sp}(2n, \mathbb{R})$. □

Each quadratic Hamiltonian on $\mathbb{R}^{2n}$ can be written as a linear combination, i.e. as a sum of multiples of the functions on $\mathbb{R}^{2n}$ given by the expressions

$$\begin{array}{rcllll} H^1_{jk}(p,q) & = & p_j p_k \ , & & & \\ H^2_{jk}(p,q) & = & q_j q_k \ , & & \text{and} & \\ H^3_{jk}(p,q) & = & p_j q_k & = \dfrac{1}{2}(p_j q_k + q_k p_j) & \text{for} & j,k = 1,\ldots,n \ . \end{array}$$

We call these functions *generating quadratic Hamiltonians.*

Lemma 3.2 *For the generating quadratic Hamiltonians the corresponding elements in* $\mathfrak{sp}(2n, \mathbb{R})$ *due to Lemma 3.1 are given in the following way.*

(1) *If* $H = H^1_{jk}$, *then* $A = -Z_{jk} = -Y^\top_{jk}$.

(2) *If* $H = H^2_{jk}$, *then* $A = Y_{jk} = Z^\top_{jk}$.

(3) *If* $H = H^3_{jk}$, *then* $A = X^\top_{jk}$.

Proof:

(1) $H = H^1_{jk}$ is given by $Q = \begin{pmatrix} B_{jk} & 0 \\ 0 & 0 \end{pmatrix}$. Thus

$$A = -\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} B_{jk} + B_{kj} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -B_{jk} - B_{kj} 0 & \end{pmatrix} = -Z_{jk} = -Y^\top_{jk} \ .$$

(2) $H = H^2_{jk}$ is given by $Q = \begin{pmatrix} 0 & 0 \\ 0 & B_{jk} \end{pmatrix}$, which implies

$$A = -\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & B_{jk} + B_{kj} \end{pmatrix} = \begin{pmatrix} 0 & B_{jk} + B_{kj} \\ 0 & 0 \end{pmatrix} = Y_{jk} = Z^\top_{jk} \ .$$

(3) Finally, $H = H^3_{jk}$ is given by $Q = \frac{1}{2}\begin{pmatrix} 0 & B_{jk} \\ B_{kj} & 0 \end{pmatrix}$. This yields

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & B_{jk} \\ B_{kj} & 0 \end{pmatrix} = \begin{pmatrix} B_{kj} & 0 \\ 0 & -B_{jk} \end{pmatrix} = X^{\top}_{jk} .$$

□

Let $\mathcal{H}$ denote the Hamilton operator which is given by H via normal ordering quantization, i.e. one obtains $\mathcal{H}$ by replacing in H formally the variable p_j by the multiplication operator ix_j and q_k by the operator $\frac{\partial}{\partial x_k}$. Thereby "normal ordering" means that the expression $p_j q_k = \frac{1}{2}(p_j q_k + q_k p_j)$ is replaced be the operator $\frac{i}{2}\left(x_j \frac{\partial}{\partial x_k} + \frac{\partial}{\partial x_k} x_j\right)$.

Corollary 3.3 *For the quadratic Hamiltonian H let A be given by Lemma 3.1 and let $\mathcal{H}$ be the Hamilton operator given via normal ordering quantization. Then, one has the relation*

$$L_* \circ \rho_*^{-1}(A^{\top}) = -i\mathcal{H} .$$

Proof: Since L_* and ρ_*^{-1} are linear and H is a linear combination of the generating quadratic Hamiltonians, it suffices to prove the assertion for the generating quadratic Hamiltonians. Then, by Lemma 1.1, Lemma 3.2, and Proposition 1.2 one has

$$\begin{aligned} L_* \circ \rho_*^{-1}(-Y_{jk}) &= ix_j x_k = -i(ix_j)(ix_k) = -i\mathcal{H} && \text{for } H = H^1_{jk}, \\ L_* \circ \rho_*^{-1}(Z_{jk}) &= -i\frac{\partial^2}{\partial x_j \partial x_k} = -i\mathcal{H} && \text{for } H = H^2_{jk}, \quad \text{and} \\ L_* \circ \rho_*^{-1}(X_{jk}) &= -\frac{i}{2}\left(ix_j \frac{\partial}{\partial x_k} + \frac{\partial}{\partial x_k} ix_j\right) = -i\mathcal{H} && \text{for } H = H^3_{jk} . \end{aligned}$$

□

We consider a quadratic Hamiltonian H on $\mathbb{R}^{2n}$ with $A \in \mathfrak{sp}(2n, \mathbb{R})$ given according to Lemma 3.1. Then, we consider the family $S_t \in Sp(2n, \mathbb{R})$ of symplectic matrices defined by $S_t = \exp(tA^{\top})$ for $t \in \mathbb{R}$. We lift this family of symplectic matrices into the double covering of $Sp(2n, \mathbb{R})$. That is, we consider the family $M_t \in Mp(2n, \mathbb{R})$ given by $\rho(M_t) = S_t$ such that M_0 is the unit element in $Mp(2n, \mathbb{R})$.

Definition 3.4 *For fixed $\psi_0 \in \mathcal{S}(\mathbb{R}^n)$ we define $\psi(t, x) := L(M_t)(\psi_0)(x)$, where $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$.*

Furthermore let $\psi(t)$ be the curve in $\mathcal{S}(\mathbb{R}^n)$ given by $\psi(t)(x) := \psi(t, x)$ for $x \in \mathbb{R}^n$, i.e. $\psi(t) = L(M_t)\psi_0$.

Proposition 3.5 *$\psi(t)$ satisfies the Schrödinger equation*

$$\frac{d}{dt}\psi(t)_{|t=0} = -i\mathcal{H}(\psi_0)\ .$$

Proof: We have

$$\frac{d}{dt}\psi(t)_{|t=0} = \frac{d}{dt}L(M_t)(\psi_0)_{|t=0} = L_*\left(\frac{d}{dt}M_{t|t=0}\right)(\psi_0)\ .$$

The definition of M_t gives

$$\rho_*\left(\frac{d}{dt}M_{t|t=0}\right) = \frac{d}{dt}\rho(M_t)_{|t=0} = \frac{d}{dt}S_{t|t=0} = \frac{d}{dt}\exp(tA^\top)_{|t=0} = A^\top\ .$$

Hence

$$\frac{d}{dt}M_{t|t=0} = \rho_*^{-1}(A^\top)$$

and finally

$$\frac{d}{dt}\psi(t)_{|t=0} = L_*\circ\rho_*^{-1}(A^\top)(\psi_0) = -i\mathcal{H}(\psi_0)$$

by the previous Lemma. □

Let us now give the announced interpretation of the Lie derivative.

3.2 Interpretation as Lie derivative

In fact, the Schrödinger equation above gives the Lie derivative of a constant symplectic spinor field φ_0 on $\mathbb{R}^{2n}$ in the direction of the Hamiltonian vector field X.

First observe that the symplectic standard basis $\{a_1,\dots,a_n,b_1,\dots,b_n\}$ gives a global section s of the symplectic frame bundle R of $\mathbb{R}^{2n}$. Then $\overline{s}$ denotes a lift of s into the canonical metaplectic structure P of $\mathbb{R}^{2n}$.

Now, if ψ_0 is any fixed function in $\mathcal{S}(\mathbb{R}^n)$ the symplectic spinor field φ_0 over $\mathbb{R}^{2n}$ is defined to be

$$\varphi_0 = [\overline{s},\psi_0]\ .$$

Further, we consider the family $\{\phi_t:\mathbb{R}^{2n}\longrightarrow\mathbb{R}^{2n}\}_{t\in\mathbb{R}}$ given by

$$\phi_t(z) := \exp(tA)z \qquad \text{for } z\in\mathbb{R}^{2n}\ ,$$

where A denotes the matrix according to Lemma 3.1. Then,

$$\frac{d}{dt}\phi_t(z)_{|t=0} = Az = X(z)\ ,$$

which says that $\{\phi_t\}_{t\in\mathbb{R}}$ is exactly the one-parametergroup of diffeomorphisms induced by the Hamiltonian vector field X. Recalling the computations in the proof of Proposition 2.2, one has

$$(\phi_t^{-1})_*\varphi_0 = [\overline{s}_t, \psi_0]$$

with $s_t = s\exp(-tA)$ and $\overline{s}_t$ its lift to P. Since P is an $Mp(2n,\mathbb{R})$-principal fibre bundle, we obtain a family of elements $N_t \in Mp(2n,\mathbb{R})$ such that

$$\overline{s}_t = \overline{s}N_t \qquad \text{with} \qquad \rho(N_t) = \exp(-tA)\ .$$

Hence,

$$(\phi_t^{-1})_*\varphi_0 = [\overline{s}_t, \psi_0] = [\overline{s}N_t, \psi_0] = [\overline{s}, L(N_t)\psi_0]\ .$$

For a fixed element $\tilde{J} \in \rho^{-1}(J)$ the metaplectic representation was given by $L(\tilde{J}) = i^{\frac{n}{2}}\mathcal{F}^{-1}$ (cf. equation (1.5)). Thus, $i^{-\frac{n}{2}}L(\tilde{J})\circ\mathcal{F} = id$. Using relation (1.1) we obtain

$$\rho(N_t)\rho(\tilde{J}) = \exp(-tA)J = (\exp(tA))^{-1}J = J(\exp(tA))^{\top} = J\exp(tA^{\top}) = \rho(\tilde{J})\rho(M_t)\ ,$$

where M_t is given above. Consequently,

$$L(N_t)\circ L(\tilde{J}) = L(\tilde{J})\circ L(M_t)\ .$$

Altogether, we arrive at

$$(\phi_t^{-1})_*\varphi_0 = i^{-\frac{n}{2}}[\overline{s}, L(N_t)\circ L(\tilde{J})\circ\mathcal{F}\psi_0] = i^{-\frac{n}{2}}[\overline{s}, L(\tilde{J})\circ L(M_t)\circ\mathcal{F}\psi_0]\ .$$

Finally, we compute the Lie derivative of φ_0 in the direction of X and obtain, by Definition 2.1,

$$\mathcal{L}_X\varphi_0 = i^{-\frac{n}{2}}\left[\overline{s}, L(\tilde{J})\left(\frac{d}{dt}L(M_t)(\mathcal{F}\psi_0)_{|t=0}\right)\right] = -i[\overline{s}, \mathcal{F}^{-1}\circ\mathcal{H}\circ\mathcal{F}(\psi_0)]\ .$$

Here, the Fourier transform $\mathcal{F}$ means the transition between position and momentum representations.

Concluding Remarks

Fixing a compatible almost complex structure for (M,ω), Andreas Klein introduced a globally defined Fourier transform acting on symplectic spinor fields. See [11]. If one would define a Hamilton operator $\hat{\mathcal{H}}$ acting on symplectic spinor fields in the way that

$$\hat{\mathcal{H}}[\overline{s}, \psi] := [\overline{s}, \mathcal{H}\psi]\ ,$$

however, this does not work in general. The reason is that $\hat{\mathcal{H}}$ is not well defined by this relation.

But, setting formally

$$\mathfrak{q}(h)\varphi := i\mathcal{L}_{X_h}\varphi \ , \tag{3.9}$$

equation (2.3) gives

$$\begin{aligned}\mathfrak{q}(\{h,g\})\varphi &= i\mathcal{L}_{X_{\{h,g\}}}\varphi = -i\mathcal{L}_{[X_h,X_g]}\varphi = -i\mathcal{L}_{X_h}\circ\mathcal{L}_{X_g}\varphi + i\mathcal{L}_{X_h}\circ\mathcal{L}_{X_h}\varphi \\ &= i\mathfrak{q}(h)\circ\mathfrak{q}(g)\varphi - i\mathfrak{q}(g)\circ\mathfrak{q}(h)\varphi = i[\mathfrak{q}(h),\mathfrak{q}(g)]\varphi \ ,\end{aligned}$$

which is in fact the "magic" Heisenberg relation

$$[\mathfrak{q}(h),\mathfrak{q}(g)]\varphi = -i\mathfrak{q}(\{h,g\})\varphi \ .$$

We do not claim that (3.9) gives a quantization procedure for arbitrary Hamiltonians over any symplectic manifold, although this expression makes sense in the general situation. We deduced the Heisenberg relation purely formal.

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Authors:

Katharina Habermann	Andreas Klein
Ernst-Moritz-Arndt-Universität Greifswald	Humboldt-Universität zu Berlin
Institut für Mathematik und Informatik	Institut für Mathematik
Jahnstrasse 15a	Unter den Linden 6
17487 Greifswald	10099 Berlin
Germany	Germany
e-mail: khaberma@mail.uni-greifswald.de	e-mail: klein@mathematik.hu-berlin.de