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A nonlinear Schrödinger equation with external magnetic field

ABSTRACT. The paper presents an existence result for a nonlinear Schrödinger equation with magnetic potential on unbounded domains.

1 Introduction

In this paper we study a Dirichlet boundary value problem for a nonlinear Schrödinger equation

$$\left(\frac{1}{i}\nabla + A(x)\right)^2 u + V(x)u = |u|^{q-1}u \tag{1}$$

with a subcritical q on $\Omega \subset \mathbb{R}^3$ an open, possibly unbounded, set. Equation (1) is a modification of the linear Schrödinger equation for standing waves $\psi(x,t) = e^{-iEt}u(x)$ with a magnetic potential and an energy term dependent on the magnitude of the wave function.

The problem (1) has been studied by [2], where existence was proved for the constant magnetic field $B = curl A$ on $\mathbb{R}^N$, $N = 2, 3$. In the case of variable magnetic field [2] gives only an implicit condition in terms of asymptotic values of the functional. The problem on $\mathbb{R}^N$ was also studied by [3]. This paper provides existence under explicit conditions on the electric and magnetic fields only in part overlapping with those [3], and for problems on more general domains than $\mathbb{R}^N$.

The proof of existence is variational: convergence of a critical sequence is obtained by application of an abstract concentration compactness framework ([7, 8]) which provides a functional-analytic generalization of original concentration compactness ([5, 6]).

2 Concentration compactness in Hilbert space and in Sobolev space

In what follows we quote results from [8].

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Definition 2.1 *Let H be a separable Hilbert space. A bounded set D of bounded linear operators on H is called a set of dislocations if the following properties are satisfied:*

(I) $g \in D \Rightarrow g^{-1} \in D,\ I \in D$;

(II) *if $g_k,\ h_k \in D$ and $g_k h_k^{-1}$ does not converge weakly to 0, then there exists a renamed subsequence of $g_k h_k^{-1}$ such that $u_k \rightharpoonup 0 \Rightarrow g_k h_k^{-1} u_k \rightharpoonup 0$;*

(III) $g_k \in D,\ u_k \rightharpoonup 0 \Rightarrow g_k^* g_k u_k \rightharpoonup 0$.

Definition 2.2 *Let $u, u_k \in H$. We will say that u_k converges to u* weakly with concentration *which we will denote as*

$$u_k \overset{cw}{\to} u,$$

if for all $\varphi \in H$,

$$\lim_{k\to\infty} \sup_{g\in D} (g(u_k - u), \varphi) = 0 \,. \tag{1}$$

Theorem 2.3 *Let $u_k \in H$ be a bounded sequence. Then there exists $w^{(n)} \in H$, $g_k^{(n)} \in D$, $k, n \in \mathbb{N}$ such that for a renumbered subsequence*

$$w^{(n)} = \text{w-lim}\, {g_k^{(n)}}^{-1} u_k \,, \tag{2}$$

$${g_k^{(n)}}^{-1} g_k^{(m)} \rightharpoonup 0 \text{ for } n \neq m \,, \tag{3}$$

$$\sum_{n\in\mathbb{N}} \|w^{(n)}\|^2 \leq \limsup \|u_k\|^2 \tag{4}$$

and

$$u_k - \sum_{n\in\mathbb{N}} g_k^{(n)} w^{(n)} \overset{cw}{\to} 0 \,. \tag{5}$$

We now specify the implications of the theorem above for the case when $H = H^1(\mathbb{R}^N)$ and D is a group of shifts:

$$D = \{g_\alpha : u \mapsto u(\cdot + \alpha)\}_{\alpha \in Z^{\mathbb{N}}} \,. \tag{6}$$

Let $N > 1$ and let $2^* = \frac{2N}{N-2}$ (for $N = 2$ we set $2^* = \infty$). It is proved in [8] that D is a set of dislocations, so that Theorem 2.3 applies. Moreover ([8, 4, 6]), for bounded sequences concentrated weak convergence is same as $L^p - convergence$

Lemma 2.4 *Let H,D be as above, let $2 < p < 2^*$ and let u_k be a bounded sequence in H. Then $u_k \in H^1(\mathbb{R}^N) \overset{cw}{\to} 0$ if and only if $\|u_k\|_{L^p(\mathbb{R}^N)} \to 0$.*

An easy corollary from the above lemma is the following statement.

Lemma 2.5 *Let u_k, $w^{(n)}$, and $g_k^{(n)}$ be as in Theorem 2.3. If $F : \mathbb{R} \to \mathbb{R}$ is a continuous function and for every $\epsilon > 0$ there is a $C_\epsilon < \infty$ and a p_ϵ such that $2 < p_\epsilon < 2^*$ and*

$$|F(s)| \leq \epsilon(|s|^2 + |s|^{p^*}) + C_\epsilon |s|^{p_\epsilon} \tag{7}$$

then, on a renamed subsequence

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} F(|u_k|) = \sum_n \int_{\mathbb{R}^N} F(|w^{(n)}|). \tag{8}$$

Definition 2.6 *An open set $\Omega \subset \mathbb{R}^N$ will be called asymptotically contractive if for every sequence $u_k \in H_0^1(\Omega)$ and every sequence $\alpha_k \in \mathbb{R}^N$ such that $u_k(\cdot + \alpha_k)$ converges weakly in $H^1(\mathbb{R}^N)$ to some w, there exists a $\gamma \in \mathbb{R}^N$ such that $w(\cdot - \gamma) \in H_0^1(\Omega)$.*

A sufficient geometric condition for asymptotic contractiveness can be formulated in the terms of lower limit for sequences of sets,

$$\liminf \Omega_k := \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} \Omega_k\,, \tag{9}$$

namely,

Lemma 2.7 *Let $\Omega \subset \mathbb{R}^N$ be an open set such that $\partial\Omega = \partial(\mathbb{R}^N \setminus \bar{\Omega})$. It is asymptotically contractive if for every sequence $\alpha_k \in \mathbb{Z}^N$ there exist a set $Y \subset \mathbb{R}^N$ of zero measure and a convergent sequence $\gamma_k \in \mathbb{R}^N$ such that, on a renumbered subsequence,*

$$\liminf(\Omega + \alpha_k - \gamma_k) \subset \Omega \cup Y\,. \tag{10}$$

3 A nonlinear magnetic Schrödinger equation

Let $\Omega \subset \mathbb{R}^3$ be an open set. Let $V \in L^1_{loc}(\Omega)$ and assume that

$$\eta := \inf_{x\in\Omega}(V(x) + \lambda_0) > 0\,, \tag{1}$$

where $\lambda_0 = \inf_{u\mathbb{C}_0^\infty(\Omega):\int |u|^2=1} \int |\nabla u|^2$. Note that λ_0 might be positive even for unbounded Ω. Let $A \in L^2_{loc}(\Omega; \mathbb{R}^3)$ satisfy

$$\exists\phi \in H^1_{loc}(\Omega) : |A(x) - \nabla\phi|^2 < \liminf_{|y|\to\infty} V(y) - V(x), x \in \Omega\,. \tag{2}$$

Theorem 3.1 *Let $q \in (1, 5)$ and assume that A, V satisfy (2, 1). If $\Omega \subset \mathbb{R}^3$ is an asymptotically contractive set, then there is a solution $u \in H_0^1(\Omega; \mathbb{C}) \setminus \{0\}$ satisfying the equation*

$$\left(\frac{1}{i}\nabla + A(x)\right)^2 u + V(x)u = |u|^{q-1}u\,. \tag{3}$$

Proof: It suffices to prove existence of the minimizer for the following variational problem. Let

$$c = \inf_{\int_\Omega |u|^p = 1} I(u),\ p = q + 1\,, \tag{4}$$

where

$$I(u) = \int_\Omega Pu \cdot \overline{Pu} + V(x)|u|^2, P = \frac{1}{i}\nabla + A(x)\,. \tag{5}$$

Without loss of generality we can assume that ϕ=0 in (2), since the value of c does not change when one replaces A by $A + \nabla\phi$. For smooth ϕ this is verified by replacing u by $e^{i\phi}u$, for general ϕ one can replace a minimizing sequence u_k by $e^{i\phi_k}u_k$ with appropriate smooth approximations ϕ_k of ϕ.

Let u_k be the maximizing sequence and let $u^{(0)} :=$ w-lim u_k. The weak limit here is understood with respect to the metric of the quadratic form $I(u)$. We will make use of the following inequality (cf. [2]):

$$\int |Pu|^2 \geq \int |\nabla|u||^2 \tag{6}$$

that together with (1) implies that $c > 0$ and that $v_k := |u_k - u^{(0)}| \in H_0^1(\Omega)$. We apply Theorem 2.3 in $H^1(\mathbb{R}^3)$ to the extensions of functions v_k (by zero on the complement of Ω). Since Ω is asymptotically contractive, the dislocated weak limits of v_k, $w^{(n)}$ (redefined by appropriate constant shifts) lie in $H_0^1(\Omega)$. Moreover, since $v_k \rightharpoonup 0$, one has $|\alpha_k^{(n)}| \to \infty$ for all n.

From Lemma 2.5 we have

$$\lim \int v_k^p = \sum_{n=1}^{\infty} \int w^{(n)p} \tag{7}$$

and, since all $|\alpha_k^{(n)}| \to \infty$,

$$1 = \lim \int |u_k|^p = \int |u^{(0)}|^p + \sum_{n=1}^{\infty} \int w^{(n)p}\,. \tag{8}$$

To continue with the proof we need the following inequality:

$$\int |Pu_k|^2 + V(x)|u_k|^2 \geq \int |Pu^{(0)}|^2 + V(x)|u^{(0)}|^2 + \sum_{n=1}^{\infty} \int (|\nabla w^{(n)}|^2 + \liminf_{|x|\to\infty} V(x) {w^{(n)}}^2)\,. \tag{9}$$

We start with:

$$\int |Pu_k|^2 = \int |Pu^{(0)}|^2 + \int |P(u_k - u^{(0)})|^2 - \int P(u_k - u^{(0)}) \cdot \overline{Pu^{(0)}} - \int Pu^{(0)} \cdot \overline{P(u_k - u^{(0)})}\,. \tag{10}$$

Passing to the limit we have

$$\liminf \int |Pu_k|^2 \geq \int |Pu^{(0)}|^2 + \liminf \int |P(u_k - u^{(0)})|^2\,. \tag{11}$$

Applying (6), we get

$$\liminf \int |Pu_k|^2 \geq \int |Pu^{(0)}|^2 + \liminf \int |\nabla v_k|^2 \,, \tag{12}$$

which in turn can be estimated from below by (4) if we apply Theorem 2.3 to the sequence v_k as a bounded sequence in $D^{1,2}$, so that

$$c = \liminf \int |Pu_k|^2 \geq \int |Pu^{(0)}|^2 + \sum_n \int |\nabla w^{(n)}|^2 \,. \tag{13}$$

A similar argument allows also to show that

$$\int V(x)|u_k|^2 \geq \int V(x)|u^{(0)}|^2 + \sum_{n=1}^{\infty} \liminf_{|x|\to\infty} V(x) \int {w^{(n)}}^2 \,. \tag{14}$$

This and (13) yield (9). Using the notations

$$t_n = \int {w^{(n)}}^p, t_0 = \int |u^{(0)}|^p \,, \tag{15}$$

$$c_n = t_n^{-2/p} \int |\nabla w^{(n)}|^2 + \liminf_{|x|\to\infty} V(x) \int {w^{(n)}}^2, c_0 = t_0^{-2/p} \int |Pu^{(0)}|^2 + V(x)|u^{(0)}|^2 \,, \tag{16}$$

one has from (8), (9)

$$\sum_{n=0}^{\infty} t_n = 1, \sum_{n=0}^{\infty} c_n t_n^{2/p} \leq c \,. \tag{17}$$

However, due to (2), $c_n > c$, while $c_0 \geq c$, so it is possible to satisfy (17) only with $t_0 = 1$, $t_n = 0$, $N = 1, 2 \ldots$ This also implies that the minimizing sequence converges to $u^{(0)}$ in norm.

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