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On Discrete Solutions of Two-Scale Difference Equations

ABSTRACT. Two-scale difference equations are considered at dyadic points. The corresponding solutions are called discrete solutions. They are calculated explicitly and accompanied by generating functions. As intermediate step there are also derived formulas which are valid for distributional solutions.

KEY WORDS. Two-scale difference equations, discrete solutions, distributional solutions, two-slanted matrices, generating functions

1 Introduction

Two scale difference equations

$$\varphi\left(\frac{t}{2}\right) = \sum_{n=0}^{N} c_n \varphi(t-n) \tag{1.1}$$

($t \in \mathbb{R}$) with $N \in \mathbb{N}$, $c_n \in \mathbb{C}$, $c_0 c_N \neq 0$, appear in wavelet theory and subdivision schemes, where nontrivial L^1-solutions are sought which are necessarily compactly supported, cf. [7], [9], [12]. A necessary condition for the existence of such solutions is

$$\sum_{n=0}^{N} c_n = 2^M \tag{1.2}$$

with $M \in \mathbb{N}$ (cf. [9]). Here, we require the two *boundary conditions* (cf. [3]):

(i) $\varphi(t) = 0$ for $t < 0$,

(ii) $\varphi(t)$ is equal to a polynomial $\pi(t)$ for $t > N$.

Then in (1.2) the foregoing condition $M \in \mathbb{N}$ weakens to $M \in \mathbb{Z}$ and, for $M \leq 0$, the polynomials π in (ii) are of degree $m = -M$, whereas they vanish identically for $M > 0$.

Vice versa, under the condition (1.2) with $M \in \mathbb{Z}$ there always exists a *distributional* solution of (1.1) with (i) and (ii), i.e. a solution which is a derivative of finite order of a continuous function. This solution is a *simple* one, i.e. it is nontrivial and uniquely determined up to a constant factor. For $M \leq 0$ the polynomial π in condition (ii) can be calculated explicitly, and the natural extension of it to the whole set $\mathbb{R}$ is itself a simple solution of (1.1), cf. [6].

The case $N = 1$ in equation (1.1) is closely connected with de Rham's singular function which already is well investigated, cf. [1] and the literature quoted there. Hence we require in this paper

(iii) $N \geq 2$ and $M \in \mathbb{Z}$ in (1.2).

The natural domain for the solutions of (1.1) is not necessarily the whole set $\mathbb{R}$ but a subset containing with a number t also $t-1$, $t+1$, $\frac{t}{2}$ and $2t$. For this reason we introduce *discrete solutions* of (1.1), which satisfy this equation at dyadic points $t = \frac{j}{2^\ell}$ $(j \in \mathbb{Z}, \ell \in \mathbb{N}_0)$ and which satisfy the boundary conditions (at dyadic points) in the corresponding closed intervals, i.e.

(i') $\varphi(t) = 0$ for $t \leq 0$,

(ii') $\varphi(t)$ is equal to a polynomial for $t \geq N$.

Discrete solutions determine uniquely corresponding continuous solutions and vice versa, but we do not require the existence of a nontrivial continuous solution. However, in the case of a discontinuous or even distributional solution there is in general no connection between such a solution and a discrete one. For discrete solutions it is convenient to restrict the arguments automatically to dyadic points. According to [6] it is also possible to calculate discrete solutions at the points $t = \frac{j}{2^\ell(2^k-1)}$ with a fixed $k \in \mathbb{N}$, but we are not concerned with this modification.

Our aim is to establish formulas for the explicit calculation of discrete solutions without claim that they are more effective than the usual subdivision algorithms. It suffices to determine the solutions in the interval $0 \leq t \leq 1$, since for $t > 1$ they can be extended by means of (1.1). The main result in this direction is the later formula (2.25). Preliminarily, we derive consequences of (1.1) which are even valid for distributional solutions, and if we speak about such solutions we tacitly assume that the sufficient conditions (i), (ii) and (iii) for their existence are satisfied. The main result concerning distributional solutions is contained in Theorem 2.4. Finally, discrete solutions are connected with corresponding generating functions.

Remark 1.1

1. Equalities in open intervals (with integer endpoints) for distributional solutions are valid for discrete solutions also in the endpoints. This follows from the differences in the boundary conditions.

2. Under the condition (i) equation (1.1) implies

$$\varphi(0) = c_0\varphi(0).$$

This means for discrete solutions that the sharpening (i') of (i) is quite natural in the case $c_0 \neq 1$. In the case $c_0 = 1$ it would be possible to generalize the notion of a discrete solution allowing $\varphi(0)$ to be an arbitrary constant. An analogous generalization concerning $\varphi(N)$ would be possible in the case $c_N = 1$. We come back to these generalizations in Example 3.3.

In this paper we refer to polynomials $z_k = z_k(p, q)$, which are recursively determined by the system of two discrete two-scale difference equations

$$z_{2k} = p\, z_k, \quad z_{2k+1} = q\, z_k + z_{k+1} \quad (k \in \mathbb{N}) \tag{1.3}$$

and the initial condition

$$z_1 = 1. \tag{1.4}$$

These polynomials are intensively investigated in [2], and concerning their properties we refer to this paper. In particular, the polynomials z_k have the explicit representation

$$z_k = \sum_{j=0}^{k-1} \alpha^{\nu(j)}\, \beta^{\nu(k-1-j)} \tag{1.5}$$

where α and β are determined by

$$p = \alpha + \beta, \quad q = \alpha\,\beta \tag{1.6}$$

and where $\nu(j)$ is the binary sum-of-digits function, i.e. the number of $''1s''$ in the dyadic representation of j.

2 The main results

We begin with some notations. Equation (1.1) is connected with the polynomial

$$Q(z) = \sum_{n=0}^{N} c_n z^n \tag{2.1}$$

which we term *characteristic polynomial* of equation (1.1) in order to distinguish it from the so-called symbol in such cases where it is used with a factor $\frac{1}{2}$ or $\frac{1}{2}\sqrt{2}$. By means of (2.1) we can write condition (1.2) as $Q(1) = 2^M$. Moreover, we introduce the infinite vector

$$\Psi(t) = (\varphi(t), \varphi(t+1), \varphi(t+2), \ldots)^{\mathrm{T}} \tag{2.2}$$

and the infinite two-slanted matrix

$$\boldsymbol{A} = (c_{2j-k}) \qquad (j, k \geq 1) \tag{2.3}$$

with $c_j = 0$ for $j \notin \{0, \ldots, N\}$, so that the solution of (1.1) is equivalent to the solution of

$$\Psi\left(\frac{t+1}{2}\right) = \boldsymbol{A}\Psi(t) \tag{2.4}$$

for $t < 1$, both equations subject to (i), cf. [10], [5] and the literature quoted there. In the case (i') equation (1.1) is equivalent to (2.4) for $t \leq 1$. We also introduce the vector

$$\psi(t) = (\varphi(t), \varphi(t+1), \ldots, \varphi(t+N-2))^{\mathrm{T}} \tag{2.5}$$

and the matrix

$$A = (c_{2j-k}) \qquad (1 \leq j, k \leq N-1). \tag{2.6}$$

In the following existence theorem a *simple eigenvalue* means an eigenvalue with geometric multiplicity one:

Theorem 2.1 *Let* (*iii*) *be satisfied. For* $M \geq 1$ *in condition* (1.2) *let* 1 *be a simple eigenvalue of* A *whereas for* $M \leq 0$ *let* 1 *be no eigenvalue of* A*. Then* (1.1) *has exactly one simple discrete solution.*

Proof: Introducing suitable block matrices $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{O}$ where $\boldsymbol{O}$ is a zero matrix, we can split Ψ and $\boldsymbol{A}$ into

$$\Psi(t) = \begin{pmatrix} \psi(t) \\ \Psi(t+N-1) \end{pmatrix}, \qquad \boldsymbol{A} = \begin{pmatrix} A & \boldsymbol{B} \\ \boldsymbol{O} & \boldsymbol{C} \end{pmatrix}.$$

Choosing $t = 1$ in (2.4), this equation turns over into the two equations

$$\psi(1) = A\psi(1) + \boldsymbol{B}\Psi(N), \qquad \Psi(N) = \boldsymbol{C}\Psi(N). \tag{2.7}$$

In the case $M \geq 1$ we have to look for a discrete solution with $\varphi(t) = 0$ for $t \geq N$. Hence $\Psi(N) = 0$, and $\psi(1)$ can be determined as a simple right eigenvector to the eigenvalue 1 of A out of the first equation in (2.7). In the case $M \leq 0$ there exists a simple polynomial solution of (1.1), cf. [6], Theorem 6.1, which determines $\Psi(N)$ as a simple vector satisfying the second equation of (2.7), and $\psi(1)$ can likewise be determined out of the first of these equations. Hence, in any case, we have a simple vector $\Psi(1)$, satisfying (2.4) for $t = 1$, and the values of Ψ at the dyadic points and therefore also the values of the discrete solution φ, follow recursively from (2.4) and (i') for $-1 \leq t \leq 1$. Finally, the simplicity of $\Psi(1)$ transfers to φ which finishes the proof ■

Remark 2.2 If 1 is an eigenvalue of A with the geometric multiplicity $r > 1$, then there exist r linearly independent eigenvectors and, in the case $M \geq 1$, therefore r linearly independent discrete solutions, but at most one of these can be extended to a continuous solution. Concerning the choice of the correct eigenvector $\psi(1)$ cf. [6]. If we speak about a discrete solution of (1.1), we tacitly assume that besides of the conditions (i') and (ii') either the conditions of Theorem 2.1 are satisfied or in the case of a multiple eigenvalue that a fixed eigenvector $\psi(1)$ is chosen for the construction of the discrete solution.

Concerning the multiplication of the matrix from the right $\boldsymbol{A}$ has always the eigenvalue c_N, since $c_N \neq 0$ is the entry of $\boldsymbol{A}$ for $j = k = N$, and $c_{2j-k} = 0$ for $j \geq N, k \leq N$ provided that $(j, k) \neq (N, N)$.

Proposition 2.3 *To the eigenvalue c_N of $\boldsymbol{A}$ there belongs a left eigenvector*

$$\boldsymbol{x} = (x_1, x_2, x_3, \ldots) \tag{2.8}$$

with $x_1 = \ldots = x_{N-1} = 0$, $x_N = 1$. The generating function

$$G(z) = \sum_{k=0}^{\infty} x_{N+k} z^k \tag{2.9}$$

with $G(0) = 0$ has the product representation

$$G(z) = \prod_{j=0}^{\infty} \frac{z^{2^j N}}{c_N} Q\left(z^{-2^j}\right), \tag{2.10}$$

and both representations converge for $|z| < 1$.

Proof: The components of the left eigenvector (2.8) must satisfy the equations

$$\sum_{j=1}^{\infty} c_{2j-k} x_j = c_N\, x_k \tag{2.11}$$

for $k \in \mathbb{N}$. In the case $k \leq N$ these equations are satisfied for $x_1 = \ldots = x_{N-1} = 0$, $x_N = 1$, and in the case $k > N$ they are recursions, which determine x_k uniquely. By multiplication with z^{k-N} and summation over k we obtain formally

$$c_N G(z) = z^N Q\left(\frac{1}{z}\right) G(z^2). \tag{2.12}$$

A continuous solution of this equation with $G(0) = 1$ must have the form (2.10) for $|z| < 1$. Vice versa, (2.10) converges uniformly for $|z| \le \varrho < 1$ and satisfies (2.12), hence also (2.9) is convergent for $|z| < 1$ ■

The next three values of x_j after $x_N = 1$ are

$$x_{N+1} = \frac{c_{N-1}}{c_N}, \quad x_{N+2} = \frac{c_{N-1} + c_{N-2}}{c_N}, \quad x_{N+3} = \left(\frac{c_{N-1}}{c_N}\right)^2 + \frac{c_{N-3}}{c_N}. \tag{2.13}$$

Besides of $\boldsymbol{x}$ from (2.8) we introduce the vector $\boldsymbol{y} = (y_1, y_2, y_3, \ldots)$, the components of which are determined by the initial values

$$y_1 = \ldots = y_{N-1} = 0, \quad y_N = 1 \tag{2.14}$$

and the recursions

$$\sum_{j=1}^{\infty} c_{N+k-2j}\, y_j = c_0\, y_k \qquad (k \in \mathbb{N}). \tag{2.15}$$

These recursions mean that

$$\left.\begin{aligned} c_{N-1}\, y_\ell + c_{N-3}\, y_{\ell+1} + \ldots &= c_0\, y_{2\ell-1}, \\ c_N\, y_\ell + c_{N-2}\, y_{\ell+1} + \ldots &= c_0\, y_{2\ell}, \end{aligned}\right\} \qquad (\ell \in \mathbb{N}). \tag{2.16}$$

The vector $\boldsymbol{y}$ arises from $\boldsymbol{x}$ if c_j is replaced by c_{N-j} for $j = 0, \ldots, N$, i.e. if (2.1) is replaced by the reversed polynomial $z^N Q(\frac{1}{z})$, cf. [3]. Hence for $|z| < 1$ we see from (2.10) that its generating function

$$F(z) = \sum_{k=0}^{\infty} y_{N+k} z^k \tag{2.17}$$

simplifies to

$$F(z) = \prod_{j=0}^{\infty} \frac{1}{c_0} Q\left(z^{2^j}\right), \tag{2.18}$$

and (2.12) simplifies to

$$F(z) = \frac{1}{c_0} Q(z) F\left(z^2\right). \tag{2.19}$$

Extending the components of the vector $\boldsymbol{y}$ by $y_j = 0$ for $j \le 0$ we can state:

Theorem 2.4 *For $\ell \in \mathbb{N}_0$ the matrix $\boldsymbol{A}^\ell$ has the entries*

$$c_0^\ell\; y_{2^\ell+N-1}, \quad c_0^\ell\; y_{2^\ell+N-2}, \ldots \tag{2.20}$$

in the first row, and the first $2^\ell + N - 1$ entries of the N^{th} row read

$$c_N^\ell\, x_1,\; c_N^\ell\, x_2,\; \ldots, c_N^\ell\, x_{2^\ell+N-1}. \tag{2.21}$$

Moreover, for non-negative integers $k < 2^\ell$ the distributional solution of (1.1) *satisfies the equations*

$$\varphi\left(\frac{k+t}{2^\ell}\right) = c_0^\ell \sum_{j=0}^{k} y_{N+k-j}\varphi(j+t) \qquad (t < 1). \tag{2.22}$$

Proof: The statement about (2.21) is valid for $\ell = 0$ (and in view of (2.13) also for $\ell = 1$). If it is valid for a fixed $\ell \in \mathbb{N}$, then $\boldsymbol{A}^\ell \boldsymbol{A} = \boldsymbol{A}^{\ell+1}$ and (2.11) with $k \leq 2^{\ell+1} + N - 1$ as well as $2j - k \leq N$, i.e. $j \leq 2^\ell + N - 1$, show its validity also for $\ell + 1$ instead of ℓ. Analogously, the statement about (2.20) is valid for $\ell = 0$ and proved in general, if

$$\sum_{j=1}^{\infty} c_{2j-k}\ y_{2^\ell+N-j} = c_0\ y_{2^{\ell+1}+N-k} \qquad (k \in \mathbb{N}).$$

But this equation is equivalent to (2.15).

The equation (2.4) implies that

$$\Psi\left(\frac{2^\ell + t - 1}{2^\ell}\right) = \boldsymbol{A}^\ell \Psi(t) \qquad (t < 1). \tag{2.23}$$

Replacing t by $t + k + 1 - 2^\ell$ with $k \leq 2^\ell - 1$ and considering the properties of (2.20), we obtain for the first component of Ψ

$$\varphi\left(\frac{k+t}{2^\ell}\right) = c_0^\ell \sum_{j=1}^{\infty} y_{2^\ell+N-j}\varphi(k+t+j-2^\ell) \tag{2.24}$$

and therefore (2.22) in view of $y_n = 0$ for $n < N$ ■

Using equation (2.22) for $t = 0$ and according to Remark 1.1/1 also for $t = 1$, we obtain

Corollary 2.5 *The discrete solution of* (1.1) (*cf. Remark* 2.2.) *has the representation*

$$\varphi\left(\frac{k}{2^\ell}\right) = c_0^\ell \sum_{j=1}^{k} y_{N+k-j}\varphi(j) \tag{2.25}$$

$(0 \leq k \leq 2^\ell, \ell \in \mathbb{N}_0)$, *where the sequence y_j is defined by* (2.14) *and* (2.16).

Remark 2.6

1. Usually, one works with the matrices $T_0 = (c_{2j-k-1})$, $T_1 = (c_{2j-k})$, j, $k = 1, \ldots, N$, instead of A, cf. [10]. The products of these matrices turn out to be submatrices of $\boldsymbol{A}^\ell$. Namely, if $K \in \mathbb{N}_0$ with $K < 2^\ell$ has the dyadic representation $K = d_\ell d_{\ell-1} \ldots d_1$, $d_i \in \{0, 1\}$, $i = 1, \ldots, \ell$ ($d_\ell = 0$ is allowed), then the product $T_{d_\ell} T_{d_{\ell-1}} \ldots T_{d_1}$ is equal to the submatrix of $\boldsymbol{A}^\ell$ with $1 \leq j \leq N$, $2^\ell - K \leq k \leq 2^\ell + N - K - 1$.

2. In the case $0 < t < 1$ the terms $\varphi(j+t)$ with $j \geq N$ in (2.22) can be replaced by the polynomials $\pi(j+t)$ known from condition (ii), so that we obtain for $k \geq N$

$$\varphi\left(\frac{k+t}{2^\ell}\right) = c_0^\ell \sum_{j=0}^{N-1} y_{N+k-j}\varphi(j+t) + c_0^\ell \sum_{j=N}^{k} y_{N+k-j}\pi(j+t) \qquad (0 < t < 1). \tag{2.26}$$

For $t = 0$ we get a modification of (2.25).

3. By means of a suitable interpretation, the foregoing results are also valid in the excluded case $N = 1$. In particular, (2.18) and (2.19) yield for $N = 1$ that $y_{k+1} = (\frac{c_1}{c_0})^{\nu(k)}$ with ν as at the end of Section 1, and (2.26) turns over into [1], (3.16).

Again for $N \geq 2$, the equations (2.22) have the disadvantage that they contain values of φ with arguments greater than 1. However, we can eliminate these values out of the first N equations or out of N arbitrary equations of (2.22) (in the version (2.26) for $k \geq N$). The result simplifies if we use additional equations as

$$\sum_{j=0}^{N-1} \xi_j \varphi(t+j) = S(t) \qquad (0 < t < 1) \tag{2.27}$$

with known coefficients ξ_j and known polynomials $S(t)$ (Concerning the existence and the number of linearly independent equations (2.27) cf. [4]). We only want to point out this in the most important case that we use only one additional relation, namely

$$\varphi(t) + \varphi(t+1) + \ldots + \varphi(t+N-1) = K \qquad (0 < t < 1), \tag{2.28}$$

which is valid in the case $M = 1$ in (1.2) almost everywhere for locally Lebesgue-integrable solutions of (1.1) where $\int_0^N \varphi(t)dt = K \neq 0$, cf. [5], [11]. In the case $M = 1$ equation (2.28), with a certain constant $K \neq 0$, is valid even for distributional solutions if and only if Q in (2.1) has a representation of the form $Q(z) = \frac{R(z^2)}{R(z)}P(z)$ where R and P are polynomials with $R(0)P(0) \neq 0$, $R(1) \neq 0$ and $P(-1) = 0$, cf. [3], Theorem 7.2. In this case it is $Q(1) = 2$ and therefore $M = 1$ in (1.2), too.

Proposition 2.7 *Let φ be a distributional solution of* (1.1) *such that* (2.28) *is satisfied. Then for fixed $k \in \mathbb{N}_0$ there exist constants Y_j $(j = 0, 1, \ldots, N-2)$, depending on k, such that for arbitrary $\ell \in \mathbb{N}$ with $2^\ell > k$ it holds*

$$\frac{1}{c_0^\ell}\varphi\left(\frac{k+t}{2^\ell}\right) = KY_{N-1} + \sum_{j=0}^{N-2} Y_j \frac{1}{c_0^{m_j}}\varphi\left(\frac{j+t}{2^{m_j}}\right) \qquad (0 < t < 1) \tag{2.29}$$

where $m_j = [\frac{\ln j}{\ln 2}] + 1$.

Proof: In view of $M = 1$ we have $\varphi(t) = 0$ for $t > N$. On the other side, $y_n = 0$ for $n < N$ and $y_N = 1$, so that for $k = 1, 2, \ldots, N-2$ and an arbitrary fixed k the equations (2.22), divided by c_0^ℓ, together with (2.28) yield the system

$$\begin{array}{rcccccccc}
c_0^{-\ell}\varphi\left(\frac{1+t}{2^\ell}\right) & = & y_{N+1}\varphi(t) & + & \varphi(t+1), & & & & \\
c_0^{-\ell}\varphi\left(\frac{2+t}{2^\ell}\right) & = & y_{N+2}\varphi(t) & + & y_{N+1}\varphi(t+1) & + & \varphi(t+2), & & \\
\vdots & & \vdots & & \vdots & & \ddots & & \\
c_0^{-\ell}\varphi\left(\frac{N-2+t}{2^\ell}\right) & = & y_{2N-2}\varphi(t) & + & y_{2N-3}\varphi(t+1) & +\ldots+ & \varphi(t+N-2), & & \\
K & = & \varphi(t) & + & \varphi(t+1) & + & \ldots & + & \varphi(t+N-1), \\
c_0^{-\ell}\varphi\left(\frac{k+t}{2^\ell}\right) & = & y_{N+k}\varphi(t) & + & y_{N+k-1}\varphi(t+1) & + & \ldots & + & y_{k+1}\varphi(t+N-1)
\end{array}$$

with a Hessenberg matrix as coefficient matrix on the right-hand side. Choosing

$$(Y_1, Y_2, \ldots, Y_{N-1}, -1) \tag{2.30}$$

orthogonal to the last $N-1$ columns of this matrix, which is always possible, we obtain by multiplication from the left

$$c_0^{-\ell}\sum_{j=1}^{N-2} Y_j\,\varphi\left(\frac{j+t}{2^\ell}\right) + KY_{N-1} - c_0^{-\ell}\,\varphi\left(\frac{k+t}{2^\ell}\right) = -Y_0\,\varphi(t)$$

where

$$Y_0 = y_{N+k} - \sum_{j=1}^{N-2} Y_j y_{N+j} - Y_{N-1}. \tag{2.31}$$

Since $c_0^{-\ell}\varphi(\frac{j+t}{2^\ell})$ is independent of ℓ for $j \le 2^\ell - 1$, we finally obtain (2.29) for $1 \le k < 2^\ell$, but this equation is also valid for $k = 0$ with $Y_j = 0$ for $j > 0$ and $Y_0 = 1$ ∎

Though k was fixed in Proposition 2.7, the result (2.29) is valid for arbitrary $k \in \{0, 1, \ldots, 2^\ell - 1\}$, where the constants Y_j are depending on k, but not on ℓ.

Proposition 2.8 *In the case $\ell > m_{N-2}$ the system (2.29) of 2^ℓ equations with $0 \le k \le 2^\ell - 1$ determines its distributional solution φ uniquely.*

Proof: The system (2.29) is an inhomogeneous one, since $K \ne 0$ and $Y_{N-1} = y_{k+1}$ is equal to 1 for $k = N-1$, and it is $N - 1 < 2^\ell$ in view of $N - 2 < 2^{m_{N-2}} < 2^\ell$. After n integrations with suitable great n (and arbitrary constants of integration) it defines a contractive operator in $C[0,1]$. Hence, the corresponding operator equation has a unique continuous solution, and the n^{th} (distributional) derivative of this is φ ■

As in the discrete case the solution φ can be extended from the interval (0,1) to greater t by means of (1.1) (only the integer values of t require an additional consideration). Note that the equations in (2.29) are nearly trivial for $0 \le k \le N-2$, since then we have $Y_k = 1$ and $Y_j = 0$ for $j \ne k$.

3 Four-coefficient equations

The foregoing results shall be illustrated by means of the equation

$$\varphi\left(\frac{t}{2}\right) = a\varphi(t) + (1-d)\varphi(t-1) + (1-a)\varphi(t-2) + d\varphi(t-3) \qquad (t \in \mathbb{R}) \tag{3.1}$$

($ad \ne 0$) with the characteristic polynomial

$$Q(z) = a + (1-d)z + (1-a)z^2 + dz^3 = (1+z)(a + (1-a-d)z + dz^2), \tag{3.2}$$

cf. (2.1), so that $N = 3$ in (1.1) and $M = 1$ in (1.2). Colella and Heil have determined the conditions for real a, d under which (3.1) has a nontrivial integrable compactly supported and, in particular, such a continuous solution, cf. Example 2 in [8]. Here we look for discrete and for distributional solutions.

The matrix

$$A = \begin{pmatrix} 1-d & a \\ d & 1-a \end{pmatrix},$$

cf. (2.6), has the eigenvalues 1 and $1-a-d$, and a right eigenvector $\psi(1)$ to the eigenvalue 1 reads

$$\begin{pmatrix} \varphi(1) \\ \varphi(2) \end{pmatrix} = \begin{pmatrix} a \\ d \end{pmatrix}. \tag{3.3}$$

It is simple and remains simple also in the case $d = -a\,(\ne 0)$. Hence the conditions of Theorem 2.1 are satisfied for arbitrary complex a, d and there exists a simple discrete solution of (3.1) without additional restrictions to the coefficients. According to (2.25) and (3.3) it has the representation

$$\varphi\left(\frac{k}{2^\ell}\right) = a^\ell(a\, y_{k+2} + d\, y_{k+1}), \tag{3.4}$$

$k \in \{0, 1, \dots, 2^\ell\}$ ($\ell \in \mathbb{N}_0$), and the recursions (2.16) specialize to

$$y_{2k-1} = \frac{1-a}{a} y_k + y_{k+1}, \qquad y_{2k} = \frac{d}{a}\, y_k + \frac{1-d}{a}\, y_{k+1} \qquad (k \in \mathbb{N}). \tag{3.5}$$

The initial values (2.14) imply in particular $y_4 = \frac{1-d}{a}$.

Moreover, equation (3.1) possesses a distributional solution. In view of $Q(1) = 2$ and $Q(-1) = 0$ it satisfies (2.28) with a certain K, so that we can apply Proposition 2.7. Since the components of the vector (2.30) with $N = 3$ determine to

$$Y_1 = y_{k+2} - y_{k+1}, \qquad Y_2 = y_{k+1},$$

and since (2.31) turns over into

$$Y_0 = y_{k+3} - \frac{1-d}{a} y_{k+2} + \frac{1-a-d}{a} y_{k+1}, \tag{3.6}$$

we obtain the

Corollary 3.1 *For $k \in \{0, 1, \ldots, 2^\ell - 1\}$ ($\ell \in \mathbb{N}$), the distributional solution φ of* (3.1) *with* (2.28) *satisfies the equations*

$$\varphi\left(\frac{k+t}{2^\ell}\right) = a^\ell \left[K\, y_{k+1} + Y_0 \varphi(t) + \frac{1}{a}(y_{k+2} - y_{k+1})\varphi\left(\frac{t+1}{2}\right)\right] \quad (0 < t < 1) \tag{3.7}$$

with y_ℓ from (3.5) *and Y_0 from* (3.6).

It is possible to simplify the recursions (3.5) by means of the substitution

$$z_k = y_{k+2} - y_{k+1}, \tag{3.8}$$

so that $z_0 = 0$, $z_1 = 1$. A short calculation yields

$$z_{2k} = \frac{1-a-d}{a} z_k, \qquad z_{2k+1} = \frac{d}{a} z_k + z_{k+1}. \tag{3.9}$$

This means that we have the special case $z_k = z_k(p, q)$ of (1.3) and (1.4) with $p = \frac{1-a-d}{a}$, $q = \frac{d}{a}$. Since (3.8) implies

$$y_{k+1} = \sum_{j=1}^{k-1} z_j, \tag{3.10}$$

and (3.6) that $Y_0 = z_{k+1} - \frac{1-a-d}{a} z_k$, we obtain from (3.7)

$$\varphi\left(\frac{k+t}{2^\ell}\right) = a^\ell \left[K \sum_{j=1}^{k-1} z_j + \left(z_{k+1} - \frac{1-a-d}{a} z_k\right)\varphi(t) + \frac{1}{a} z_k \varphi\left(\frac{1+t}{2}\right)\right] \tag{3.11}$$

for $0 < t < 1$. According to Remark 1.1/1 the equations (3.11) can be used for a discrete solution also in the cases $t = 0$ and $t = 1$, where in view of $\varphi(\frac{1}{2}) = a^2$ and (3.9), if we choose $K = a + d$ (matching to (3.3)), we obtain the representation

$$\varphi\left(\frac{k}{2^\ell}\right) = a^{\ell+1} \sum_{j=1}^{k} z_{2j-1}, \tag{3.12}$$

$k \in \{0, 1, \dots, 2^\ell\}$ ($\ell \in \mathbb{N}$), which is equivalent to (3.4) owing to (3.10). For $k = 2^\ell$ a simple consequence of (3.12) is

$$(p+q+1)^\ell = \sum_{j=1}^{2^\ell} z_{2j-1}(p,q)$$

since $\varphi(1) = a$ and $\frac{1}{a} = p+q+1$. Further properties of z_k are contained in [2].

Remark 3.2

1. In the case $d = -a$ we have $K = 0$, but (2.28) with this K is only satisfied at dyadic points, i.e. on a set with Lebesgue measure zero (whereas for a nontrivial distributional solution we recall that $K \neq 0$ there). The representation (3.12) simplifies to $\varphi(\frac{k}{2^\ell}) = a^{\ell+1} z_k$.

2. In the case $d = a$ equation (3.1) is symmetric (or self-reversed) so that $\varphi(t) = \varphi(3-t)$ for all $t \in \mathbb{R}$ (cf. [3], Corollary 8.6). Hence, (3.1) and (2.28) with $N = 3$ and $K = 2a$ yield the relation

$$\varphi\left(\frac{1+t}{2}\right) = 2a^2 + (1-2a)\varphi(t) - a\varphi(1-t) \quad (0 < t < 1)$$

which can be substituted into (3.11) with $d = a$.

Example 3.3 The case

$$\varphi\left(\frac{t}{2}\right) = \varphi(t) + \varphi(t-3) \qquad (t \in \mathbb{R}) \tag{3.13}$$

is a specialization of (3.1) with $a = d = 1$. The recursions (3.5) read

$$y_{2\ell-1} = y_{\ell+1}, \qquad y_{2\ell} = y_\ell,$$

and subject to the initial values (2.14) they have the solution

$$y_{3k} = 1, \qquad y_{3k-1} = y_{3k-2} = 0 \qquad (k \in \mathbb{N}).$$

According to (3.3) we have $\varphi(1) = \varphi(2) = 1$, and (2.25) turns over into

$$\varphi\left(\frac{k}{2^\ell}\right) = y_{k+2} + y_{k+1} = \begin{cases} 0 & \text{for} \quad k \equiv 0 \mod 3 \\ 1 & \text{for} \quad k \not\equiv 0 \mod 3. \end{cases}$$

Equation (3.13) does not have a nontrivial continuous solution, but it has the discontinuous solution

$$\varphi(t) = \begin{cases} c & \text{for} \quad t = 0 \\ 1 & \text{for} \quad 0 < t < 3 \\ 1-c & \text{for} \quad t = 3 \\ 0 & \text{elsewhere,} \end{cases}$$

with an arbitrary c. On the other side, it has a generalized discrete solution with conditions (i)-(ii), $\varphi(0) = c$, $\varphi(1) = \varphi(2) = 1$ and arbitrary $\varphi(3)$, which is interesting only in the case $\varphi(3) = 1 - c$.

Examples 3.4 Finally, we consider two examples with $N = 2$, which formally arise from the case $N = 3$ as limit case $c_3 \to 0$.

1. The three-coefficient equation

$$\varphi\left(\frac{t}{2}\right) = a\varphi(t) + \varphi(t-1) + (1-a)\varphi(t-2) \qquad (t \in \mathbb{R})$$

is the limit case of (3.1) as $d \to 0$. For $0 < a < 1$ it has a nontrivial compactly supported continuous solution, cf. [13]. According to (1.5) with (1.6) the system (3.9) has the explicit solution

$$z_k = \left(\frac{1}{a} - 1\right)^{\nu(k-1)}.$$

The corresponding representation (3.12) is already known from [1] and [13], disregarding one factor a which stems from a different normalization.

2. The three-coefficient equation

$$\varphi\left(\frac{t}{2}\right) = a\varphi(t) + (1-2a)\varphi(t-1) + a\varphi(t-2) \qquad (t \in \mathbb{R}) \tag{3.14}$$

is neither a limit case of (3.1) nor it can be treated by means of Proposition 2.7 according to $M = 0$ in (1.2), but we can apply Theorem 2.4. Equation (3.14) possesses the constant solution $\varphi(t) = 1$, which we only use for $t \geq 2$ according to condition (ii'). For $t = 2$ we find from (3.14) and condition (i') that $\varphi(1) = \frac{1}{2}$ and therefore $\Psi(1) = (\frac{1}{2}, 1, 1, 1, \ldots)^{\mathrm{T}}$. From (2.25) we obtain the discrete solution

$$\varphi\left(\frac{k}{2^\ell}\right) = a^\ell \left(\frac{1}{2} y_{k+1} + y_k + \ldots + y_2\right), \tag{3.15}$$

and the recursions (2.16) read

$$y_{2\ell-1} = \left(\frac{1}{a} - 2\right) y_\ell, \qquad y_{2\ell} = y_\ell + y_{\ell+1}$$

with the initial values $y_1 = 0$, $y_2 = 1$. In this case we also have a connection to the sequence defined by (1.3) and (1.4), namely $y_{\ell+1} = z_\ell(p,q)$ with $p = \frac{1}{a} - 2$ and $q = 1$. Let us mention the curiosity that $z_k(p,1)$ has the property

$$z_{3^n} = (p+1)^n \qquad \text{for } n = 0,\ 1,\ 2,\ 3 \text{ and } 6,$$

whereas for other integers n such a relation is unknown.

For the solutions of (3.14) we do not have a relation of the form (2.28). However, in view of the symmetry of (3.14) as well as (i), (ii) we have $\varphi(1+t)+\varphi(1-t)=1$ for $t\in\mathbb{R}$ (cf. [3], Corollary 8.6), so that we immediately obtain

$$\varphi\left(\frac{1+t}{2}\right)=a+(1-2a)\varphi(t)-a\varphi(1-t)$$

and after replacing t by $1-t$

$$\varphi\left(1-\frac{t}{2}\right)=a-a\varphi(t)+(1-2a)\varphi(1-t).$$

By means of these two equations and (3.14) for $0<t<1$ it can iteratively be shown for fixed $k\in\{0,1,\ldots,2^\ell-1\}$ ($\ell\in\mathbb{N}$) that $\varphi(\frac{k+t}{2^\ell})$ is a linear combination of 1, $\varphi(t)$ and $\varphi(1-t)$. But it is not necessary to carry out this procedure, since it is easy to see that $\varphi_3(t)=a\,[\varphi_2(t)-\varphi_2(t-1)]$ for all $t\in\mathbb{R}$, where φ_2 shall be the solution of (3.14) and φ_3 the solution of (3.1) with $d=a$. This means that $\varphi_2(t)=\frac{1}{a}\varphi_3(t)$ for $0<t<1$, so that the formulas (3.7) in the case $d=a$ immediately yield the formulas in question. In particular, the formulas (3.12) and (3.15) are modifications of each other.

4 Generating functions

Let φ be a discrete solution of (1.1). Condition (i') implies $\varphi\left(\frac{t}{2}\right)=c_0\varphi(t)$ for $t\le 1$. We define a function φ^* by

$$\varphi^*(t)=\varphi(t)\quad\text{for}\quad t\le 1,\tag{4.1}$$

and for $t>1$ we extend φ^* successively in the intervals $(2^{\ell-1},2^\ell]$ ($\ell\in\mathbb{N}$) by $\varphi^*(t)=\frac{1}{c_0}\varphi^*(\frac{t}{2})$, so that the equation

$$\varphi^*\left(\frac{t}{2}\right)=c_0\varphi^*(t)\tag{4.2}$$

is satisfied for all real (dyadic) t. Obviously, for $t\le 2^\ell$ ($\ell\in\mathbb{N}_0$) we have

$$c_0^{-\ell}\varphi\left(\frac{t}{2^\ell}\right)=\varphi^*(t),\tag{4.3}$$

and equation (2.25) can be written as

$$\varphi^*(k)=\sum_{j=1}^{k}y_{N+k-j}\varphi(j)\qquad(k\in\mathbb{N}_0).\tag{4.4}$$

In the following we consider the generating function

$$\Phi(z)=\sum_{k=1}^{\infty}\varphi(k)\,z^{k-1}\tag{4.5}$$

of the values $\varphi(k)$ (cf. [3], Lemma 2.9), as well as the generating function

$$\Phi^*(z) = \sum_{k=1}^{\infty} \varphi^*(k)\, z^{k-1} \tag{4.6}$$

of the values $\varphi^*(k)$ defined by (4.1) and (4.2). If $M \in \mathbb{N}$ in (1.2), then $\varphi(t) = 0$ for $t \geq N$, and (4.5) is a polynomial of degree $N-2$. Otherwise, (4.5) converges for $|z| < 1$ in view of (ii'). The convergence of (4.6) for the same z will follow from the next proposition.

Proposition 4.1 *The generating function* (4.6) *is representable in the form*

$$\Phi^*(z) = \Phi(z) F(z) \tag{4.7}$$

where F is given by (2.17), *and the generating function* (4.5) *satisfies the equation*

$$\Phi(z^2) = \frac{\Phi(z)\, Q(z) - \Phi(-z)\, Q(-z)}{2z} \tag{4.8}$$

with (2.1).

Proof: By multiplication of equation (4.4) with z^{k-1} and summation over k we find (4.7) with the functions (2.17) and (4.5). In view of $\varphi(t) = 0$ for $t \leq 0$, equation (1.1) with $t = 2k$ implies that

$$\varphi(k) = \sum_{j=1}^{2k} c_{2k-j}\, \varphi(j) \qquad (k \in \mathbb{N}).$$

By multiplication with z^{2k-2}, summation over k, and by means of

$$\Phi(z)Q(z) = \sum_{\ell=1}^{\infty} \sum_{j=1}^{\ell} c_{\ell-j} \varphi(j) z^{\ell-1}\ ,$$

we obtain (4.8) after short calculations ■

Remark 4.2

1. The generating function Φ^* satisfies the equation

$$\frac{1}{c_0}\, \Phi^*(z^2) = \frac{\Phi^*(z) - \Phi^*(-z)}{2z} \tag{4.9}$$

which is equivalent to the specialization

$$\varphi^*(k) = c_0\, \varphi^*(2k)$$

of (4.2) for $t = 2k$, but which also follows directly from (2.19), (4.7) and (4.8).

2. Note the connection of the product representation (2.18) for the generating function (2.17) of the coefficients in (2.24) with respect to the product representation

$$\mathcal{L}\{\varphi\} = z^{M-1} \prod_{j=1}^{\infty} \frac{1}{2^M} Q\left(e^{-\frac{z}{2^j}}\right)$$

(cf. [9], p. 175) for the Laplace transform of the solution φ of (1.1) satisfying (2.22).

Example 4.3 In the simple case $Q(z) = 2\left(\frac{1+z}{2}\right)^{m+1}$ with $m = N-1 \in \mathbb{N}$ we have $y_j = x_j$ ($j \in \mathbb{N}$) in view of the symmetry of the coefficients. Formula (2.18) yields

$$F(z) = \prod_{j=0}^{\infty} \left(1 + z^{2^j}\right)^{m+1} = \frac{1}{(1-z)^{m+1}}. \tag{4.10}$$

Using the normalization $\varphi(1) = 1$, the solution of (1.1) is $\varphi(t) = m!\, N_m(t)$, where $N_m(t)$ is the B-spline of degree m. Hence, (4.5) turns over into the Euler-Frobenius polynomials

$$\Phi(z) = m! \sum_{k=1}^{m} N_m(k) z^{k-1} = E_m(z), \tag{4.11}$$

cf. [7], [14]. On the other side, we have $\varphi(t) = t^m$ for $0 \le t \le 1$ and therefore $\varphi^*(t) = t^m$ for all $t \ge 0$ so that (4.6) turns over into

$$\Phi^*(z) = \sum_{k=1}^{\infty} k^m z^{k-1}. \tag{4.12}$$

According to (4.10) and (4.11) the relation (4.7) expresses the well known fact that for $|z| < 1$ the series on the right-hand side of (4.12) has the sum

$$\frac{E_m(z)}{(1-z)^{m+1}}.$$

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