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Guo-Jing Jiang

Common Fixed Points in Compact Metric Spaces

ABSTRACT. Common fixed point theorems for contractive type mappings of compact metric spaces are given. Our works generalize known results of Edelstein, Fisher, Leader, Jungck and Liu.

KEY WORDS AND PHRASES. Contractive type mappings, common fixed points, compact metric spaces.

1 Introduction

Throughout this paper, we assume that (X,d) is a compact metric space and that f,g,a and b are self mappings of (X,d). N and ω denote the sets of positive integers and nonnegative integers, respectively. For $x,y\in X$, define $O(x,f)=\{f^nx\,|\,n\in\omega\}$ and $O(x,y,f,g)=O(x,f)\cup O(y,f)\cup O(x,g)\cup O(y,g)$. Put $\delta(A)=\sup\{d(a,b)\,|\,a,b\in A\}$ for $A\subseteq X$. Define $C_f=\{h\,|\,h:X\to X \text{ and } hf=fh\}$, $H_f=\{h\,|\,h:X\to X \text{ and } h\bigcap_{n\in N}f^nX\subseteq\bigcap_{n\in N}f^nX\}$. Clearly $H_f\supseteq C_f\supseteq\{f^n\,|\,n\in\omega\}$.

In 1962, Edelstein [1] proved the following

Theorem E *Let f satisfy $d(fx,fy)<d(x,y)$ for all distinct $x,y\in X$. Then f has a unique fixed point.*

Fisher [2, 3, 4, 5], Leader [6], Jungck [7] and Liu [8] et al. gave a number of generalizations of Theorem E, some of which deal with contractive type mappings as follows:

1) $$d(fx,fy)<\max\{d(x,y),\frac{1}{2}[d(x,fx)+d(y,fy)],\frac{1}{2}[d(x,fy)+d(y,fx)]\}$$
   for all distinct $x,y\in X$ (Fisher [2]).

2) There exist $p,q\in N$ such that
   $$d(f^px,f^qy)<\max\{d(f^mx,f^ny),d(f^mx,f^ix),d(f^ny,f^jy)\,|\,0\le m,i\le p,0\le n,j\le q\}\tag{I}$$
   for all $x,y\in X$ for which the right-hand side of (I) is positive (Fisher [3]).

3) There exist $p, q, r, s \in \omega$, $p + q \geq 1$, $r + s \geq 1$ such that

$$d(f^p g^q x, f^r g^s y) < \max\{d(f^i g^j x, f^t g^k y), d(f^i g^j x, f^m g^n x), d(f^t g^k y, f^u g^v y) \mid 0 \leq i, m \leq p, 0 \leq j, n \leq q, 0 \leq t, u \leq r, 0 \leq k, v \leq s\} \quad \text{(II)}$$

for all $x, y \in X$ for which the right-hand side of (II) is positive (Fisher [4]).

4)
$$d(fx, gy) < \max\{d(ax, by), d(ax, fx), d(by, gy), d(ax, gy), d(by, fx)\} \quad \text{(III)}$$

for all $x, y \in X$ for which the right-hand side of (III) is positive (Fisher [5]).

5) There exists $p \in N$ such that

$$d(f^p x, f^p y) < \delta(O(x, f) \cup O(y, f))$$

for all distinct $x, y \in X$ (Leader [6]).

6)
$$d(fx, fy) < \delta(\bigcup_{h \in C_{gf}} h\{x, y\})$$

for all $x, y \in X$ with $fx \neq gy$ (Jungck [7]).

7)
$$d(fx, gy) < \delta(\bigcup_{h \in H_{gf}} h\{x, y\})$$

for all $x, y \in X$ with $fx \neq gy$ (Liu [8]).

The main purpose of this paper is to investigate the existence of common fixed points of self mappings f, g of (X, d) satisfying the following contractive type conditions:

8) There exist $p, q, r, s \in \omega$, $p + q \geq 1$, $r + s \geq 1$, such that

$$d(f^p g^q x, f^r g^s y) < \delta(\bigcup_{i,j,m,n \in \omega, h \in H_{gf}} hO(x, y, g^i f^j, f^n g^m))$$

for all $x, y \in X$ with $f^p g^q x \neq f^r g^s y$.

9) There exist $p, q \in \omega$, $p + q \geq 1$, such that

$$d(f^p g^q x, f^p g^q y) < \delta(\bigcup_{i,j,m,n \in \omega, h \in H_{gf}} hO(x, y, g^i f^j, f^n g^m)) \quad \text{(IV)}$$

for all distinct $x, y \in X$.

In Section 2, we prove some contractive type conditions are equivalent and establish several common fixed point theorems which extend and unify some known results in [1]-[8]. In Section 3, we give five examples to show that our results are more general than the results in [1]-[8].

2 Common fixed points

We first define contractive type conditions as follows:

2)′ There exist $p, q \in N$ such that (I) holds for all $x, y \in X$ with $f^p x \neq f^q y$.

3)′ There exist $p, q, r, s \in \omega$, $p+q \geq 1$, $r+s \geq 1$, such that (II) holds for all $x, y \in X$ with $f^p g^q x \neq f^r g^s y$.

4)′ (III) holds for all $x, y \in X$ with $fx \neq gy$.

9)′ There exist $p, q \in \omega$, $p+q \geq 1$, such that (IV) holds for all $x, y \in X$ with $f^p g^q x \neq f^p g^q y$.

We now prove the following

Lemma 1 *Conditions* 2) *and* 2)′ *are equivalent.*

Proof: Let $R(x, y)$ and $L(x, y)$ denote the right-hand and left-hand sides of 2) respectively. Assume that 2) holds. For $x, y \in X$, if $L(x, y) > 0$, then $R(x, y) > 0$. By assumption it follows that (I) holds for $x, y \in X$ with $L(x, y) > 0$. This proves that 2) implies 2)′.

Conversely, for $x, y \in X$, if $R(x, y) > 0$, we consider two cases:

(i) $L(x, y) = 0$. Clearly $L(x, y) = 0 < R(x, y)$.

(ii) $L(x, y) > 0$. Since 2)′ holds, $L(x, y) < R(x, y)$. It follows that 2)′ implies 2).

Similarly we have

Lemma 2 *Conditions* 3) *and* 3)′ *are equivalent.*

Lemma 3 *Conditions* 4) *and* 4)′ *are equivalent.*

Lemma 4 *Conditions* 9) *and* 9)′ *are equivalent.*

The following lemma was given in [6, 7].

Lemma 5 *Let f be continuous and $B = \bigcap_{n \in N} f^n X$. Then B is compact, $B = fB \neq \emptyset$ and $\delta(f^n X) \to \delta(B)$ as $n \to \infty$. Further $hB \subseteq B$ for $h \in C_f$.*

Our main results are as follows:

Theorem 1 *Let gf be continuous and $f \bigcap_{n \in N} (gf)^n X = \bigcap_{n \in N} (gf)^n X$. If f and g satisfy* 8)*, then f and g have a unique common fixed point u and $(gf)^n x$ converges to u uniformly on X. Further, $u = hu$ for $h \in H_{gf}$.*

Proof: Let $B = \bigcap_{n\in N}(gf)^n X$. It follows from Lemma 5 that B is a nonempty compact subset of X and $gfB = B$. Consequently $gB = gfB = B$ since $fB = B$. Suppose that $\delta(B) > 0$. By the compactness of B, there exist $u, v \in X$ such that $\delta(B) = d(u, v)$. Since $fB = gB = B$, there exist $x, y \in B$ such that $u = f^p g^q x$, $v = f^r g^s y$. Obviously $\bigcup_{i,j,m,n\in\omega,h\in H_{gf}} hO(x, y, g^i f^j, f^n g^m) \subseteq B$. Using 8) we get

$$\begin{aligned}\delta(B) &= d(f^p g^q x, f^r g^s y)\\ &< \delta\Big(\bigcup_{i,j,m,n\in\omega,h\in H_{gf}} hO(x, y, g^i f^j, f^n g^m)\Big)\\ &\le \delta(B)\end{aligned}$$

which is impossible and hence $\delta(B) = 0$; i.e., B is a singleton, say, $B = \{u\}$ for some $u \in X$. It follows from $fB = gB = B$ that $fu = gu = u$; i.e., f and g have a common fixed point u.

Suppose that f and g have a second common fixed point v. Then $fv = gv = v = (gf)^n v \in (gf)^n X$ for $n \in N$. This means that $v \in \bigcap_{n\in N}(gf)^n X = B = \{u\}$. Therefore f and g have a unique common fixed point u. Note that $hB \subseteq B$ for $h \in H_{gf}$. Consequently $u = hu$ for $h \in H_{gf}$.

By Lemma 5, we have for $x \in X$ and $n \in N$

$$d((gf)^n x, u) \le \delta((gf)^n X) \to \delta(B) = 0 \text{ as } n \to \infty\,,$$

i.e., $(gf)^n x$ converges to u uniformly on X. This completes the proof.

From Theorem 1 we get immediately

Corollary 1 [8, Theorem 1]. *Let gf be continuous and $f\bigcap_{n\in N}(gf)^n X = \bigcap_{n\in N}(gf)^n X$. If f and g satisfy* 7)*, then the conclusion of Theorem* 1 *remains unchanged.*

Proof: In case $p = s = 1$, $q = r = 0$, Corollary 1 follows from Theorem 1.

As a consequence of Corollary 1 we have

Corollary 2 [7, Theorem 4.2]. *Let gf be continuous and f and g commute. If f and g satisfy* 6)*, then f and g have a unique common fixed point u. Further, $u = hu$ for $h \in C_{gf}$.*

Remark 1 Example 1 reveals that Theorem 1 is indeed a proper extension of Theorem 1 of Liu [8] and Theorem 4.2 of Jungck [7].

Corollary 3 *Let gf be continuous and $f\bigcap_{n\in N}(gf)^nX = \bigcap_{n\in N}(gf)^nX$. If f and g satisfy*

10) $$d(f^pg^qx, f^rg^sy) < \delta(\bigcup_{i,j,m,n\in\omega} O(x, y, g^if^j, f^ng^m))$$

for all $x, y \in X$ with $f^pg^qx \neq f^rg^sy$, where $p, q, r, s \geq 0$ are fixed integers with $p+q \geq 1$, $r+s \geq 1$,

then f and g have a unique common fixed point.

Proof: Since 10) implies 8), Corollary 3 follows from Theorem 1.

Corollary 4 *Let gf be continuous and $f\bigcap_{n\in N}(gf)^nX = \bigcap_{n\in N}(gf)^nX$. If f and g satisfy* 3)*, then f and g have a unique common fixed point u. Further, if $q = 0$ or $s = 0$, then u is the unique fixed point of f and if $p = 0$ or $r = 0$, then u is the unique fixed point of g.*

Proof: Note that 3)$'$ implies 10). It follows from Lemma 2 and Corollary 3 that f and g have a unique common fixed point u. Suppose that $q = 0$ and that v is a second distinct fixed point of f. From 3) we have

$$0 < d(v, u) = d(f^pv, f^rg^su) < d(v, u)$$

giving a contradiction. This proves the uniqueness of u.

Similarly, we can prove that u is the unique fixed point of f if $s = 0$, and that u is the unique fixed point of g if $p = 0$ or $r = 0$.

Remark 2 Theorem 5 of Fisher [4] is a special case of Corollary 4. Example 2 proves that Corollary 4 is a substantial generalization of the result of Fisher.

Theorem 2 *Let fg be continuous and $g\bigcap_{n\in N}(fg)^nX = \bigcap_{n\in N}(fg)^nX$. If f and g satisfy*

11) $$d(f^pg^qx, f^rg^sy) < \delta(\bigcup_{i,j,m,n\in\omega,h\in H_{fg}} hO(x, y, g^if^j, f^ng^m))$$

for all $x, y \in X$ with $f^pg^qx \neq f^rg^sy$, where $p, q, r, s \geq 0$ are fixed integers and $p+q \geq 1$, $r+s \geq 1$,

then f and g have a unique common fixed point u and $(fg)^nx$ converges to u uniformly on X. Further, $u = hu$ for $h \in H_{fg}$.

The proof of this theorem goes in a similar fashion as that of Theorem 1, so we omit the proof.

Corollary 5 *Let fg be continuous, $g\bigcap_{n\in N}(fg)^nX = \bigcap_{n\in N}(fg)^nX$ and $a, b \in H_{fg}$. If f, g, a and b satisfy* 4)*, then f, g, a and b have a unique common fixed point u. Further, u is the unique common fixed point of f and a and of g and b.*

Proof: By Lemma 3 we conclude easily that 4) is a particular case of 11). It follows from Theorem 2 that f, g, a and b have a unique common fixed point u. Suppose that f and a have a second distinct common fixed point v. Using 4)

$$0 < d(v, u) = d(fv, gu) < d(v, u)$$

which is a contradiction. This proves the uniqueness of u. Similarly we can prove that u is the unique common fixed point of g and b.

Remark 3 By setting $g =$identity mapping in Theorem 2, we get an improved version of Theorem 4 of Fisher [3]. Example 3 demonstrates that Theorem 2 extends properly Theorem 4 of Fisher [3]. Example 4 reveals that Corollary 5 is indeed a generalization of Theorem 2 of Fisher [5].

From Theorem 1 and Lemma 4 we have

Theorem 3 *Let gf be continuous and $f\bigcap_{n\in N}(gf)^nX = \bigcap_{n\in N}(gf)^nX$. If f and g satisfy* 9)*, then the conclusion of Theorem* 1 *remains unchanged.*

Remark 4 Theorem E, Theorems 7, 8, 9 and 10 of Fisher [2] and Corollaries 2 and 3 of Leader [6] are particular cases of Theorem 3. Example 5 proves that Theorem 3 extends properly the results of Edelstein, Fisher and Leader.

3 Examples

Example 1 Let $X = \{1, 2, 3, 4\}$ with the usual metric, $f1 = f4 = 1$, $f2 = g2 = g4 = 2$, $f3 = g3 = 4$ and $g1 = 3$. Take $p = r = s = 2$, $q = 1$. It is easy to check that the conditions of Theorem 1 are satisfied. But Theorem 1 of Liu [8] and Theorem 4.2 of Jungck [7] are not applicable since 6) and 7) do not hold for $x = 1$ and $y = 3$.

Example 2 Let $X = [0, 1]$ with the usual metric, $fx = x/3$ for $x \in X$, $g1 = 0$ and $gx = x/2$ for $x \in [0, 1)$. Clearly $\bigcap_{n\in N}(gf)^nX = \bigcap_{n\in N}[0, 1/6^n] = \{0\}$. Set $p = s = 1$, $q = r = 0$. It is simple to verify that the assumptions of Corollary 4 are satisfied. But Theorem 5 of Fisher [4] is not applicable since $fg1 = 0 \neq 1/6 = gf1$.

Example 3 Let X and f be as in Example 1. For any $p, q \in N$, f does not satisfy 2) since

$$d(f^p 1, f^q 2) = 1 = \max\{d(f^m 1, f^n 2), d(f^m 1, f^i 1), d(f^n 2, f^j 2) \mid 0 \le m, i \le p, 0 \le n, j \le q\}$$

and hence Theorem 4 of Fisher [3] is not applicable. Define a self mapping g of (X, d) by $g1 = 3$, $g2 = g4 = 2$ and $g3 = 4$. Take $p = r = s = 2$, $g = 1$. It is easily seen that the hypotheses of Theorem 2 are valid.

Example 4 Let $X = [0, 1]$ with the usual metric, $f1 = 0$, $fx = x^2/2$ for $x \in [0, 1)$, $gx = x/3$, $ax = bx = x$ for $x \in [0, 1]$. Then $\bigcap_{n \in N} (fg)^n X = \bigcap_{n \in N} [0, 1/18^{2^n - 1}] = \{0\}$. It is easy to check that the conditions of Corollary 5 are satisfied. Theorem 2 of Fisher [5] however is not applicable since $fg1 = 1/18 \ne 0 = gf1$.

Example 5 Let $X = \{1, 2, 3, 7\}$ with the usual metric, $f1 = 1$, $f2 = f7 = 2$ and $f3 = 7$. Then Theorem E, Theorems 7, 8, 9 and 10 of Fisher [2] and Corollaries 2 and 3 of Leader [6] are not applicable since 5) does not hold for every $p \in N$ and $x = 1$, $y = 2$. Define a self mapping g of X by $g1 = g2 = g3 = 2$ and $g7 = 1$. Choose $p = q = 1$. It is easy to show that the conditions of Theorem 3 are satisfied.

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Author:

G.-J. Jiang
Dalian Management Cadre's College
Dalian
Liaoning 116033
People's Republic of China