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Pavel G. Todorov

Taylor Series of the Pick Function and the Loewner Variation within the Class S and Applications[0]

ABSTRACT. We find the unknown Taylor series of the Pick function (4) (or (8)) and the Loewner variation (5). As an application we prove the local Bieberbach conjecture for the Loewner variation (5) in a simple way (see Corollaries 3.2 and 3.3 and their consequences).

KEY WORDS. Pick function, Loewner variation, Gauss hypergeometric function, Taylor series, local Bieberbach conjecture, Bombieri proof in the class S, our proof for the Loewner variation within the class S.

1 Introduction

Let S denote the class of all functions

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_1 = 1, \tag{1}$$

analytic and univalent in the unit disk $D(z) = \{z : |z| < 1\}$. The family $f_\lambda(z)$ ($0 \leqq \lambda < \lambda_0$ for some $\lambda_0 > 0$) of functions is called a variation of $f(z)$ within S if $f_0(z) = f(z)$ and $f_\lambda(z) \in S$ and if

$$\dot{f}_0 = \frac{\partial}{\partial \lambda} f_\lambda(z)\Big|_{\lambda=0} = \lim_{\lambda \to 0} \frac{f_\lambda(z) - f(z)}{\lambda} \tag{2}$$

exists locally uniformly in $D(z)$ (Pommerenke [1, p. 185]). Let the function (1) be the rotated Koebe function

$$g(z) = \frac{z}{(1 + \overline{\zeta} z)^2} = \sum_{n=1}^{\infty} n(-\overline{\zeta})^{n-1} z^n \in S, \quad |\zeta| = 1, \tag{3}$$

and G be its inverse function, i.e. $G(g(z)) = z$. If $\lambda > 0$, then the function

$$w = G(e^{-\lambda} g(z)) = e^{-\lambda} z + \dots \tag{4}$$

[0] Änderung zur Druckvorlage! Korrigiert am 19.4.2002 Formel (4) und Formel (21) letzte Zeile

maps $D(z)$ onto $D(w)$ minus some slit. The function (4) and its normalization are known as the Pick function [2] or as the bounded Koebe function (Goodman [3, pp. 36-38]). It follows from (1) and (4) that the function

$$f_\lambda(z) = e^\lambda f(G(e^{-\lambda}g(z))) = z + \cdots \in S. \tag{5}$$

Further it follows from (2), (5) and (3) that

$$\dot{f}_0(z) = f(z) - \frac{g(z)}{g'(z)}f'(z) = f(z) - zf'(z)\frac{\zeta + z}{\zeta - z}. \tag{6}$$

The familiy (5) with $0 \leqq \lambda < \lambda_0$ is called the Loewner variation within S (Pommerenke [1, p. 185, Example 7.3]). Having in mind (1), the Taylor series of (6) is easily obtained (Pommerenke [1, p. 191, Formula (10)]):

$$\dot{f}_0(z) = -\sum_{n=2}^{\infty}\left[(n-1)a_n + 2\sum_{j=1}^{n-1}(n-j)a_{n-j}\overline{\zeta}^j\right]z^n, \quad a_1 = 1, \tag{7}$$

for $z \in D(z)$ and $|\zeta| = 1$. But the explicit form of the Taylor series of (4) and (5) are unknown. Now we will give a method with the help of which we discover the Taylor series of (4) and (5) even in a wider range $0 \leqq \lambda < +\infty$.

2 Taylor series of the natural powers of the Pick function (4)

It is clear from (3) and (4) that the Pick function (4) can also be determined by the equation

$$\frac{z}{(1+\overline{\zeta}z)^2} = \frac{e^\lambda w}{(1+\overline{\zeta}w)^2}, \quad z \in D(z), \quad 0 \leqq \lambda < +\infty, \quad |\zeta| = 1. \tag{8}$$

The function w, determined by (8), maps the disk $D(z)$ onto the disk $D(w)$ except for a slit along the rectilinear segment from the point

$$G\left(\frac{e^{-\lambda}\zeta}{4}\right) = \zeta\left[2e^\lambda - 1 - 2\sqrt{e^\lambda(e^\lambda - 1)}\right]$$

to the point

$$G\left(\zeta\left[1 - 2e^{-\lambda} \pm 2i\sqrt{e^{-\lambda}(1-e^{-\lambda})}\right]\right) = \zeta,$$

where

$$|w| \leqq |z|, \quad z \in D(z), \quad 0 \leqq \lambda < +\infty, \tag{9}$$

where, for $0 < \lambda < +\infty$, the equality holds only for $z = 0$. Further we will use the Pochhammer symbol

$$(a)_\nu = a(a+1)\dots(a+\nu-1), \quad \nu = 1,2,\dots; \quad (a)_0 = 1, \tag{10}$$

for an arbitrary number a, the Gauss hypergeometric series (function, respectively)

$$F(\alpha,\beta;\gamma;x) = \sum_{\nu=0}^{\infty} \frac{(\alpha)_\nu(\beta)_\nu}{(\gamma)_\nu \nu!} x^\nu \tag{11}$$

for arbitrary parameters (or elements) α, β and γ with $\gamma \neq 0, -1, -2, \dots$, and a variable x with $|x| < 1$, the summation formula

$$F(\alpha,\beta;\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}, \quad Re(\gamma-\alpha-\beta) > 0, \tag{12}$$

where Γ is the Gamma function, and the transformation formula

$$F(\alpha,\beta;\gamma;x) = (1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha,\gamma-\beta;\gamma;x) \tag{13}$$

(see, for example, in [4, pp. 1065-1071]). For $\alpha = -n$, $n = 0,1,2,\dots$, the series (11) is reduced to the hypergeometric polynomial

$$F(-n,\beta;\gamma;x) = \sum_{\nu=0}^{n} \frac{(-n)_\nu(\beta)_\nu}{(\gamma)_\nu \nu!} x^\nu \tag{14}$$

for all x. If $Re(\gamma+n-\beta) \leqq 0$, then the formula (12) is not always applicable to (14) when $x = 1$. But there is another general summation formula which is given without a proof in the Gould tables for binomial coefficent summations [5, p. 58, Identity (7.1) and p. 61, Identity (7.20)] and in the Bailey monograph [6, p. 3], namely:

$$F(-n,\beta;\gamma;1) = \frac{(\gamma-\beta)_n}{(\gamma)_n}, \quad n = 1,2,\dots, \quad \gamma \neq 0,-1,-2,\dots,-n+1 \tag{15}$$

having in mind the notation (10). A simple proof of the identity (15) can be obtained with the help of the following procedure. We have the identity

$$(-n)_\nu = (-n+1)_\nu - \nu(-n+1)_{\nu-1}, \quad n = 1,2,\dots, \quad \nu = 1,2,\dots,n. \tag{16}$$

From (14) for $x = 1$ and (16) we obtain the identity

$$F(-n,\beta;\gamma;1) = \frac{\gamma-\beta}{\gamma} F(-n+1,\beta;\gamma+1;1), \quad n = 1,2,\dots. \tag{17}$$

By induction on the first and third elements in (17) we attain to (15).

Theorem 1 *The natural powers w^p, $p = 1, 2, \ldots$, of the Pick function w, determined by (4) (or (8)), have the following Taylor series*

$$w^p = e^{-p\lambda} \sum_{n=p}^{\infty} (-\overline{\zeta})^{n-p} \binom{n+p-1}{2p-1} F(p-n, p+n; 2p+1; e^{-\lambda}) z^n \tag{18}$$

for $|z| < 1$, $|\zeta| = 1$ and $0 \leqq \lambda < +\infty$, where

$$F(p-n, p+n; 2p+1; e^{-\lambda}) = \sum_{\nu=0}^{n-p} \frac{(p-n)_\nu (p+n)_\nu}{(2p+1)_\nu \nu!} e^{-\nu\lambda} \tag{19}$$

are the hypergeometric polynomials in $e^{-\lambda}$, determined according to (10)-(11) and (14).

Proof: Let the complex number z and the real number r be fixed with $0 \leqq |z| < r < 1$. Then, on the basis of (9) for the Pick function w, determined by (4) (or(8)), and the Cauchy theorem on residues, we have the integral representation

$$w^p = \frac{1}{2\pi i} \int\limits_{|u|=r} u^p \frac{e^\lambda g'(u)}{e^\lambda g(u) - g(z)} du \tag{20}$$

for $0 \leqq \lambda < +\infty$, and $p = 1, 2, \ldots$, where the integration along the circle $|u| = r$ is performed in positive direction. From (20) and (3) we obtain the Taylor series as follows

$$\begin{aligned} w^p &= \frac{1}{2\pi i} \int\limits_{|u|=r} \frac{u^{p-1}(1-\overline{\zeta}u)}{1+\overline{\zeta}u} \frac{du}{1 - \frac{(1+\overline{\zeta}u)^2}{e^\lambda u} \frac{z}{(1+\overline{\zeta}z)^2}} = \\ &= \sum_{\nu=p}^{\infty} \frac{z^\nu}{(1+\overline{\zeta}z)^{2\nu}} \frac{e^{-\nu\lambda}}{2\pi i} \int\limits_{|u|=r} \frac{(1-\overline{\zeta}u)(1+\overline{\zeta}u)^{2\nu-1}}{u^{\nu-p+1}} du = \\ &= \sum_{\nu=p}^{\infty} (-1)^\nu \overline{\zeta}^{-p} \frac{p}{\nu} \binom{2\nu}{\nu-p} e^{-\nu\lambda} \sum_{n=\nu}^{\infty} \binom{n+\nu-1}{n-\nu} (-\overline{\zeta})^n z^n = \\ &= e^{-p\lambda} \sum_{n=p}^{\infty} (-\overline{\zeta})^{n-p} z^n \sum_{\nu=0}^{n-p} c_\nu(n,p) e^{-\nu\lambda} \end{aligned} \tag{21}$$

for $|z| < 1$, $0 \leqq \lambda < +\infty$, $p = 1, 2, \ldots$, where

$$c_\nu(n,p) = (-1)^\nu \frac{p}{\nu+p} \binom{2\nu+2p}{\nu} \binom{n+\nu+p-1}{n-p-\nu} \tag{22}$$

for $0 \leqq \nu \leqq n-p$, $n \geqq p$, $p = 1, 2, \ldots$. It follows from (22) that

$$\frac{c_\nu(n,p)}{c_{\nu-1}(n,p)} = \frac{(p-n+\nu-1)(p+n+\nu-1)}{(2p+\nu)\nu} \tag{23}$$

for $1 \leqq \nu \leqq n-p$, $n \geqq p+1$, $p = 1, 2, \ldots$, with

$$c_0(n,p) = \binom{n+p-1}{2p-1}, \quad n \geqq p, \quad p = 1, 2, \ldots. \tag{24}$$

From (23) and (24) we deduce that

$$c_\nu(n,p) = \binom{n+p-1}{2p-1} \frac{(p-n)_\nu (p+n)_\nu}{(2p+1)_\nu \nu!} \tag{25}$$

for $0 \leqq \nu \leqq n-p$, $n \geqq p$, $p = 1, 2, \ldots$, having in mind the notation (10). With the help of (25), having in mind (11) and (14), the expansion (21) takes the form (18)-(19).

This completes the proof of Theorem 1.

Corollary 2.1 *The natural powers w^p, $p = 1, 2, \ldots$, of the Pick function w, determined by (4) (or (8)), have the following Taylor series*

$$\begin{aligned} w^p = {} & e^{-p\lambda} z^p + e^{-p\lambda}(1 - e^{-\lambda}) \cdot \\ & \cdot \sum_{n=p+1}^{\infty} (-\overline{\zeta})^{n-p} \binom{n+p-1}{2p-1} F(p+1-n, p+1+n; 2p+1; e^{-\lambda}) z^n \end{aligned} \tag{26}$$

for $|z| < 1$, $|\zeta| = 1$ and $0 \leqq \lambda < +\infty$, where

$$F(p+1-n, p+1+n; 2p+1; e^{-\lambda}) = \sum_{\nu=0}^{n-p-1} \frac{(p+1-n)_\nu (p+1+n)_\nu}{(2p+1)_\nu \nu!} e^{-\nu\lambda} \tag{27}$$

are the hypergeometric polynomials in $e^{-\lambda}$, determined according to (10)-(11) and (14).

Proof: By (13) we have the factorization

$$F(p-n, p+n; 2p+1; e^{-\lambda}) = (1 - e^{-\lambda}) F(p-n+1, p+n+1; 2p+1; e^{-\lambda}) \tag{28}$$

for $n \geqq p+1$. Thus from (28) and (18) we obtain (26)-(27).

3 Taylor series of the Loewner variation (5)

These series originate from (18)-(19), namely:

Theorem 2 *The Loewner variation $f_\lambda(z)$, determined by (5), has the following Taylor series*

$$f_\lambda(z) = \sum_{n=1}^{\infty} a_n(\lambda) z^n \in S, \quad |z| < 1, \quad 0 \leqq \lambda < +\infty, \quad a_1(\lambda) = 1, \tag{29}$$

where

$$a_n(\lambda)=\sum_{p=1}^{n}a_p(-\overline{\zeta})^{n-p}e^{-(p-1)\lambda}\binom{n+p-1}{2p-1}F(p-n,p+n;2p+1;e^{-\lambda}), \tag{30}$$

where a_p, $1\leqq p\leqq n$, *are the coefficients of the function* $f(z)$ *in* (1), $|\zeta|=1$, *and* F *are the hypergeometric polynomials in* $e^{-\lambda}$, *determined by* (19) *for* $p=1,2,\ldots,n$, $n\geqq 1$.

Proof: From (5), (4) and (1) we have

$$f_\lambda(z)=e^{\lambda}f(w)=e^{\lambda}\sum_{p=1}^{n}a_pw^p. \tag{31}$$

Now we use (18)-(19) to (31) to obtain (29)-(30).

Corollary 3.1 *The Loewner variation* $f_\lambda(z)$, *determined by* (5) *has the following representation*

$$f_\lambda(z)=e^{\lambda}f(ze^{-\lambda})+(1-e^{-\lambda})\sum_{n=2}^{\infty}b_n(\lambda)z^n\in S,\quad |z|<1,\quad 0\leqq\lambda<+\infty, \tag{32}$$

where

$$b_n(\lambda)=\sum_{p=1}^{n-1}a_p(-\overline{\zeta})^{n-p}e^{-(p-1)\lambda}\binom{n+p-1}{2p-1}F(p+1-n,p+1+n;2p+1;e^{-\lambda}) \tag{33}$$

where a_p, $1\leqq p\leqq n-1$, $n\geqq 2$, *are the coefficients of the function* $f(z)$ *in* (1), $|\zeta|=1$, *and* F *are the hypergeometric polynomials in* $e^{-\lambda}$, *determined by* (27) *for* $p=1,2,\ldots,n-1,n\geqq 2$.

Proof: The transformation of (30) for $1\leqq p\leqq n-1$, $n\geqq 2$, by means of (28) leads us to (32)-(33). The representation (32)-(33) can also be obtained from (31) and (26).

Corollary 3.2 *The Loewner variation* (5) *tends to the rotated Koebe function* (3) *as* $\lambda\to+\infty$, *i.e.*

$$\lim_{\lambda\to+\infty}f_\lambda(z)=g(z),\quad |z|<1. \tag{34}$$

Proof: As $\lambda\to+\infty$ it follows that the series (29), determined by (30) and (19), is reduced to the series (3) and hence the relation (34) holds. Also the relation (34) follows from the representation (32), determined by (33) and (27).

Corollary 3.3 *Let* $n\geqq 2$ *be an arbitrary fixed integer. Then the Taylor coefficient* (30) *of the Loewner variation* (5) *has the following asymptotic representation*

$$a_n(\lambda)=n(-\overline{\zeta})^{n-1}+e^{-\lambda}(-\overline{\zeta})^{n-2}\binom{n+1}{3}(a_2+2\overline{\zeta})+O(e^{-2\lambda}),\quad 0\leqq\lambda<+\infty, \tag{35}$$

where $|\zeta|=1$, a_2 *is determined by* (1) *and* $O(e^{-2\lambda})$ *denotes a magnitude, the ratio of which to* $e^{-2\lambda}$ *for* $0\leqq\lambda<+\infty$ *is bounded by some positive constant depending only on* n.

Proof: This follows from (30) and the Bieberbach result [7] that the coefficient body $(a_2, a_3, \ldots, a_n)$ is bounded in the class S (see also Schaeffer and Spencer [8, p. 12, Lemma II]).

In particular, for $\zeta = -1$ it follows from (35) that

$$a_n(\lambda) = n + e^{-\lambda}\binom{n+1}{3}(a_2 - 2) + O(e^{-2\lambda}), \quad 0 \leqq \lambda < +\infty. \tag{36}$$

By the Bieberbach inequality $|a_2| \leqq 2$ [7] with equality only for the Koebe function (3) (see also Pommerenke [1, p. 24]) it follows from (36) that

$$Re\, a_n(\lambda) \leqq n \tag{37}$$

for sufficiently large positive λ where the equality holds if and only if the function f in (5) is the Koebe function g in (3). Thus it follows from (37) that the Bieberbach conjecture $|a_n(\lambda)| \leqq n$ for the Loewner variation (5) within the class S is true for sufficiently large positive λ. Hence in accordance with Corollary 3.2 we proved the local Bieberbach conjecture for the Loewner variation (5) in a new, simpler and direct method in comparison with the Bombieri proof [9] of this conjecture in the class S (see also Pommerenke [1, p. 26] and Hummel [10]). We must note that the global Bieberbach conjecture $|a_n| \leqq n$ in the full class S with equality only for the Koebe function (3) was proved by Louis de Branges and others in different methods (see the Grinshpan reviews [11]).

Remark 1 In particular, for $\lambda = 0$, the series (29)-(30) is reduced to the series (1). In fact, for $\lambda = 0$ and $1 \leqq p \leqq n-1$, $n \geqq 2$, we have the values

$$F(p-n, p+n; 2p+1; 1) = 0 \tag{38}$$

according to the formula (12) since $\Gamma(2p+1) = (2p)!$, $\Gamma(1) = 1$, $\Gamma(p+n+1) = (p+n)!$ and $\Gamma(p-n+1) = \infty$. Of course, the values (38) follow from the formula (15) as well. Also it is evident that the equation $f_0(z) = f(z)$ is immediately obtained from (32).

Remark 2 In particular, the derivative (2) applied to the expansions (29)-(30) for $\lambda = 0$ yields the equation (7). In fact, for $1 \leqq p \leqq n-1$, $n \geqq 2$, from (19) we have the relations

$$\frac{d}{d\lambda}F(p-n, p+n; 2p+1; e^{-\lambda}) = \frac{(n-p)(n+p)}{(2p+1)e^{\lambda}}F(p-n+1, p+n+1; 2p+2; e^{-\lambda}) \tag{39}$$

and for $\lambda = 0$ we have the values

$$F(p-n+1, p+n+1; 2p+2; 1) = (-1)^{n-p-1}\frac{(n-p-1)!}{(2p+2)_{n-p-1}} \tag{40}$$

according to the formula (15) since the formula (12) is not applicable in this case. Now it follows from (30) and (38)-(40) that

$$\frac{d}{d\lambda}a_n(\lambda)\Big|_{\lambda=0} = -(n-1)a_n - 2\sum_{p=1}^{n-1}(n-p)a_{n-p}\overline{\zeta}^p, \quad n \geqq 2, \quad |\zeta| = 1. \tag{41}$$

The formula (41) in accordance with (29) leads us to (7). The equation (7) can also be obtained from the derivative of (32) with respect to λ and (33) for $\lambda = 0$, having in mind the values

$$F(p+1-n, p+1+n; 2p+1; 1) = (-1)^{n-p-1}\frac{(n-p)!}{(2p+1)_{n-p-1}}$$

for $1 \leqq p \leqq n-1$, $n \geqq 2$, according to (15).

Remark 3 The Theorems 1 and 2 and their Corollaries appear to be basic results in the theory of the Pick function (4) (or (8)) and the Loewner variation (5). Using the Gauss hypergeometric function theory (see, for example, [4], [6], [12]-[13]) for these results, we can obtain other formulas that suit our purposes better.

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Author:

Pavel G. Todorov
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bontchev Str., Block 8
1113 Sofia
Bulgaria