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Active boundaries of upper semicontinuous and compactoid relations; closed and inductively perfect maps

Dedicated to professor Harry Poppe on his 65th birthday

ABSTRACT. It is shown that upper semicontinuity (of a relation) is a weak variant of compactoidness; in particular, for the relations that are the inverses of maps, the upper semicontinuity amounts to closedness of the map, and the compactoidness to the perfectness of the map. Therefore the Choquet theorem on the compactness of active boundaries, and the Vainštein lemma on the compactness of boundaries of fibers are instances of the same quest on conditions that make coincide upper semicontinuity and compactoidness. The role of Fréchetness and its variants in this quest is discussed.

1 Introduction

A filter $\mathcal{F}$ is *compactoid* (resp., *finitely compactoid*) *in* A whenever every open cover (resp., finite open cover) of A admits a finite subfamily the union of which belongs to $\mathcal{F}$. The upper semicontinuity of a relation Ω (at a point y) amounts to the finite compactoidness of the filter $\mathcal{F} = \Omega\mathcal{N}(y) \vee (\Omega y)^c$ in the set $A = \Omega y$. The latter property can be rephrased as

$$A^c \in \mathcal{F} \geq \mathcal{N}(A)\,. \tag{1.1}$$

In this paper I show how some compactness properties of A (like paracompactness), and some set-theoretic compactness properties of $\mathcal{F}$ (like strong Fréchetness) can improve original finite compactoidness to compactoidness. We have to do with the composition of a chain of three compactness-like properties. A general theory of such properties in the framework of convergence spaces is developed in [4]. Applied to relations of the type f^-, where f is a continuous closed map, the mentioned results entail the compactness of the boundaries of fibers.

Let X, Y be topological spaces. A relation $\Omega : Y \rightrightarrows X$ is *upper semicontinuous at* y if for every open set P such that $\Omega y \subset P$, there is a neighborhood W of y such that $\Omega W \subset P$. A relation Ω is *compactoid at* y if for every family $\mathcal{P}$ of open sets fulfilling $\bigcup \mathcal{P} \supset \Omega y$, there exists a finite $\mathcal{P}_0 \subset \mathcal{P}$ and a neighborhood W of y such that $\Omega W \subset \bigcup \mathcal{P}_0$.

It is known (and follows immediately from the definitions above) that a relation which is compactoid at y, is upper semicontinuous at y ([1]). It is clear that the converse is true if Ωy is compact, and that this is in some sense also a necessary condition. However under some additional conditions on the spaces (that need not look like variants of compactness), a partial converse is true. G. Choquet announced without proof [2] the following

Theorem 1.1 (Choquet) *If X and Y are metrizable ([2]), and if $\Omega : Y \rightrightarrows X$ is upper semicontinuous at y, then there exists a compact set $K \subset \Omega y$ such that for each $U \in \mathcal{N}(K)$, there exists $W \in \mathcal{N}(y)$ for which $\Omega W \subset \Omega y \cup U$.*

Later it was observed [3] that the least set K fulfilling the conditions of Theorem 1.1, is equal to $\partial_{\#}\Omega y$, the *active boundary of* Ω *at* y:

$$\partial_{\#}\Omega y = \operatorname{adh}(\Omega\mathcal{N}(y) \vee (\Omega y)^c), \tag{1.2}$$

where the *adherence* $\operatorname{adh}\mathcal{F}$ of a filter $\mathcal{F}$ is the union of the limits of all the filters that are finer than $\mathcal{F}$; if $\mathcal{F}$ is a filter in a topological space (which is the framework of this paper), then

$$\operatorname{adh}\mathcal{F} = \bigcap_{F \in \mathcal{F}} \operatorname{cl} F.$$

In [9] S. Rolewicz and the present author proved, in the language of measures of non compactness, that if Ω is valued in a completely metrizable space, and upper semicontinuous at a point of countable character ([3]) y, then $\Omega\mathcal{N}(y) \vee (\Omega y)^c$ is compactoid. A. Lechicki and the present author proved in [7] the same result, but under merely the Dieudonné completeness of X.

At about the same time as the appearance of Theorem 1.1 of G. Choquet, I. A. Vainštein proved in [20] that

Theorem 1.2 (Vainštein) *If f is a (continuous) closed map from a metrizable space onto a topological space Y, then for every y of countable character, $\partial f^{-}(y)$ is compact,*

[1] For example, R. E. Smithson [18] calls Ω *subcontinuous* at y if for each cover $\mathcal{P}$ of X, there exist a neighborhood W of y and a finite subfamily that covers ΩW, and shows that if Ω is subcontinuous and graph-closed at y (hence, compactoid at y), then it is upper semicontinuous at y.

[2] It was noticed [3] that it is enough that Y be first-countable.

[3] A point is of *countable character* if its neighborhood filter is countably based.

where ∂A denotes the boundary of A.

It is well-known (and immediate) that the relation $f^- : Y \Longrightarrow X$ ([4]) is upper semicontinuous if and only if the map f is closed. On the other hand (if f is continuous), f^- is compactoid if and only if f is perfect, that is, closed with compact fibers; hence if the relation ∂f^- is compactoid (and f is continuous), then f is inductively perfect ([5]). Therefore Y is metrizable by the theorem of Hanai-Morita-Stone on preservation of metrizability by perfect maps [19, 17]. Now,

Proposition 1.3 *If f is continuous, then the boundary of $f^-(y)$ is included in the active boundary of f^- at y ([6]).*

Proof: If x is a boundary point of $f^-(y)$, then $Q \setminus f^-(y) \neq \emptyset$ for every $Q \in \mathcal{N}(x)$. If f is continuous, then for every $W \in \mathcal{N}(y)$, there exists $Q \in \mathcal{N}(x)$ such that $Q \subset f^-(W)$. Thus $Q \cap f^-(W) \setminus f^-(y) \neq \emptyset$. $\square$

Therefore Theorem 1.2 is a consequence of Theorem 1.1 (strengthened as in the footnote).

A space X is called a *q-space* if every $x \in X$ is a *q-point*, i.e., such that there exists a sequence $(Q_n)_n$ of neighborhoods of x with the property that if $x_n \in Q_n$, then $\mathrm{adh}(x_n) \neq \emptyset$ [16]. E. Michael proved [16, Theorem 2.1] that can be stated as follows:

Theorem 1.4 (Michael) *Let X be T_1, and let $f : X \to Y$ be continuous, closed and onto. If y is a q-point, then $\partial f^-(y)$ is pseudocompactoid in X ([7]).*

In [11] R. Hansell, J. Jayne, I. Labuda and C. A. Rogers extended Theorem 1.4 from the relations f^- to arbitrary relations Ω (of course, the active boundary of Ω at y generalizes the boundary of $f^-(y)$), but under the provision that Y is regular.

As every point of countable character is a q-point, and since in paracompact spaces pseudocompactness implies compactness, Theorem 1.4 generalizes Theorem 1.2.

However, the argument of Vainštein uses only the fact that f is closed at a given point y (that is, that f^- is upper semicontinuous at y), while the argument of Michael requires the upper semicontinuity of f^- on the whole Y. Therefore, the quest of Choquet and of Vainštein (but not that of Michael) can be reduced to that on compactness of the adherence of a filter $\mathcal{F}$ that is finer than the neighborhood filter of a set A as formulated in (1.1), so

[4]For a relation Γ, the inverse relation is denoted by Γ^-; in particular, f^- stands for the inverse relation of a map f.

[5]A continuous map is *inductively perfect* if there exists its restriction (preserving the range) which is perfect.

[6]On the other hand, if the graph of f is closed, then the active boundary is included in $\partial f^-(y)$.

[7]A subset A of X is *pseudocompactoid* if for each $f \in C(X)$, $\sup |f(A)| < \infty$.

that I can reformulate the (strengthened version of) Theorem 1.1, in terms of filters on a single space ([8]).

Theorem 1.5 *If X is metrizable, $\mathcal{F}$ is a countably based filter on X, $A \subset X$, and* (1.1) *holds, then*

$$\operatorname{adh}\mathcal{F} \subset A; \tag{1.3}$$

$$\operatorname{adh}\mathcal{F} \text{ is compact}; \tag{1.4}$$

$$\mathcal{F} \geq \mathcal{N}(\operatorname{adh}\mathcal{F}). \tag{1.5}$$

Moreover (1.4), (1.5) imply immediately that $\mathcal{F}$ is compactoid in $\operatorname{adh}\mathcal{F}$.

In this paper I investigate the conditions on X, A, and $\mathcal{F}$ that guarantee, separately, (1.3), (1.4), the compactoidness of $\mathcal{F}$, or (1.5).

2 $\mathfrak{J}$-compactoid filters

Recall that two families $\mathcal{A}, \mathcal{B}$ of subsets (of a set) *mesh* (in symbols, $\mathcal{A}\#\mathcal{B}$) if $A \cap B \neq \emptyset$ for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$; the *grill* $\mathcal{A}^{\#}$ of $\mathcal{A}$ is the family of all subsets that intersect every element of $\mathcal{A}$. Recall also that $\mathcal{A}$ is *finer* than $\mathcal{B}$ (in symbols, $\mathcal{A} \geq \mathcal{B}$) if for every $B \in \mathcal{B}$, there exists $A \in \mathcal{A}$ such that $A \subset B$.

Let $\mathfrak{J}$ be a class of filters, and let X be a topological space. Specializing the definitions of [6, 4], I say that a filter $\mathcal{F}$ on X is $\mathfrak{J}$-*compactoid in* a subset A of X if

$$\forall_{\mathcal{F}\#\mathcal{H}\in\mathfrak{J}} \quad \operatorname{adh}\mathcal{H} \cap A \neq \emptyset. \tag{2.1}$$

If $\mathfrak{J}$ is the class of, respectively, all filters, countably based filters, functional filters ([9]), sequences ([10]), and principal filters, then $\mathfrak{J}$-*compactoid* becomes *compactoid, countably compactoid, pseudocompactoid, sequence-compactoid, and finitely compactoid.*

One should not confuse sequence-compactoidness and sequential compactoidness, the latter being the compactoidness with respect to the sequential modification of the underlying topology.

[8] If the topology is T_1, then (1.1) implies that $\mathcal{F}$ is free.

[9] By a *functional filter*, I understand a filter generated by $(\{|f| \geq n\})$, where f is a continuous real-valued function.

[10] I use the term *sequence* both in the usual sense and for each filter generated by a sequence; (x_n) is used rather than $(x_n)_n$ if no ambiguity is feared. By a *subsequence* of a sequence (x_n), I understand (x_{n_k}) with (n_k) tending to ∞ (not necessarily increasing), so that the filter generated by (y_m) is finer than the filter generated by (x_n) if and only if (y_{m_0+m}) for some m_0, is a subsequence of (x_n) in our sense.

$\mathfrak{J}$ (class of filters)	$\mathfrak{J}$-compactoid
all	compactoid
countably based	countably compactoid
functional	pseudocompactoid
sequences	sequence-compactoid
principal	finitely compactoid

Tabelle 2.1: Some notions of compactoidness of filters

In particular, the principal filter of a set F is compactoid (resp., countably compactoid, pseudocompactoid) in A if and only if F is compact (resp., countably compact, pseudocompact) relative to A in the usual sense (but without separation hypotheses).

The notion of cover in the theory of convergence spaces specializes to topological spaces as follows. A family $\mathcal{P}$ is a *cover* of a subset A of a topological space whenever

$$\bigcup_{P \in \mathcal{P}} \operatorname{int} P \supset A.$$

It coincides with the classical concept of cover under the provision that its elements are open. The cover definitions of compactness, countable compactness, and so on, with respect to the two concepts of cover, are tantamount. I use here the convergence-theoretic cover, because this makes several subsequent arguments and formulations simpler.

Given a family $\mathcal{P}$ of subsets of a set, let us denote by $\mathcal{P}_c = \{P^c : P \in \mathcal{P}\}$. If $\mathfrak{P}$ is a class of families of sets, then we set

$$\mathfrak{P}_c = \{\mathcal{P}_c : \mathcal{P} \in \mathfrak{P}\},$$

and denote by $\mathfrak{P}_\star$ the class of all (possibly) degenerate filters generated by the elements of $\mathfrak{P}_c$. It is straightforward that

Theorem 2.1 [4] *A family $\mathcal{F}$ is $\mathfrak{P}_\star$-compactoid in A if and only if for every cover $\mathcal{P} \in \mathfrak{P}$ of A, there exists a finite subfamily $\mathcal{P}_0$ of $\mathcal{P}$ such that $\bigcup \mathcal{P}_0 \in \mathcal{F}$.*

In particular, a filter $\mathcal{F}$ is finitely compactoid in a subset A of a topological space if and only if every open set Q that includes A belongs to $\mathcal{F}$; in other words, whenever $\mathcal{F} \geq \mathcal{N}(A)$.

A *functional cover* of A is a cover of the form $(\{|f| < n\})$, where f is a real-valued map continuous at every point of A. Hence, a filter $\mathcal{F}$ on X is *pseudocompactoid* in A [4] if for every real-valued function f on X that is continuous on A (that is, $f \in C_A(X)$), there exists $F \in \mathcal{F}$ such that $\sup |f(F)| < \infty$ ([11]).

[11] This notion is different from the one introduced in [6] under the same name.

Of course, compactoidness implies countable compactoidness, and the latter implies pseudocompactoidness, finite compactoidness and sequence-compactoidness, and the implications are in general strict. Indeed, the principal filter of the whole space X is compactoid (resp., countably compactoid, pseudocompactoid, finitely compactoid, sequence-compactoid) whenever the topological space X is compact (resp., countably compact, pseudocompact, finitely compact, sequence-compact). Notice that every space is finitely compact, and X is sequence-compact if and only if it is countably compact.

Example 3.4 shows that there exists a filter that is simultaneously finitely compactoid and sequence-compactoid, but not countably compactoid (actually not even pseudocompactoid). In the next section I will consider conditions on filters that enable one to reverse these implications.

I shall say that $\mathcal{F}$ is *nearly* $\mathfrak{J}$-*compactoid in* A if the following variant of (2.1) holds ([12])

$$\underset{\mathcal{F}\leq\mathcal{H}\in\mathfrak{J}}{\forall} \quad \operatorname{adh}\mathcal{H}\cap A\neq\emptyset. \tag{2.2}$$

It is clear that (2.1) implies (2.2). Of course, near compactoidness coincides with compactoidness. It was observed in [6, Proposition 6.4] that countable near compactoidness coincides with near sequence-compactoidness, and is strictly weaker than countable compactoidness [6, Proposition 6.5]; in fact, it is strictly weaker than sequence-compactoidness (a free ultrafilter on a countable discrete space X is nearly sequence-compactoid in X, but not sequence-compactoid).

The following proposition slightly extends [13, Lemma 1].

Proposition 2.2 *If X is T_1 and if $A^c\in\mathcal{F}\geq\mathcal{N}(A)$, then $\mathcal{F}$ is nearly sequence-compactoid in A.*

Proof: Suppose that, on the contrary, there exists a sequence $(x_n)\geq\mathcal{F}$ such that $\operatorname{adh}(x_n)\cap A=\emptyset$. As $A^c\in\mathcal{F}$, we can suppose without loss of generality that $\{x_n:n\in\mathbb{N}\}\cap A=\emptyset$. Consequently, $\operatorname{cl}\{x_n:n\in\mathbb{N}\}=\{x_n:n\in\mathbb{N}\}\cup\operatorname{adh}(x_n)$ is disjoint from A, thus $\big(\operatorname{cl}\{x_n:n\in\mathbb{N}\}\big)^c$ is a one-element cover of A, hence by finite compactoidness, $\operatorname{cl}\{x_n:n\in\mathbb{N}\}\notin\mathcal{F}^{\#}$, thus in particular $\{x_n:n\in\mathbb{N}\}\notin\mathcal{F}^{\#}$ in contradiction with $(x_n)\geq\mathcal{F}$. $\square$

As for classical variants of compactness, certain compactoidness-type properties of a space combined (we can even say, composed) with compactoidness properties of filters improve the latter [4]. For example, a countably compactoid filter on a Lindelöf space is compactoid, or

Proposition 2.3 *A pseudocompactoid filter on a normal topological space is sequence-compactoid.*

[12]In [6] *quasi* was used for what I call now *nearly*; because *quasi compact* has had frequently a different meaning in the literature, I decided to change the terminology of [6].

Proof: If $\mathcal{F}$ is not sequence-compactoid, then there exists (x_n) such that $(x_n)\#\mathcal{F}$ and $\mathrm{adh}(x_n) = \emptyset$. Consequently, the set $\{x_n : n \in \mathbb{N}\}$ is closed and discrete. Therefore the function $f(x_n) = n$, that is continuous on $\{x_n : n \in \mathbb{N}\}$, can be continuously extended to the whole space. Since $\{x_k : k \geq n\} \in \mathcal{F}^\#$ for every n, one has $\sup |f(F)| = \infty$ for every $F \in \mathcal{F}$. □

However, many other classical results in this direction do not extend (from principal) to larger classes of filters. Example 3.4 shows that a sequence-compactoid filter in a completely metrizable space need not be even pseudocompactoid. In the case of principal filters, sequence-compactoid is countably compactoid, and thus compactoid in completely metrizable spaces.

3 Set-theoretic properties of filters improving compactoidness

If a filter enjoys both one of the compactoidness properties, and a set theoretic property, like Fréchetness or strong Fréchetness, then the type of its compactoidness can be improved.

Let $\mathfrak{D}, \mathfrak{J}$ be classes of filters. A filter $\mathcal{F}$ is said to be a *$\mathfrak{J}/\mathfrak{D}$-filter* if for every $\mathcal{H} \in \mathfrak{J}$ such that $\mathcal{H}\#\mathcal{F}$, there exists a filter $\mathcal{D} \in \mathfrak{D}$ such that $\mathcal{D} \geq \mathcal{H}$ and $\mathcal{D}\#\mathcal{F}$. Notice that a filter $\mathcal{F}$ is a $\mathfrak{J}/\mathfrak{D}$-filter if and only if for every set-theoretic cover of an element of $\mathcal{F}$ by a family $\mathcal{P} \in \mathfrak{D}_c$, there exists a refinement $\mathcal{R}$ of $\mathcal{P}$ which is a set-theoretic cover of an element of $\mathcal{F}$, and such that $\mathcal{R} \in \mathfrak{J}_c$.

Properties of the type $\mathfrak{J}/\mathfrak{D}$ can be considered as variants of compactness of filters $\mathcal{F}$ with respect to the discrete topology ([13]). Conjugate with compactoidness properties of $\mathcal{F}$ in A, and with compactness-like properties of A, they imply stronger compactoidness properties of $\mathcal{F}$ in A.

A filter $\mathcal{F}$ is said to be a *super-$\mathfrak{J}/\mathfrak{D}$-filter* if for every $\mathcal{H} \in \mathfrak{J}$ such that $\mathcal{H}\#\mathcal{F}$, there exists a filter $\mathcal{D} \in \mathfrak{D}$ such that $\mathcal{D} \geq \mathcal{F} \vee \mathcal{H}$. It is obvious that

Theorem 3.1 *If a $\mathfrak{J}/\mathfrak{D}$-filter is $\mathfrak{D}$-compactoid in A, then it is $\mathfrak{J}$-compactoid in A. If a super-$\mathfrak{J}/\mathfrak{D}$-filter is nearly $\mathfrak{D}$-compactoid in A, then it is $\mathfrak{J}$-compactoid in A.*

A filter $\mathcal{F}$ is called *Fréchet* [5] if for every $A \in \mathcal{F}^\#$, there is a sequence (x_n) such that $(x_n) \geq \mathcal{F} \vee A$, that is, whenever

$$\mathcal{F}^\# = \bigcup_{(x_n)\geq\mathcal{F}} (x_n),$$

[13] More general schemes are studied in [4].

where (x_n) is a shorthand for the filter generated by (x_n). In other words, a filter is Fréchet if and only if it is a super-principal/sequence filter (super-$\mathfrak{J}/\mathfrak{D}$-filter, where $\mathfrak{J}$ is the class of principal filters, and $\mathfrak{D}$ is the class of sequences) ([14]).

On the other hand, Fréchet filters belong to the class of super-sequence/sequence filters (that is, super-$\mathfrak{D}/\mathfrak{D}$-filters for the class $\mathfrak{D}$ of sequences). Indeed,

Proposition 3.2 *If $\mathcal{F}$ is a Fréchet filter and a sequence (x_n) meshes $\mathcal{F}$, then there exists a subsequence (x_{n_k}) that is finer than $\mathcal{F}$.*

Proof: We decompose $\mathcal{F} = \mathcal{F}_\bullet \wedge \mathcal{F}_\circ$, where $\mathcal{F}_\bullet$ is the principal filter of the kernel $\bigcap \mathcal{F}$ of $\mathcal{F}$ and $\mathcal{F}_\circ$ is the trace of $\mathcal{F}$ on the complement of the kernel. Of course, $\mathcal{F}_\bullet$ and $\mathcal{F}_\circ$ are Fréchet filters. Let (x_n) be a sequence that meshes $\mathcal{F}$; hence (x_n) meshes either with $\mathcal{F}_\bullet$ or with $\mathcal{F}_\circ$. In the first case the trace of (x_n) on $\mathcal{F}_\bullet$ is a subsequence of (x_n) that is finer than $\mathcal{F}$. In the second case, (x_n) must be free, and $\mathcal{F}_\circ \vee \{x_n : n \in \mathbb{N}\}$ is a Fréchet filter, hence there is a sequence $(y_k) \geq \mathcal{F}_\circ$, and since $\mathcal{F}_\circ$ is free, (y_k) is finer than the cofinite filter of the set $\{x_n : n \in \mathbb{N}\}$, that is, (x_n). $\square$

Remark that no ultrafilter is a super-sequence/sequence filter.

A filter $\mathcal{F}$ is called *strongly Fréchet* if for every countably based filter $\mathcal{H}$ with $\mathcal{H}\#\mathcal{F}$, there is a sequence $(x_n) \geq \mathcal{H} \vee \mathcal{F}$; in other words, strongly Fréchet filters coincide with super-countably-based/sequence filters (super-$\mathfrak{J}/\mathfrak{D}$-filters, where $\mathfrak{J}$ is the class of countably based filters, and $\mathfrak{D}$ is the class of sequences) ([15]). It immediately follows from Theorem 3.1 that

Proposition 3.3 *A Fréchet filter that is nearly sequence-compactoid in A is sequence-compactoid in A and finitely compactoid in A.*

A strongly Fréchet filter that is nearly sequence-compactoid in A is countably compactoid in A.

Example 3.4 (*A finitely compactoid, sequence-compactoid non pseudocompactoid filter*) Consider $\mathcal{F} = \mathcal{N}(\mathbb{Z}) \vee \mathbb{Z}^c$ on the real line equipped with the natural topology. No sequence is finer than $\mathcal{F}$, hence $\mathcal{F}$ is nearly countably compactoid. Since $\mathcal{F}$ is a Fréchet filter, by Proposition 3.3, $\mathcal{F}$ is sequence-compactoid. $\mathcal{F}$ is not pseudocompactoid, because the function $f(r) = r$ is continuous and unbounded on every element of $\mathcal{F}$.

By Propositions 2.2 and 3.3,

Corollary 3.5 *If $\mathcal{F}$ is a strongly Fréchet filter on a T_1 space, such that $A^c \in \mathcal{F} \geq \mathcal{N}(A)$, then $\mathcal{F}$ is countably compactoid in A.*

[14] It is known that a filter is Fréchet [5] if and only if it is an intersection of sequential filters.

[15] A topological space is *Fréchet* if and only if its every neighborhood filter is Fréchet; a topological space is *strongly Fréchet* if and only if its every neighborhood filter is strongly Fréchet.

In a regular topological space every compactoid filter has compact adherence, but from the countable compactoidness of a filter we cannot conclude that the adherence is countably compact (think of a relatively countably compact set with non countably compact closure) ([16]). The latter is true in normal spaces, because pseudocompactoid closed sets are countably compact therein.

As countable compactoidness implies pseudocompactoidness of a filter, and the latter implies the pseudocompactoidness of the adherence of the filter, we have

Corollary 3.6 *If $\mathcal{F}$ is a strongly Fréchet filter on a T_1 space, such that $A^c \in \mathcal{F} \geq \mathcal{N}(A)$, then* $\operatorname{adh} \mathcal{F}$ *is pseudocompactoid in A.*

Recall [10, page 568] that a topology is *Dieudonné complete* if it is homeomorphic to a closed subset of a product of metrizable spaces; equivalently, if its topology admits a complete uniformity $\mathfrak{U}$. Every paracompact space is Dieudonné complete. A filter $\mathcal{F}$ in a uniform space is *totally bounded* if for every $U \in \mathfrak{U}$, there exists a finite set A such that $UA \in \mathcal{F}$. It is known [6, Proposition 7.1] that every compactoid filter is totally bounded, and that the converse statement is true for complete uniformities. Here is a refinement of [6, Theorem 7.2] that required $\mathcal{F}$ to be countably based.

Proposition 3.7 *A strongly Fréchet, countably compactoid filter in a Dieudonné complete space, is compactoid.*

Proof: It is enough to show that a strongly Fréchet filter $\mathcal{F}$ which is countably compactoid on a Dieudonné complete space, is totally bounded. Let $\mathfrak{U}$ be a complete uniformity of X, and suppose that $\mathcal{F}$ is not totally bounded: there exists $U \in \mathfrak{U}$ such that $UA \notin \mathcal{F}$ for every finite subset A of X. Consider $W \in \mathfrak{U}$ for which $W^5 \subset U$.

I claim that there exists a sequence $(x_n)_n$ and sequences $(x_{(n,k)})_k \geq \mathcal{F}$ such that $x_n \in \operatorname{adh}(x_{(n,k)})_k$, and $\{x_{(n,k)} : k \in \mathbb{N}\} \subset Wx_n$, and moreover $W^2x_n \cap W^2x_m = \emptyset$ for $n \neq m$. Indeed, as $\mathcal{F}$ is Fréchet and countably compactoid, there exists a sequence $(x_{0,k})_k \geq \mathcal{F}$ and x_0 such that $x_0 \in \operatorname{adh}(x_{0,k})_k$, and $(x_{0,k})_k$ is included in Wy_0, so that the induction hypothesis of order 0 is fulfilled. Suppose that the induction hypothesis holds up to the order n. Then, by assumption, $U\{x_0, x_1, \ldots, x_n\} \notin \mathcal{F}$, so that by Fréchetness and countable compactoidness, there exist x_{n+1} and in $Wx_{n+1} \setminus U\{x_0, x_1, \ldots, x_n\}$ a sequence $(x_{n+1,k}) \geq \mathcal{F}$ such that $x_{n+1} \in \operatorname{adh}(x_{n+1,k})_k$. Clearly, $W^2x_m \cap W^2x_{n+1} = \emptyset$ for $0 \leq m \leq n$.

Now, by strong Fréchetness, there exists a sequence $(x_{(n_p,k_p)})_p \geq \mathcal{F}$, for which $(n_p)_p$ tends to ∞, and $\operatorname{adh}(x_{(n_p,k_p)})_p \neq \emptyset$ by countable compactoidness. But it follows from the construction

[16] In order to avoid this inconvenience I. Labuda assumes in several arguments of [13] that each relatively countably compact set is relatively compact. Dieudonné complete spaces have this property.

that $\{x_{(n_p,k_p)} : p \in \mathbb{N}\}$ is discrete, hence its every subset is closed, and thus $\operatorname{adh}(x_{(n_p,k_p)})_p = \emptyset$: a contradiction. □

Corollary 3.8 *If $\mathcal{F}$ is a strongly Fréchet filter on a Dieudonné complete space, and $A^c \in \mathcal{F} \geq \mathcal{N}(A)$, then $\mathcal{F}$ is compactoid in A.*

Since each a Dieudonné complete space is regular, it follows that $\operatorname{adh}\mathcal{F}$ is compact.

A. V. Arhangel'skii and A. Bella proved [1, Theorem 9] that (for T_1 topologies) the boundary of each fiber of a continuous closed map to a countably fan-tight space is pseudocompactoid in the domain space. I say that a filter $\mathcal{F}$ on a set X is *countably fan-tight* if for every (decreasing) sequence (A_n) of sets such that $A_n \in \mathcal{F}^{\#}$, there exists a sequence (K_n) of finite sets such that $K_n \subset A_n$ and $\bigcup_{n\in\mathbb{N}} K_n \in \mathcal{F}^{\#}$. A topological space is countably fan-tight if and only if its every neighborhood filter is countably fan-tight. A Fréchet filter is countably fan-tight if and only if it is strongly Fréchet [1]. Therefore the following theorem generalizes Corollary 3.5 and, because of Corollary 3.6, it generalizes also [1, Theorem 9].

Theorem 3.9 *If $\mathcal{F}$ is a countably fan-tight filter on a T_1 space, and if $A^c \in \mathcal{F} \geq \mathcal{N}(A)$, then $\mathcal{F}$ is countably compactoid in A.*

Proof: Let $\mathcal{H}$ be a countably based filter that meshes $\mathcal{F}$. As $A^c \in \mathcal{F}$, the filter $\mathcal{H} \vee A^c$ also meshes $\mathcal{F}$. Consider (H_n) be a (decreasing) base of $\mathcal{H} \vee A^c$ such that $H_n \cap A = \emptyset$. By countable fan-tightness, there exists a sequence (K_n) of finite sets such that $K_n \subset H_n$ such $B = \bigcup_{n\in\mathbb{N}} K_n \in \mathcal{F}^{\#}$. By T_1, the filter $\mathcal{F}$ is free, and hence $B_n = \bigcup_{m\geq n} K_m \in \mathcal{F}^{\#}$ for every n. Therefore $\operatorname{cl} B_n \cap A \neq \emptyset$, because $\mathcal{F}$ is finitely compactoid in A.

I claim that $\operatorname{adh}(B_n) \cap A \neq \emptyset$. If this were not the case, then on one hand $\operatorname{adh}(B_n) \cap A = \emptyset$, and by construction $B \cap A = \emptyset$. As the filter generated by (B_n) is a sequence in a T_1-space, $\operatorname{cl} B = B \cup \operatorname{adh}(B_n)$, that is a contradiction. As $(B_n) \geq \mathcal{H}$, we conclude that $\mathcal{F}$ is countably compactoid in A. □

In order to come back to the starting point, that of upper semicontinuous relations, let me notice that it is straightforward that if $\mathcal{N}(y)$ is, respectively, countably based, Fréchet, strongly Fréchet, countably fan-tight, then $\Omega\mathcal{N}(y) \vee (\Omega y)^c$ is, respectively, countably based, Fréchet, strongly Fréchet, countably fan-tight. As a result, the following corollary generalizes [1, Theorem 9] in several respects.

Corollary 3.10 *Let X be T_1, and let y be a countably fan-tight point of Y. If $\Omega : Y \Longrightarrow X$ is upper semicontinuous at y, then $\Omega\mathcal{N}(y) \vee (\Omega y)^c$ is countably compactoid, and thus its active boundary $\partial_{\#}\Omega y$ is pseudocompactoid. If moreover X is Dieudonné complete, then $\partial_{\#}\Omega y$ is compact.*

4 Active boundaries

The *active boundary* $\partial_{\mathcal{F}} A$ of a filter $\mathcal{F}$ with respect to a set A is defined by $\partial_{\mathcal{F}} A = \operatorname{adh}\left(\mathcal{F} \vee A^c\right)$ [6]. It follows right away that $\partial_{\mathcal{F}} A$ is disjoint from $\operatorname{int} A$. Of course, if $A^c \in \mathcal{F}$, then the active boundary of $\mathcal{F}$ with respect to A is equal to $\operatorname{adh} \mathcal{F}$. By Theorem 1.5, if $\mathcal{F}$ is a countably based filter on a metrizable space, and if A is a subspace such that

$$A^c \in \mathcal{F} \geq \mathcal{N}(A), \tag{4.1}$$

then the active boundary of $\mathcal{F}$ with respect to A is a subset of A, hence of ∂A. Observe that A is not assumed to be closed. In general, (4.1) does not imply that $\operatorname{adh} \mathcal{F} \subset A$, even if A is closed.

Example 4.1 Let X be a non regular Hausdorff topological space. Then there exist a closed set A and $x \notin A$ such that the filter $\mathcal{F} = \mathcal{N}(x) \vee \mathcal{N}(A)$ is non degenerate. This is a free filter finer than $\mathcal{N}(A)$ and than $\mathcal{N}(x)$, hence convergent to x. Therefore $\{x\} = \operatorname{adh} \mathcal{F}$, the active boundary of $\mathcal{F}$ with respect to A, is disjoint from A.

Example 4.1 is a sort of characterization, because if A is a closed subset of a regular space, then $\operatorname{adh} \mathcal{N}(A) \subset A$. Therefore in this case (4.1) implies $\operatorname{adh} \mathcal{F} \subset A$.

A subset of a topological space is said to be G_δ-*closed* if it is closed for the topology obtained from the original topology by taking G_δ sets as a base of open sets. I. Labuda proved in [13, Lemma 6] a result to the effect that if a countably based filter $\mathcal{F}$ on a regular (Hausdorff) space fulfills (4.1) and if A is G_δ-closed, then $\operatorname{adh} \mathcal{F} \subset A$.

Theorem 4.2 *Let A be a G_δ-closed subset of a regular T_1 space and let $\mathcal{F}$ be strongly Fréchet. If* (4.1) *holds, then* $\operatorname{adh} \mathcal{F} \subset A$.

Proof: Let $x \notin A$ and let (P_n) be a (decreasing) sequence of closed neighborhoods of x such that $\bigcap_n P_n \cap A = \emptyset$. If $x \in \operatorname{adh} \mathcal{F}$, then $P_n \in \mathcal{F}^\#$ for each n. Hence there exists a sequence $(x_n) \geq \mathcal{F}$ such that $x_n \in P_n$, because $\mathcal{F}$ is strongly Fréchet. It follows that $\operatorname{adh}(x_n) \subset \bigcap_n P_n$. On the other hand by Proposition 2.2, $\mathcal{F}$ is nearly sequence-compactoid in A, thus $\operatorname{adh}(x_n) \cap A \neq \emptyset$. A contradiction. $\square$

In the case of Fréchet filters $\mathcal{F}$ in Hausdorff spaces, (4.1) implies that the sequential adherence of $\mathcal{F}$ is included in A. Therefore in this case we prove that $\operatorname{adh} \mathcal{F} \subset A$ if we can prove that the sequential adherence of $\mathcal{F}$ coincides with the adherence of $\mathcal{F}$.

Recall that $\operatorname{adh}_{\mathrm{Seq}} \mathcal{F}$ stands the *sequential adherence* of $\mathcal{F}$, that is,

$$\operatorname{adh}_{\mathrm{Seq}} \mathcal{F} = \bigcup_{(x_n) \# \mathcal{F}} \lim(x_n)\,.$$

By Proposition 2.2, each sequence (x_n) on A^c finer than $\mathcal{N}(A)$ is nearly sequence-compactoid in A. This fact slightly refines [13, Lemma 1]. It follows that

Proposition 4.3 *Suppose that each sequence has at most one limit point. If $\mathcal{F}$ is Fréchet and if $A^c \in \mathcal{F} \geq \mathcal{N}(A)$, then*

$$\mathrm{adh}_{\mathrm{Seq}}\,\mathcal{F} \subset A\,.$$

5 Minimal kernels

I say that a set A is a *kernel* of a filter $\mathcal{F}$ if (4.1) holds ([17]). If $\mathcal{F}$ is a countably based filter on a metrizable space, then by Theorem 1.5, (4.1) implies that $\mathrm{adh}\,\mathcal{F} \subset A$ and that $(\mathrm{adh}\,\mathcal{F})^c \in \mathcal{F} \geq \mathcal{N}(\mathrm{adh}\,\mathcal{F})$, so that $\mathrm{adh}\,\mathcal{F}$ is a the least kernel of $\mathcal{F}$. In general, (4.1) does not imply that $\mathrm{adh}\,\mathcal{F}$ is a kernel of $\mathcal{F}$.

Example 5.1 (A filter with empty adherence that admits kernels). Let $X = \mathbb{R}$ and $\mathcal{F} = \mathcal{N}(\mathbb{N}) \vee \mathbb{N}^c \vee \mathcal{B}$, where the filter $\mathcal{B}$ is generated by $\{\{x : |x| \geq n\} : n \in \mathbb{N}\}$. The adherence of $\mathcal{F}$ is empty, thus $\mathcal{N}(\emptyset)$ is the degenerate filter, so that $\mathcal{F}$ is not finer than $\mathcal{N}(\emptyset)$.

On the other hand, it follows from Example 4.1 that if $\mathrm{adh}\,\mathcal{F}$ is a kernel of $\mathcal{F}$, it need not be the least one. Indeed in that example $\{x\} = \mathrm{adh}\,\mathcal{F}$ is a kernel of $\mathcal{F}$ disjoint from another kernel: A.

It is known [6] that if $\mathcal{F}$ is compactoid and $(\mathrm{adh}\,\mathcal{F})^c \in \mathcal{F}$, then $\mathrm{adh}\,\mathcal{F}$ is a kernel of $\mathcal{F}$. This follows from the fact that if $\mathcal{F}$ is compactoid (in A), then $\mathrm{adh}\,\mathcal{H} \cap \mathrm{adh}\,\mathcal{F} \neq \emptyset$ for every $\mathcal{H}\#\mathcal{F}$, because there exists a (convergent) ultrafilter $\mathcal{U} \geq \mathcal{H} \vee \mathcal{F}$, so that $\mathcal{F}$ is compactoid in $\mathrm{adh}\,\mathcal{F}$. If $\mathcal{F}$ is merely countably compactoid (in A) and $\mathcal{H}$ is a countably based filter such that $\mathcal{H}\#\mathcal{F}$, then $\mathrm{adh}\,\mathcal{H}$ need not intersect $\mathrm{adh}\,\mathcal{F}$, because there need not be a convergent filter $\mathcal{G}$ finer than $\mathcal{H} \vee \mathcal{F}$.

For a given filter $\mathcal{F}$, consider the set $\mathfrak{U}(\mathcal{F}) = \{\mathcal{U} \in \beta(\mathcal{F}) : \lim \mathcal{U} \neq \emptyset\}$ of convergent ultrafilters. If $\mathfrak{U}(\mathcal{F})$ is Stone closed, that is, if the filter $\mathcal{F}^+ = \bigwedge \mathfrak{U}(\mathcal{F})$ is compactoid, then $\mathcal{F}$ is called *solid*. The infimum of a compactoid filter and of a non adherent filter is an example of a solid non compactoid filter.

Proposition 5.2 *If $\mathcal{F}$ is a free solid non compactoid filter in a regular topological space, then* $\mathrm{adh}\,\mathcal{F}$ *is not a kernel of $\mathcal{F}$.*

Proof: Take any $\mathcal{U} \in \beta(\mathcal{F}) \setminus \beta(\mathcal{F}^+)$. Since by definition $\mathrm{adh}\,\mathcal{U} = \emptyset$, for every $x \in \mathrm{adh}\,\mathcal{F} = \mathrm{adh}\,\mathcal{F}^+$, there exists an open set O_x that contains x and such that $\mathcal{O}_x \notin \mathcal{U}$. Of course, $O_x \in \mathcal{W}$ for every $\mathcal{W} \in \beta(\mathcal{F})$ with $x \in \lim \mathcal{W}$. Since $\mathcal{F}^+$ is compactoid and X is regular,

[17] A similar notion of an $\mathcal{F}$-kernel of a set B was introduced in [6] as an abstraction of a concept from [7]; It is a kernel of $\mathcal{F}$ in the present sense that is moreover a subset of B.

$\operatorname{adh}\mathcal{F}^+ = \operatorname{adh}\mathcal{F}$ is compact, and thus there exist $x_1, \dots x_n$ such that $O_{x_1} \cup \dots \cup O_{x_n} \in \mathcal{N}(\operatorname{adh}\mathcal{F})$. On the other hand, $O_{x_1} \cup \dots \cup O_{x_n} \notin \mathcal{U}$, thus $O_{x_1} \cup \dots \cup O_{x_n} \notin \mathcal{F}$. □

The *Fréchet adherence* of a filter $\mathcal{F}$:

$$\operatorname{adh}^{\varepsilon}\mathcal{F} = \bigcup_{(x_n)\geq\mathcal{F}} \operatorname{adh}(x_n) \tag{5.1}$$

has been introduced by A. Lechicki [14] ([18]) as an abstraction of an object considered by I. Labuda [13]. This is not a notion of adherence with respect to some convergence modification of the underlying topology; $\operatorname{adh}^{\varepsilon}\mathcal{F}$ depends only on the *Fréchet core* of $\mathcal{F}$, that is, on

$$\mathcal{F}^{\varepsilon} = \bigcap_{(x_n)\geq\mathcal{F}} (x_n),$$

where as usual (x_n) is a shorthand for the filter generated by (x_n). A filter is Fréchet if and only if it is equal to its Fréchet core. If $\mathcal{F}$ is Fréchet, then $\operatorname{adh}^{\varepsilon}\mathcal{F} = \bigcup_{(x_n)\#\mathcal{F}} \operatorname{adh}(x_n)$, hence $\operatorname{adh}_{\mathrm{Seq}}\mathcal{F} \subset \operatorname{adh}^{\varepsilon}\mathcal{F}$. The following theorem has been proved for countably based filters by A. Lechicki in [14, 3.1].

Theorem 5.3 *If $\mathcal{F}$ is a Fréchet filter and if $A^c \in \mathcal{F} \geq \mathcal{N}(A)$, then*

$$\mathcal{F} \geq \mathcal{N}\big(A \cap \operatorname{adh}^{\varepsilon}\mathcal{F}\big). \tag{5.2}$$

Proof: Because $\mathcal{F}$ is Fréchet, it is enough to show that $(x_n) \geq \mathcal{N}\big(A \cap \operatorname{adh}^{\varepsilon}\mathcal{F}\big)$ for every $(x_n) \geq \mathcal{F}$. If this were not the case, then there would exist an open set $O \supset A \cap \operatorname{adh}^{\varepsilon}\mathcal{F}$ and a subsequence (x_{n_k}) such that $x_{n_k} \notin O$ for each k. By Proposition 2.2, $\operatorname{adh}(x_{n_k}) \cap A \neq \emptyset$, but on the other hand, $\operatorname{adh}(x_{n_k}) \cap O = \emptyset$, which is a contradiction. □

It follows from Theorem 5.3 and from Proposition 4.3 that in Hausdorff spaces, $\operatorname{adh}_{\mathrm{Seq}}\mathcal{F}$ is included in every kernel of $\mathcal{F}$. If $\mathcal{F}$ is Fréchet, then $\operatorname{adh}_{\mathrm{Seq}}\mathcal{F} \subset \operatorname{adh}^{\varepsilon}\mathcal{F}$, thus it follows [13, Lemma 2] that if $\operatorname{adh}_{\mathrm{Seq}}\mathcal{F}$ is a kernel of $\mathcal{F}$, then it is the smallest one.

A topological space is said to have the *Fréchet property with respect to a class* $\mathcal{B}$ of subsets if for every $B \in \mathcal{B}$ if $x \in \operatorname{cl} B$, then there exists a sequence (x_n) in B such that $x \in \lim(x_n)$. Of course, if a topological space has the Fréchet property with respect to all its subsets, then the topology is Fréchet by definition.

I. Labuda proved (for Hausdorff spaces, and in different terms) [13] the following

Proposition 5.4 *If $\mathcal{F}$ is a filter on a T_1 space with the Fréchet property with respect to countably compactoid sets, then*

$$\operatorname{adh}^{\varepsilon}\mathcal{F} \subset \operatorname{adh}_{\mathrm{Seq}}\mathcal{F}.$$

[18] Without any name.

Proof: If $x \in \operatorname{adh}^{\varepsilon} \mathcal{F}$, then by definition there exists a sequence $(x_n) \geq \mathcal{F}$ such that $x \in \operatorname{adh}(x_n) \subset \operatorname{cl}\{x_n : n \in \mathbb{N}\}$. By the Fréchet property with respect to countably compactoid sets, there exists a sequence on $\{x_n : n \in \mathbb{N}\}$ that converges to x, and since the topology is T_1, this is a subsequence of (x_n). Therefore $x \in \operatorname{adh}_{\mathrm{Seq}} \mathcal{F}$. □

Corollary 5.5 *Let $\mathcal{F}$ be a Fréchet filter on a Hausdorff space with the Fréchet property with respect to countably compactoid sets. If* (4.1) *holds, then* $\operatorname{adh}_{\mathrm{Seq}} \mathcal{F} = \operatorname{adh}^{\varepsilon} \mathcal{F}$ *is the least kernel of $\mathcal{F}$.*

We conclude that $\operatorname{adh} \mathcal{F}$ is the least kernel of a Fréchet filter $\mathcal{F}$ in a Hausdorff topological space if and only if $\mathcal{F}$ admits a kernel, and if

$$\operatorname{adh} \mathcal{F} = \operatorname{adh}_{\mathrm{Seq}} \mathcal{F} . \tag{5.3}$$

It turns out that the equality (5.3) can be characterized in terms of the Fréchetness of the product of certain spaces. It is known [15] that if the product of X with a non discrete space is Fréchet, then X must be strongly Fréchet. Consider $\mathfrak{Fr}$ the class of Fréchet filters, and let $\mathfrak{E}, \mathfrak{F}$ classes of filters included in the class of strongly Fréchet filters; I assume that moreover $\mathfrak{F}$ has the property

$$\mathcal{F} \in \mathfrak{F}(V), A \subset V \times X \Longrightarrow \; A\mathcal{F} \in \mathfrak{F}(X) .$$

Theorem 5.6 *The following assertions are equivalent:*

$$\mathcal{N}(x) \in \mathfrak{E}, \mathcal{F} \in \mathfrak{F} \Longrightarrow \; \big(x \in \operatorname{adh}_X \mathcal{F} \Longrightarrow \; x \in \operatorname{adh}_{\mathrm{Seq}\,X} \mathcal{F}\big) ; \tag{5.4}$$

$$\mathcal{N}(x) \in \mathfrak{E}, \mathcal{N}(y) \in \mathfrak{F} \Longrightarrow \; \mathcal{N}(x,y) \in \mathfrak{Fr} . \tag{5.5}$$

Proof: Assume (5.4) and consider $A \subset X \times Y$ and $(x,y) \in \operatorname{cl} A$. This amounts to $\mathcal{N}(x) \# A^{-}\mathcal{N}(y)$, that is, to $x \in \operatorname{adh} A^{-}\mathcal{N}(y)$. By the assumption, there exists a sequence $(x_n) \# A^{-}\mathcal{N}(y)$ that converges to x. It follows that $A(x_n)_n \# \mathcal{N}(y)$ and since $\mathcal{N}(y)$ is strongly Fréchet, there exist sequences $(y_k) \geq \mathcal{N}(y)$ and x_{n_k} such that $(x_{n_k}, y_k) \in A$. We conclude that $(x,y) \in \operatorname{cl}_{\mathrm{Seq}} A$.

Assume (5.5) and let $x \in \operatorname{adh} \mathcal{G}$, that is $\mathcal{N}(x) \# \mathcal{G}$. Define on $Y = \{\infty\} \cup X$ the finest topology in which $\mathcal{G}$ converges to ∞. The set $\Delta = \{(x,x) : x \in X\}$ belongs to the grill of $\mathcal{N}(x) \times \mathcal{G}$ (hence to the grill of $\mathcal{N}(x) \times \mathcal{N}(y)$), thus by the assumption there exists a sequence (x_n, y_n) on Δ that is finer than $\mathcal{N}(x) \times \mathcal{G}$. Consequently, $x_n = y_n$ for each n, and $x \in \operatorname{adh}_{\mathrm{Seq}} \mathcal{G}$. □

Theorem 5.6 constitutes a return to the original setting, that of a relation between two spaces.

Proposition 5.7 *If X is Hausdorff, if $X \times Y$ is Fréchet, and if $\Omega : \; \rightrightarrows X$ is upper semicontinuous at y, then the active boundary of Ω at y is a kernel of Ω at y.*

Proof: Indeed, under the assumptions, the adherence of $\Omega\mathcal{N}(y) \vee (\Omega y)^c$ is equal to the sequential adherence: $x \in \mathrm{adh}\left(\Omega\mathcal{N}(y) \vee (\Omega y)^c\right)$ if and only if $(x, y) \in \mathrm{cl}\,\Omega \setminus \Omega$. By Fréchetness, there exists a free sequence (x_n, y_n) in Ω that converges to (x, y). Hence $(x_n) \geq \Omega\mathcal{N}(y) \vee (\Omega y)^c$ and converges to x. As the active boundary of Ω at y is equal to the adherence of $\Omega\mathcal{N}(y) \vee (\Omega y)^c$, by Proposition 4.3, the proof is complete. □

The two-fold theorem of [8, Theorem 4] (by T. Nogura and the present author) is a common generalization and refinement of the known theorems on the Fréchetness of products. A convergent bisequence

$$x_{(n,k)} \underset{k}{\longrightarrow} x_n \underset{n}{\longrightarrow} x \tag{5.6}$$

is *free* if $x_n \neq x$, and *stationary* if $x_n = x$ for every n. A space X is an α_3-space (respectively, a β_3-*space*) [8] if for every stationary (respectively, free) convergent bisequence, there exists a compact metrizable subset C of $\{x\} \cup \{x_n : n \in \omega\} \cup \{x_{n,k} : n, k \in \omega\}$ such that

$$|\{n : |C \cap \{x_{n,k} : k \in \omega\}| = \omega\}| = \omega. \tag{5.7}$$

The property α_3 (but not β_3) is a property of single neighborhood filters. Therefore, it can be formulated for filters without relation to any topological structure. A filter $\mathcal{F}$ is α_3 if for every countable collection of sequences $(x_{n,k})_k$ finer than $\mathcal{F}$, there exists a set C such that (5.7) holds, and the cofinite filter of C is finer than $\mathcal{F}$. This property is preserved by relations.

As a consequence of Proposition 4.3, of the two-fold theorem and of Theorem 5.6, we have

Theorem 5.8 *If X is Hausdorff and $\mathcal{F}$ admits kernels, then* $\mathrm{adh}\,\mathcal{F}$ *is the least kernel, provided that one of the following holds:*

1. *X is first-countable and $\mathcal{F}$ is strongly Fréchet;*
2. *X is strongly Fréchet and $\mathcal{F}$ is first-countable;*
3. *X is a regular β_3 Fréchet q-space, and $\mathcal{F}$ is strongly Fréchet;*
4. *X is a regular Fréchet q-space, and $\mathcal{F}$ is α_3 Fréchet.*

Each first-countable space is a β_3 Fréchet q-space, but need not be regular; a regular Fréchet q-space is strongly Fréchet; a first-countable filter is α_3 Fréchet, and the latter is strongly Fréchet. However, because no regularity is required in the first two conditions, they are not a consequence of the two latter.

By virtue of Theorem 5.7, one can formulate other conditions for the adherence to be the least kernel. For example,

Corollary 5.9 *If Y is a regular Fréchet q-space, X is a Hausdorff α_3 Fréchet space, and $\Omega : Y \Longrightarrow X$ is upper semicontinuous, then (at each point) the active boundary of Ω is its least kernel. Moreover, if X is Dieudonné complete, then the active boundary at every point is compact.*

The last conclusion of the the corollary follows from the results of [11] by R. Hansell, J. Jayne, I. Labuda and C. A. Rogers, that I have mentioned in the introduction.

6 Disjoint kernels and non normality number

A filter can have many disjoint kernels. By definition a topological space X is not normal if for there exist closed disjoint sets F_0, F_1 such that $\mathcal{N}(F_0)\#\mathcal{N}(F_1)$. The supremum of cardinal numbers κ for which there exists a family $\mathcal{A}$ of disjoint closed subsets of X such that $|\mathcal{A}| = \kappa$ and the filter

$$\mathcal{F} = \bigvee_{A \in \mathcal{A}} \mathcal{N}(A) \tag{6.1}$$

is non degenerate is called the *non normality number* of X. It is clear that in each space of non normality κ, there is a filter that admits κ disjoint closed kernels ([19]).

Example 6.1 (Non normality of the Niemytzki plane is $\mathfrak{c}$). Recall [10, Example 1.2.4] that the Niemytzki plane can be defined as the upper half of the Euclidean plane, where the neighborhood filters of the elements of the strict upper half of the plane are induced from the natural topology, while for each $(x, 0)$, the neighborhood filter admits a base consisting of the Euclidean-closed discs tangent to $\mathbb{R} \times \{0\}$ at $(x, 0)$. The non normality of the Niemytzki plane is $\mathfrak{c}$.

The argument below slightly refines that employed in [12, p. 514] to prove the Bernstein theorem. Let $\{C_\beta : \beta < \mathfrak{c}\}$ be the set of the Cantor subsets of the real line. Set $C_{(\alpha,\beta)} = C_\beta$ for each $\beta \leq \alpha < \mathfrak{c}$, and order $\{(\alpha, \beta) : \beta \leq \alpha < \mathfrak{c}\}$ lexicographically (denote $l(\alpha, \beta)$ the ordinal that corresponds to (α, β) in the lexicographic order). Arrange $\mathbb{R}$ in the set $\{x_\gamma : \gamma < \mathfrak{c}\}$ of distinct terms.

Let $x_{\gamma(0,0)}$ be the first element that belongs to C_0. If $\delta < \mathfrak{c}$ and $\{x_{\gamma(\alpha,\beta)} : l(\alpha, \beta) < \delta\}$ is such that $x_{\gamma(\alpha,\beta)} \in C_\beta$, let $\gamma(\alpha_\delta,\beta_\delta)$ be the first γ such that x_γ has not been already selected and belongs to C_{β_δ}. Such an element exists, because C_{β_δ} is of cardinality $\mathfrak{c}$.

Let $B_\alpha = \{x_{\gamma(\alpha+\beta,\beta)} : \beta < \mathfrak{c}\}$. The family $\{B_\alpha : \alpha < \mathfrak{c}\}$ is of cardinality $\mathfrak{c}$, all its elements are disjoint, dense, of cardinality $\mathfrak{c}$. For each α_0, the union $\bigcup_{\alpha_0 \neq \alpha < \mathfrak{c}} B_\alpha$ is a totally imperfect

[19]Dr Roberto Peirone (University of Rome, Tor Vergata) proved that for every regular cardinal κ, there exists a completely regular space which the non normality number is equal κ.

set (includes no Cantor set), hence each B_α is of second category in every open interval [12, Section 40, Theorem 3].

The sets $B_\alpha \times \{0\}$ are closed disjoint subsets of the Niemytzki space. Let $O_1, \dots, O_n$ be Niemytzki open sets such that $B_{\alpha_k} \times \{0\} \subset O_k$ for $1 \le k \le n$; for every $x \in B_{\alpha_k}$, let $r(x)$ stand for the radius of a disc tangent to $(x, 0)$ and included in O_k. There are open subsets $W_1 \supset W_k \supset \ldots \supset W_n$ of $\mathbb{R}$ and $r_k > 0$ such that $\{x \in B_{\alpha_k} : r(x) > r_k\}$ is dense in W_k. Therefore $\bigcap_{1 \le k \le n} O_k \ne \emptyset$.

On the other hand, the cardinality of the Niemytzki plane is $\mathfrak{c}$, so that there are no more than $\mathfrak{c}$ mutually disjoint closed sets.

References

[1] **Arhangel'skii, A. V.**, and **Bella, A.** **:** *Countable fan-tightness versus countable tightness.* Comment. Math. Univ. Carolinae **37**, 565–576 (1996)

[2] **Choquet, G.** **:** *Convergences.* Ann. Univ. Grenoble **23**, 55–112 (1947-48)

[3] **Dolecki, S.** **:** *Constraints stability and moduli of upper semicontinuity.* unpublished, 1977

[4] **Dolecki, S.** **:** *Convergence-theoretic characterizations of compactness.* to appear

[5] **Dolecki, S.**, and **Greco, G. H.** **:** *Cyrtologies of convergences, II: Sequential convergences.* Math. Nachr. **127**, 317–334 (1986)

[6] **Dolecki, S.Greco, G. H.**, and **Lechicki, A.** **:** *Compactoid and compact filters.* Pacific J. Math. **117**, 69–98 (1985)

[7] **Dolecki, S.**, and **Lechicki, A.** **:** *On structure of upper semicontinuity.* J. Math. Anal. Appl. **88**, 547–554 (1982)

[8] **Dolecki, S.**, and **Nogura, T.** **:** *Two-fold theorem on Fréchetness of products.* Czech. Math. J., to appear

[9] **Dolecki, S.**, and **Rolewicz, S.** **:** *Metric characterizations of upper semicontinuity.* J. Math. Anal. Appl. **69**, 146–152 (1979)

[10] **Engelking, R.** **:** *Topology.* PWN, 1977

[11] **Hansell, R.Jayne, J., Labuda, I.**, and **Rogers, C. A. :** *Boundaries and selectors for upper semicontinuous multi-valued maps.* Math. Z. **189**, 297–318 (1985)

[12] **Kuratowski, K. :** *Topology.* Academic Press - PWN, 1966

[13] **Labuda, I. :** *On a theorem of Choquet and Dolecki.* J. Math. Anal. Appl. **126**, 1–10 (1987)

[14] **Lechicki, A. :** *Supplement to "Compact and compactoid filters".* Preprint, 1997

[15] **Michael, E. :** *A quintuple quotient quest.* Gen. Topology Appl. **2**, 91–138 (1972)

[16] **Michael, E. :** *A note on closed maps and compact sets.* Israel J. Math. **2**, 173–176 (1994)

[17] **Morita, K.**, and **Hanai, S. :** *Closed mappings and metric spaces.* Proc. Japan Acad. **32**, 10–14 (1956)

[18] **Smithson, R. E. :** *Subcontinuity for multifunctions.* Pacific J. Math. **61**, 283–288 (1975)

[19] **Stone, A. H. :** *Metrizability of decomposition spaces.* Proc. Amer. Math. Soc. **7**, 690–700 (1956)

[20] **Vainštein, I. A. :** *On closed mappings of metric spaces.* Dokl. Akad. Nauk SSSR **57**, 319–321 (1947) (in Russian)

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