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Mapping Theorems of Some Topological Spaces*

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* *Dedicated to Prof. Harry Poppe on his 65th anniversary*

Abstract: The aim of this paper is to study images and preimages of some topological spaces under open finite -to-one (compact) mappings and perfect mappings. Some questions of Gittings are answered negatively.

KEY WORDS and PHRASES. Open finite-to-one mapping, perfect mapping, S_1-space, quasi-metrizable, $w\Delta_2$.

1 Introduction

In the field of general topology, mappings and spaces are main themes. One of the main problems is: What tpological prpoperties are preserved under certain classes of mappings? Since the survey article [1] of Arhangel'skiĭ in the 1960s, a lot of nice mapping theorems have been established. Classification of spaces by mappings and classification of mappings by spaces hve been investgated by many topologists. Among all mappings, two interesting classes are open finite-to-one mappings and perfect mappings. The main purpose of the present paper is to study the behaviour of $w\Delta_2$-spaces, $\sigma^{\#}$-spaces, spaces with the S_1-property, G_δ-diagonal and first countability under open finite-to-one, cpmpact open , or perfect mappings. Several new mapping theorems are established.

Let X be a topological space, and le $\mathcal{G}$ be a collection of subsets of X. The topology on X is denoted by Top(X) and $St(x,\mathcal{G})$ stands for $\cup\{G : x \in G \in \mathcal{G}\}$. We say that X has a *G_δ-diagonal* if its diagonal $\Delta(X)$ is a G_δ-set in the product space X^2, i.e. $\Delta(X) = \cap_{n\in\omega} G_n$, where G_n are open sets of X^2. It is well-known that a space X has a G_δ-diagonal if and only if there is a sequence $\{\mathcal{G}_n : n \in \omega\}$ of open covers such that $\cap_{n\in\omega} St(x,\mathcal{G}_n) = \{x\}$ for each point $x \in X$. A space X has a quasi-G_δ-diagonal if there is a sequence $\{\mathcal{G}_n : n \in \omega\}$ of open collections (not necessarily covers) such that $\cap_{n\in\omega} St(x,\mathcal{G}_n) = \{x\}$ for each point

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$x \in X$, where $c(x) = \{n \in \omega : St(x, \mathcal{G}_n) \neq \emptyset\}$. Note that quasi-$G_\delta$-diagonals are called weak G_δ-diagonals in [14].

A collection $\mathcal{G}$ of subsets of a space X is *semi-open* [11] if for all $x \in X, x \in \text{Int } St(x, \mathcal{G})$. In addition, X is called a $w\Delta_2 - space$ [7] if there is a sequence $\{\mathcal{G}_n : n \in \omega\}$ of semi-open covers such that for each $x \in X$ if $x_n \in St(x, \mathcal{G}_n)$ for each $n \in \omega$, then the sequence $\{x_n : n \in \omega\}$ has a cluster point (not necessarily x).

Recently, Hiremath [11] introduced some properties similar to the G_δ-diagonal property, namely S_1- and S_2-properties. Recall that a space X has the $S_1 - property$ (resp. $S_2 - property$) [11] if there is a mapping $g : \omega \times X \to Top(X)$ such that $\cap_{n\in\omega} g(n, x) = \{x\}$ (resp. $\cap_{n\in\omega} \overline{g(n, x)} = \{x\}$) for each piont $x \in X$, and for all $x_1, x_2 \in X$ and $n \in \omega, x_2 \in g(n, x_1)$ iff $x_1 \in g(n, x_2)$.

A cover $\mathcal{C}$ of a space X is called *point-separating* if for every pair of distinct points $x, y \in X$ there exists a $C \in \mathcal{C}$ such that $x \in C$ and $y \notin C$. A space with a σ-closure-preserving point-separating closed cover is called a $\sigma^\# - space$ [13].

By a compact (resp. quasi-compact) mapping $f : X \to Y$ from a soace X into a space Y, we mean $f^{-1}(y)$ is compact (resp. countably compact) for each $y \in Y$. Moreover, we say f is a finite-to-one mapping if $|f^{-1}(y)| < \omega$ for each $y \in Y$. A mapping is called *perfect* (resp. *quasi-perfect*) if it is continuous, closed and compact (resp. quasi-compact). For other undefined notations, refer to [4] and [5].

2 Open finite-to-one images

Recall that a mapiing $f : X \to Y$ is *pseudo-open* if for $y \in Y, y \in \text{Int}(f(G))$ whenever G is a neighbourhood of $f^{-1}(y)$. Clearly, both open mappings and closed mapiings are pseudo-open.

Theorem 2.1 *Let $f : X \to Y$ be a continuous, finite-to-one and pseudo-open mapping from a space X onto a space Y. If X is $w\Delta_2$, then so is Y.*

Proof: Let $\{\mathcal{G}_n : n \in \omega\}$ be a sequence of semi-open covers of X satisfying all conditions in the definition of a $w\Delta_2$-space. For each $b \in \omega$, define $f(\mathcal{G}_n) = \{f(G) : G \in \mathcal{G}_n\}$. For each $y \in Y$ and $n \in \omega$, since $St(y, f(\mathcal{G})\} = f(St(f^{-1}(y), \mathcal{G}))$ and f is a pseudo-open mapping, $St(y, f(\mathcal{G}_n))$ is a neighbourhood of y. Thus, $\{f(\mathcal{G}_n) : n \in \omega\}$ is a sequence of semi-open covers of Y. In addition, if $y_n \in St(y, f(\mathcal{G}_n))$, then there is a $G_n \in \mathcal{G}_n$ such that $y, y_n \in f(G_n)$ for each $n \in \omega$. Choose $x_n, z_n \in G_n$ such that $f(x_n) = y$ and $f(z_n) = y_n$. Since f is finite-to-one, there is an $x \in f^{-1}(y)$ and a subsequence $\{x_{n_i} : i \in \omega\}$ of $\{x_n : n \in \omega\}$ such that $x_{n_i} = x$ for all $i \in \omega$. Furthermore, we have $z_{n_i} \in St(x, \mathcal{G}_{n_i})$ for each $i \in \omega$. Since X is a $w\Delta_2$-space, $\{z_{n_i} : i \in \omega\}$ has a cluster point, and so does the sequence $\{z_n : n \in \omega\}$. It follows zhat $\{y_n : n \in \omega\}$ has a cluster point. Hence Y is a $w\Delta_2$-space. $\square$

Corollary 2.2 *Let $f : X \to Y$ be a continuous and finit-to-open mapping from a $w\Delta_2$-space X onto a space Y. If f is either open or closed, then Y is a $w\Delta_2$-space.*

It is well-known that the G_δ-diagonal is an invariant under open finite-to-one (not necessarily continuous) mappings. Xia [15] proved that the quasi-G_δ-diagonal property is preserved by open finite-to-one mappings. Next we show that the S_1-property is also preserved by open finite-to-one mappings.

Theorem 2.3 *Let $f : X \to Y$ be a finite-to-one and open mapping from a space X onto a space Y. If X has the S_1-property, then so does Y.*

Proof: Suppose that X has the S_1-property. Then there is a mapping $g : \omega \times X \to Top(X)$ such that

(1) $\{x\} = \cap_{n\in\omega} g(n, x)$ for each $x \in X$;

(2) for all $x_1, x_2 \in X$ and $n \in \omega, x_2 \in g(n, x_1)$ iff $x_1 \in g(n, x_2)$.

Without loss of generality, we may assume that $g(n + 1, x) \subseteq g(n, x)$ for all $n \in \omega$ and all $x \in X$. For each $y \in Y$ and each $n \in \omega$, define $h(n, y) = \cup\{f(g(n, x)) : x \in f^{-1}(y)\}$.

We first show that $\cap_{n\in\omega} h(n, y) = \{y\}$ for each $y \in Y$. If $z \in Y$ and $z \neq y$, then $f^{-1}(z) \cap f^{-1}(y) = \emptyset$. Since $f^{-1}(z)$ and $f^{-1}(y)$ are finite subsets, there is an $n \in \omega$ such that $f^{-1}(z) \cap (\cup\{g(n, x) : x \in f^{-1}(y)\}) = \emptyset$. It follows that $z \notin h(n, y)$.

To complete the proof, we show that for all $y_1, y_2 \in Y$ and each $n \in \omega$, $y_1 \in h(n, y_2)$ iff $y_2 \in h(n, y_1)$. By the definition of h, $y_1 \in h(n, y_2)$ iff there exist $x_2 \in f^{-1}(y_2)$ and $x_1 \in g(n, x_2)$ such that $y_1 = f(x_1)$ which is equivalent to: there exist $x_2 \in f^{-1}(y_2)$ and $x_1 \in f^{-1}(y_1)$ such that $x_2 \in g(n, x_1)$. The last statement is equivalent to $y_2 \in h(n, y_1)$. □

By an argument similar to that of Theorem 2.3, we have the following result.

Theorem 2.4 *Let $f : X \to Y$ be a finite-to-one, open and closed mapping from a space X onto a space Y. If X has the S_2-property, then so does Y.*

In [7], Gittings asked whether the G_δ-diagonal property and $\sigma^\#$-spaces are inverse invariants under continuous, open finite-to-one mappings. The following simple example provides negative answers to these two questions. It also shows that the quasi-G_δ-diagonal property and the S_1-property are not inverse invariants under continuous, open finite-to-one mappings.

Example 2.5 A continuous, open (and closed) two-to-one mapping from a non-T_1 space X onto a metrizable space. Let Z be the set of all integers. Let $X = Z$ be the space with the topology generated by $\mathcal{B} = \{\emptyset\} \cup \{\{2n-1, 2n\} : n \in Z\}$. Let Y be the space of all even integers with the discrete topology. Define a mapping $f : X \to Y$ by $f(2n-1) = f(2n) = 2n$ for each $n \in Z$. It is easy to check that f is two-to-one, open and continuous, and Y is metrizable. Since X is not T_1, it is not a $\sigma^\#$-space, and does not have the quasi-G_δ-diagonal nor the S_1-property. Since f is closed, this example also shows that the S_2-property is not an inverse invariant under continuous, closed and open finite-to-one mappings. □

Remark 2.6 In [15], Xia proved that all quasi-perfect mappings preserve $\sigma^\#$-spaces. Note that the mapping defined in Example 2.5 is perfect. Therefore, $\sigma^\#$-spaces are not inverse invariants under perfect mappings.

3 Perfect preimages of some spaces

It is easy to give an example to show that first countability is not an inverse invariant of perfect mappings. Our next theorem, which generalizes some classical results, claims this problem has an affirmative answer under the additional hypothesis of having a domain with a quasi-G_δ-diagonal.

Lemma 3.1 [3,14] *A countably compact space X with quasi-G_δ-diagonal is metrizable.*

Theorem 3.2 *Let $f : X \to Y$ be a quasi-perfect mapping from a regular space X onto a first countable space Y. If X has a quasi-G_δ-diagonal, then X is first countable.*

Proof: Let $g : \omega \times Y \to Top(Y)$ be a mapping such that for each $y \in Y$, $\{g(n, y) : n \in \omega\}$ is a local base for y. Since X has a quasi-G_δ-diagonal, then for each $y \in Y$ the subspace $f^{-1}(y)$ has a quasi-G_δ-diagonal. By Lemma 3.1, each $f^{-1}(y)$ is a metrizable subspace of X. Thus, there is a mapping $k_y : \omega \times f^{-1}(y) \to Top(f^{-1}(y))$ such that for each $x \in f^{-1}(y)$, $\{k_y(n, x) : n \in \omega\}$ is a local base for x in $f^{-1}(y)$. By the regularity of X, we can construct a mapping $h : \omega \times X \to Top(X)$ such that

(1) $h(n, x) \subseteq f^{-1}(g(n, f(x)))$ for each $x \in X$;

(2) $x \in h(n, x) \cap f^{-1}(f(x)) \subseteq k_{f(x)}(n, x)$ for each $x \in X$;

(3) $\overline{h(n+1, x)} \subseteq h(n, x)$ for each $x \in X$ and each $n \in \omega$.

We show that for each point $x \in X$, $\{h(n, x) : n \in \omega\}$ is a local base for x. Suppose the contrary. There are a point $x \in X$, a neighbourhood U of x and a sequence $\{x_n : n \in \omega\}$ with $x_n \in h(n, x) \backslash U$ for all $n \in \omega$. Since $\cap_{n \in \omega} h(n, x) \subseteq k_{f(x)}(n, x)$, then $\cap_{n \geq 1} \overline{\{x_m : m \geq n\}} \subseteq$

$\cap_{n\geq 1}\overline{h(x,n)\backslash U} \subseteq (\cap_{n\in\omega} k_{f(x)}(n,x))\backslash U = \emptyset$. This implies that $\{x_n : n \in \omega\}$ is a discrete subspace of X. Since f is closed, $\{f(x_n) : n \in \omega\}$ has no cluster points in Y which contradicts the fact that $f(x)$ is a cluster point of $\{f(x_n) : n \in \omega\}$ in Y. □

Corollary 3.3 [12] *Let $f : X \to Y$ be a perfect mapping from a regular space X onto a first countable space Y. If X has a G_δ-diangonal, then X is first countable.*

For a topological space X, let $nw(X)$ stand for the network weight of X (see[4]).

Corollary 3.4 [4] *Let $f : X \to Y$ be a perfect mapping from a regular space X onto a first countable space Y. If $nw(X) \leq \omega$, then X is first countable.*

Proof: If $nw(X) \leq \omega$, then X is a σ-space (see [9] for definition). Since every σ-space has a G_δ-diagonal, it follows immediately from Theorem 3.2 that X is first countable. □

Recall that a *quasi-metric* (resp. *non-Archimedean quasi-metric*) [5] on a set X is a mapping $d : X \times X \to \mathbf{R}^+$ such that (1) $d(x,y) = 0$ iff $x = y$ for all $x, y \in X$; (2) $d(x,z) \leq d(x,y) + d(y,z)$ (resp. $d(x,z) \leq \max\{d(x,y), d(y,z)\}$) for all $x, y, z \in X$. A space X is called *quasi-metrizable* (resp. *non-Archimedeanly quasi-metrizable*) if it admits a compatible quasi-metric (resp. non-Archimedean quasi-metric). In [2], Borges showed that a perfect preimage of a metrizable space is metrizable if and only if it has a G_δ-diagonal. Next we provide an example which says that there are no such analogues for quasi-metrizable spaces, non-Archimedeanly quasi-metrizable spaces and γ-spaces.

Example 3.5 A non-γ-space with a G_δ-diagonal which is a perfect preimage of a non-Archimedeanly quasi-metrizable space. Let $X = \mathbf{R}^2$. Each point $(x,y) \in X$ with $y \neq 0$ has the usual neighbourhoods. Each point $(x,0) \in X$ has basic neighbourhoods of the form

$$U_\epsilon = \{(x,0)\} \cup \{(x_1,y_1) \in \mathbf{R}^2 : (x_1 - x)^2 + (y_1 \pm \epsilon)^2 < \epsilon^2\}.$$

Let Y be the Niemytzki upper half tangent plane. Since Y has a σ-interior-preserving base, it is non-Archimedeanly quasi-metrizable. Define $f : X \to Y$ by $f((x,y)) = (x,|y|)$. It is easy to see that f is a finite-to-one, continuous and closed mapping. Therefore, X is a perfect preimage of a non-Archimedeanly quasi-metrizable space. It follows from [5, Example 7.10] that X is a semi-stratifiable and non-development space. Since any semi-stratifiable γ-space is developable, X has a G_δ-diagonal, and it is not a γ-space. □

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