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Almost fixed point property of nonexpansive mappings

1 Introduction

Recently Kijima [2] and Shimizu and Takahashi [5] proved some fixed point theorems for singlevalued and multivalued mappings respectively in a metric space with some kind of convexity. The purpose of this note is to extend these results to a Hausdorff gauge space and prove fixed point theorems in the spirit of the conditions considered in [5].

A topological space (X, T) is called a gauge space if the topology T is generated by a family $P = \{p_\alpha : \alpha \in I\}$ of pseudometrics on X, in the sense that the family $\beta = \{V(\alpha, r) : \alpha \in I, r > 0\}$, where $V(\alpha, r) = \{(x, y) : x, y \in X,\ p_\alpha(x, y) < r\}$ is a subbase for T. It is well-known that in order for X to be such a space it is necessary and sufficient that X be a uniform space (equivalently, a completely regular space). Further, X is Hausdorff if and only if for each $x, y \in X$, there is an $\alpha \in I$ such that $p_\alpha(x, y) > 0$. For details, we refer to Dugundji [1].

A nonempty subset A of X is called P–bounded (cf. Mishra [4]) if $\delta = \sup\{p_\alpha(x, y) : x, y \in A\} < \infty$ for each $\alpha \in I$. Further, let $B(X) = \{A \subset X : A \neq \phi$ and A is P–bounded$\}$, $CL(X) = \{A \subset X : A \neq \phi$ and A is closed$\}$, $CB(X) = CL(X) \cap B(X)$ and $C(X) = \{A \subset X : A \neq \phi$ and A is compact$\}$. It is well–known that the family $P = \{p_\alpha : \alpha \in I\}$ induces a family $P^* = \{H_\alpha : \alpha \in I\}$ of Hausdorff pseudometrics on $CB(X)$, where $H_\alpha(A, B) = \max\{\sup p_\alpha(a, B) : a \in A, \sup p_\alpha(A, b) : b \in B\}$, $A, B \in CB(X)$. The family P^* itself induces a topology T^* on $CB(X)$ in the sense that the family $\beta^* = \{V^*(\alpha, r) : \alpha \in I, r > 0\}$, where $V^*(\alpha, r) = \{(A, B) : A, B \in CB(X), H_\alpha(A, B) < r\}$ is a subbase for T^*. Thus $CB(X)$ endowed with the topology T^* is a gauge space, called the hyperspace of X (cf. Michael [3]).

Now onward, X will denote a Hausdorff gauge space. The term bounded will mean P–bounded.

Definition 1 *Let $F : X \to CB(X)$ be a multivalued mapping. Then*

(i) *F is said to be nonexpansive if and only if for each $\alpha \in I$, $p_\alpha(F(x), F(y)) \leq p_\alpha(x, y)$ for all $x, y \in X$.*

(ii) *F is said to have the almost fixed point property in X if $\inf\{p_\alpha(x, F(x)) : x \in X\} = 0$ for each $\alpha \in I$.*

(iii) *X is said to satisfy property (A) if for each $x, y \in X$ and for each $\alpha \in I$, there exists a $z \in X$ such that $p_\alpha(z, u) \leq \frac{1}{2}[p_\alpha(x, u) + p_\alpha(y, u)]$ for all $u \in X$.*

A typical example of such a space is a midpoint convex subset of a normed space.

Definition 2 *Let $f : X \to X$ and $F : X \to CB(X)$ be mappings. Then*

(i) *f is said to be asymptotically regular if for each $\alpha \in I$, $p_\alpha(f^{n+1}(x), f^n(x)) \to 0$ as $n \to \infty$ for each $x \in X$.*

(ii) *f and F are said to commute if for each $x \in X$, $f(F(x)) \subset F(f(x))$.*

(iii) *A point $u \in X$ is said to be a common fixed point of f and F if $f(u) = u \in F(u)$.*

2 Results

Theorem 3 *Let X be a bounded Hausdorff gauge space satisfying property (A) and let $F : X \to C(X)$ be a multivalued nonexpansive mapping. Then F has the almost fixed point property in X.*

Proof: To show that F has the almost fixed point property in X, we need to prove that $\inf\{p_\alpha(x, F(x)) : x \in X\} = 0$ for each $\alpha \in I$. Suppose the contrary by assuming that $\inf\{p_\alpha(x, F(x)) : x \in X\} = 2\delta > 0$ for some $\alpha \in I$. Let $\varepsilon > 0$ be arbitrary. Then there exists $x_0 \in X$ such that $p_\alpha(x_0, F(x_0)) \leq 2\delta + \varepsilon$. Now since $F(x_0)$ is nonempty, there exists $y_0 \in F(x_0)$ such that $p_\alpha(x_0, y_0) \leq 2(\delta + \varepsilon)$. Let us define inductively the sequences $\{x_n\}$ and $\{y_n\}$ with $y_n \in F(x_n)$ in X as follows.

Suppose x_k and y_k with $y_k \in F(x_k)$ are known. Then since X satisfies the condition (A), we may choose $x_{k+1} \in X$ such that $p_\alpha(x_{k+1}, u) \leq \frac{1}{2}\left[p_\alpha(x_k, u) + p_\alpha(y_k, u)\right]$ for all $u \in X$. Further, since $F(x_{k+1})$ is nonempty and compact, we may choose $y_{k+1} \in X$ such that $y_{k+1} \in F(x_{k+1})$ and $p_\alpha(y_k, y_{k+1}) = p_\alpha(y_k, F(x_{k+1}))$. Therefore $p_\alpha(y_k, y_{k+1}) \leq p_\alpha(x_k, x_{k+1})$. We claim that for each integer $k \geq 0$,

$$p_\alpha(x_k, x_{k+1}) \leq \delta + \varepsilon \ , \quad p_\alpha(x_k, y_k) \leq 2(\delta + \varepsilon) \ . \tag{1}$$

For $k = 0$, we have $p_\alpha(x_0, x_1) \leq \frac{1}{2}\left[p_\alpha(x_0, x_0) + p_\alpha(x_0, y_0)\right] = \frac{1}{2}p_\alpha(x_0, y_0) \leq \delta + \varepsilon$ and hence the result is true for $k = 0$. Next, let it be true for $k \geq 0$. Then $p_\alpha(x_k, x_{k+1}) \leq \delta + \varepsilon$ and $p_\alpha(x_k, y_k) \leq 2(\delta + \varepsilon)$. Therefore we have

$$\begin{aligned} p_\alpha(x_{k+1}, y_{k+1}) &\leq p_\alpha(x_{k+1}, y_k) + p_\alpha(y_k, y_{k+1}) \\ &\leq \tfrac{1}{2}p_\alpha(x_k, y_k) + p_\alpha(x_k, x_{k+1}) \\ &\leq (\delta + \varepsilon) + (\delta + \varepsilon) = 2(\delta + \varepsilon) \end{aligned}$$

and

$$p_\alpha(x_{k+1}, x_{k+2}) \leq \tfrac{1}{2}p_\alpha(x_{k+1}, y_{k+1}) \leq \delta + \varepsilon \ .$$

Therefore (1) holds for all $k \geq 0$. Next, we shall show that for all integers $k, n \geq 0$

$$p_\alpha(x_k, x_{k+n}) \geq (n+2)(\delta + \varepsilon) - 2^{n+1}\varepsilon \ . \tag{2}$$

For $n = 0$, we note that $p_\alpha(x_k, y_k) \geq p_\alpha(x_k, F(x_k)) \geq 2\delta$. Now suppose that (2) holds for $n \geq 0$ and for all $k \geq 0$. Then we have

$$\begin{aligned} p_\alpha(x_k, y_{k+n+1}) &\geq 2p_\alpha(x_{k+1}, y_{k+n+1}) - p_\alpha(y_k, y_{k+n+1}) \\ &\geq 2p_\alpha(x_{k+1}, y_{k+n+1}) - \textstyle\sum_{i=0}^{n} p_\alpha(x_{k+i}, x_{k+i+1}) \\ &\geq 2\left\{(n+2)(\delta + \varepsilon) - 2^{n+1}\varepsilon\right\} - (n+1)(\delta + \varepsilon) \\ &= (n+3)(\delta + \varepsilon) - 2^{n+2}\varepsilon \ . \end{aligned}$$

Hence the result is true for $n + 1$ and all $k \geq 0$. Therefore (2) holds for all integers $k, n \geq 0$.

Now let m be a nonnegative integer and set $\varepsilon = \delta/2^m$. Then as in the previous case, we may choose sequences $\{x_n^m\}$ and $\{y_n^m\}$ in X such that

$$p_\alpha\left(x_0^m, F(x_0^m)\right) \leq 2\delta + \delta/2^m$$

and

$$p_\alpha\left(x_k^m, y_{k+n}^m\right) \geq (n+2)(\delta + \delta/2^m) - 2^{n+1}(\delta/2^m)$$

for all integers $k, n \geq 0$. In particular, we have

$$\begin{aligned} p_\alpha\left(x_0^m, y_m^m\right) &\geq (m+2)(\delta + \delta/2^m) - 2^{m+1}(\delta/2^m) \\ &= m\delta + (m/2^m)\delta + \delta/2^{m-1} > m\delta \to \infty \text{ as } m \to \infty \ , \end{aligned}$$

contradicting the boundedness of X. Hence $\inf\{p_\alpha(x, F(x)) : x \in X\}$ for all $\alpha \in I$.

Theorem 4 *Let X be a bounded Hausdorff gauge space satisfying the property (A), and let $F : X \to C(X)$ be a multivalued nonexpansive mapping such that $\boldsymbol{cl}(F(X))$, the closure of $F(X)$ is compact. If $f : X \to X$ is a continuous asymptotically regular mapping commuting with F, then f and F have a common fixed point in X.*

Proof: Let $\alpha \in I$ be arbitrary. By Theorem 1, there is a sequence $\{x_n\}$ in X such that $p_\alpha(x_n, F(x_n)) \to 0$ as $n \to \infty$. Now since $F(x_n)$ is nonempty and compact for each n, there exists $y_n \in X$ with $y_n \in F(x_n)$ and $p_\alpha(x_n, y_n) = p_\alpha(x_n, F(x_n))$. Again since $\{y_n\} \subset \boldsymbol{cl}(F(X))$ and $\boldsymbol{cl}(F(X))$ is compact, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ converging to a point x_0 of X. Therefore we have $\lim p_\alpha(x_{n_k}, y_{n_k}) \to 0$ as $k \to \infty$.

Now let us denote by $\mathrm{Fix}(F)$ the set of all fixed points of F. Clearly, $\mathrm{Fix}(F) \subset F(X)$. We shall show that $\mathrm{Fix}(F)$ is nonempty closed and is invariant under f. We have for any $\alpha \in I$

$$\begin{aligned} p_\alpha(x_0, F(x_0)) &\leq p_\alpha(x_0, x_{n_k}) + p_\alpha(x_{n_k}, F(x_{n_k})) + H_\alpha(F(x_{n_k}), F(x_0)) \\ &\leq 2p_\alpha(x_0, x_{n_k}) + p_\alpha(x_{n_k}, F(x_{n_k})) \to 0 \quad \text{as} \quad k \to \infty \ . \end{aligned}$$

Therefore $x_0 \in F(x_0)$ and hence x_0 is a fixed point of F. Thus $\mathrm{Fix}(F) \neq \phi$. To prove that $\mathrm{Fix}(F)$ is closed, let $\{z_n\} \subset \mathrm{Fix}(F)$ with $z_n \to z$ as $n \to \infty$. Then for any $\alpha \in I$, we have $p_\alpha(z_n, F(z)) \leq H_\alpha(F(z_n), F(z)) \leq p_\alpha(z_n, z)$. Making $n \to \infty$, it follows that

$p_\alpha(z, F(z)) = 0$. Hence $z \in \mathrm{Fix}(F)$ and $\mathrm{Fix}(F)$ is closed. Further, since f and F commute, for each $x \in \mathrm{Fix}(F)$, $f(x) \in f(F(x)) \subset F(f(x))$ and hence $\mathrm{Fix}(F)$ is invariant under f.

We note that $\mathrm{Fix}(F)$ is compact (as it is a closed subset of $\boldsymbol{cl}(F(x))$ which is compact). Thus for any $x \in \mathrm{Fix}(F)$, $\{f^n(x)\}$ has a convergent subsequence say $\{f^{n_k}(x)\}$. Suppose $f^{n_k}(x) \to u$ for some $u \in \mathrm{Fix}(F)$ as $k \to \infty$. Then for any $\alpha \in I$, we have

$$p_\alpha(u, f(u)) \leq p_\alpha(u, f^{n_k}(x)) + p_\alpha(f^{n_k}(x), f^{n_k+1}(x)) + p_\alpha(f^{n_k+1}(x), f(u)) \ .$$

Since f is asymptotically regular and continuous, making $k \to \infty$, we have $f(u) = u \in F(u)$. Thus u is a common fixed point of f and F.

Theorems 1 and 2 as proved above, extend and unify the results of [2] and [5].

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