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Common fixed points of multivalued mappings

Abstract. *Common fixed point theorems for contractive type multivalued mappings are proved which generalize some results of Fisher* [1,2,3].
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1. Introduction

Let (X,d) be a complete metric space, $B(X)$ denote the set of all nonempty, bounded subsets of X. For A,B in X, define $\delta(A,B)=\sup\{d(a,b):a\in A,b\in B\}$ and $\delta(A,A)=\delta(A)$. If A is a singleton, we write $\delta(A,B)=\delta(a,B)$ and if $B=\{b\}$, then we put $\delta(A,B)=d(a,b)$.

Now let $\{A_n : n=1,2,\dots\}$ be a sequence of nonempty subsets of X. We say that the sequence $\{A_n\}$ converges to the subset A of X if

(i) each point a in A is the limit of a convergent sequence $\{a_n\in A_n : n=1,2\dots\}$, and

(ii) for arbitrary $\epsilon>0$, there exists an integer k such that $A_n\subseteq A_\epsilon$, for $n>k$, where A_ϵ is the union of all open spheres with centres in A and radius ϵ.

Fisher [3] proved the following result.

Lemma 1 *If $\{A_n\}$ and $\{B_n\}$ are sequences of bounded subsets of a complete metric space (X,d) which converges to the bounded subsets A and B, respectively, then the sequence $\{\delta(A_n,B_n)\}$ converges to $\delta(A,B)$.*

Let F and G be mappings of (X,d) into $B(X)$ and f a self mapping of (X,d). For $A\subseteq X$ and $n\geq 2$, let $FA=\cup_{a\in A}Fa$, $GFA=G(FA)$ and $F^nA=F(F^{n-1}A)$. F and G are said to be commuting if $FGx=GFx$ for all x in X; F and f are said to be commuting if $Ffx=fFx$ for all x in X. We say that F is continuous at the point x in X if whenever $\{x_n\}$ is a sequence of points in X converging to x, the sequence $\{Fx_n\}$ in $B(X)$ converges

to Fx in $B(X)$. We say that F is a continuous mapping of X into $B(X)$ if F is continuous at each point x in X. Define $C_F = \{f : f$ is a self mapping of X and commutes with $F\}$.

Let R^+ denote the set of all nonnegative real numbers, Φ the family of mappings φ from $(R^+)^5$ into R^+ such that each φ is upper semicontinuous, nondecreasing in each coordinate variable, and for any $t > 0, ht = \varphi(t,t,t,t,t) < t$. N denote the set of all positive integers.

The followings lemma was proved in [4].

Lemma 2 *For every $t > 0, ht < t$ if and only if $\lim_{n\to\infty} h^n t = 0$, where h^n denotes the n-times composition of h.*

Fisher [1, 2, 3] established common fixed point theorems for two commuting multivalued mappings. The purpose of this paper is to extend Fisher's results to a more general case.

2. Common fixed point theorems.

We now prove the following theorem.

Theorem 1 *Let F and G be multivalued mappings of a complete bounded metric space (X, d) into $B(X)$, f and g continuous mappings of X into itself such that F and G are commuting, f, g in $C_F \cap C_G$. If there exists φ in Φ satisfying*

$$\delta(Fx, Gy) \le \varphi(\delta(fx, Fx), \delta(gy, Gy), \delta(fx, Gy), \delta(gy, Fx), d(fx, gy)) \tag{1}$$

for all x, y in X, then f, g, F and G have a unique common fixed point z in X. Further, $Fz = Gz = \{z\}$.

Proof: Let A, B in $B(X)$. By (1) we have

$$\delta(Fa, Gb) \le \varphi(\delta(fa, Fa), \delta(gb, Gb), \delta(fa, Gb), \delta(gb, Fa), d(fa, gb))$$

for all a in A, b in B, which implies

$$\delta(FA, GB) \le \varphi(\delta(fa, FA), \delta(gB, GB), \delta(fA, GB), \delta(gB, FA), \delta(fA, gB)). \tag{2}$$

Since X is bounded, then $r = \delta(X) < \infty$. We first show by induction that

$$\delta(F^n G^n X) \le h^n r \tag{3}$$

for n in N. Note that F and G commute and f, g in $C_F \cap C_G$: Using (2)

$$\begin{aligned}\delta(FGX, GFX) &\le \varphi(\delta(GfX, FGX), \delta(FgX, GFX), \delta(GfX, GFX),\\ &\qquad \delta(FgX, FGX), \delta(GfX, FgX))\\ &\le hr\end{aligned}$$

and hence (3) is true when $n = 1$. Now assume that (3) holds for some n in N. Again using (2)

$$\begin{aligned}\delta(F^{n+1}G^{n+1}X, G^{n+1}F^{n+1}X) \leq \varphi(&\delta(F^nG^nGfX, G^nF^nFGX), \delta(F^nG^nFgX, G^nF^nGFX),\\ &\delta(F^nG^nGfX, G^nF^nGFX), \delta(F^nG^nFgX, G^nF^nFGX),\\ &\delta(F^nG^nGfX, G^nF^nFgX))\\ &\leq h\delta(F^nG^nX) \leq h^{n+1}r\end{aligned}$$

by our assumption. Now (3) follows by induction.

For each n in N, choose a point x_n in F^nG^nX. It follows that

$$fx_n \in fF^nG^nX \subseteq F^nG^nfX \subseteq F^nG^nX.$$

Similarly, $gx_n \in F^nG^nX$. From (3) we get

$$d(x_n, x_m) \leq \delta(F^nG^nX, G^nF^nG^{m-n}F^{m-n}X) \leq h^nr$$

for m, n in N with $m \geq n$. By Lemma 2 and the above inequality it follows that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists a point z in X such that $x_n \to z$ as $n \to \infty$. Note that f and g are continuous. Thus $fx_n \to fz$, $gx_n \to gz$ as $n \to \infty$. Consequently, we have

$$d(fz, gz) \leq d(fz, fx_n) + d(fx_n, gx_n) + d(gx_n, gz) \leq d(fz, fx_n) + h^nr + d(gx_n, gz).$$

Letting n tend to infinity, we obtain $d(fz, gz) \leq 0$; i. e., $fz = gz$.

We next show that z is a common fixed point of f, g, F and G. Clearly, we have

$$d(z, fz) \leq d(z, x_n) + h^nr + d(fx_n, fz).$$

As $n \to \infty$, we have $d(z, fz) \leq 0$; i. e., $z = fz$ and hence $z = gz$. We now assert that $\delta(z, Gz) = 0$. Otherwise $\delta(z, Gz) > 0$. It follows from (2) and (3) that

$$\begin{aligned}\delta(z, Gz) &\leq d(z, gx_n) + \delta(F^nG^nX, Gz)\\ &\leq d(z, gx_n) + \varphi(\delta(F^{n-1}G^nfX, F^nG^nX), \delta(z, Gz), \delta(F^{n-1}G^nfX, Gz),\\ &\qquad \delta(z, F^nG^nX), \delta(F^{n-1}G^nfX, z))\\ &\leq d(z, gx_n) + \varphi(h^{n-1}r, \delta(z, Gz), h^{n-1}r + d(gx_{n-1}, z) + \delta(z, Gz),\\ &\qquad d(gx_n, z) + h^nr, h^{n-1}r + d(gx_{n-1}, z)).\end{aligned}$$

Letting n tend to infinity, we obtain

$$\delta(z, Gz) \leq \varphi(0, \delta(z, Gz), \delta(z, Gz), 0, 0) \leq h\delta(z, Gz) < \delta(z, Gz)$$

which is a contradiction and hence $\delta(z, Gz) = 0$. Consequently, $Gz = \{z\}$. Similarly, $Fz = \{z\}$.

We finally show that z is the only common fixed point of f, g, F and G. Suppose that f, g, F and G have a second common fixed point w. From (1) have

$$\delta(Fw, Gw) \leq \varphi(\delta(w, Fw), \delta(w, Gw), \delta(w, Gw), \delta(w, Fw), 0) \leq h\delta(Fw, Gw)$$

which implies $\delta(Fw, Gw) = 0$. Note that $w \in Fw \cap Gw$. Consequently, $Fw = Gw = \{w\}$. Using (1) again

$$d(z, w) = \delta(Fz, Gw) \leq \varphi(0, 0, d(z, w), d(w, z), d(z, w)) \leq hd(z, w).$$

Therefore $z = w$; i. e., z is the unique common fixed point of f, g, F and G. This completes the proof.

Corollary 1 *Let S, T, f and g be self mappings of a complete bounded metric space (X, d) such that S and T are commuting and f, g are in $C_S \cap C_T$. If there exists φ in Φ satisfying*

$$d(Sx, Ty) \leq \varphi(d(fx, Sx), d(gy, Ty), d(fx, Ty), d(gy, Sx), d(fx, gy))$$

for all x, y in X, then f, g, S and T have a unique common fixed point.

Proof: Define mappings F and G of X into $B(X)$ by $Fx = \{Sx\}, Gx = \{Tx\}$ for x in X. Then Corollary 1 follows from Theorem 1.

Remark 1 If f and g are the indentity mapping on X and

$$\varphi(u, v, w, x, y) = c \cdot \max\{u, v, w, x, y\},$$

then Theorem 1 is reduced to Theorem 2 of [1] and Theorem 3 of [2]; Corollary 1 is reduced to the Corollary of [1].

We next prove a theorem for compact metric spaces.

Theorem 2 *Let F and G be continuous commuting mappings of a compact metric space (X, d) into $B(X)$, f and g continuous mappings of X into itself satisfying*

$$\delta(Fx, Gy) < \max\{\delta(fx, Fx), \delta(gy, Gy), \delta(fx, Gy), \delta(gy, Fx), d(fx, gy)\} \tag{4}$$

for all x, y in X for which the right-hand side of (4) is positive. If f, g are in $C_F \cap C_G$, then f, g, F and G have a unique common fixed point z in X. Further, $Fz = Gz = \{z\}$.

Proof: Let $s(x, y) = \max\{\delta(fx, Fx), \delta(gy, Gy), \delta(fx, Gy), \delta(gy, Fx), d(fx, gy)\}$. We now assert that $s(x, y) = 0$ for some x, y in X. Otherwise $s(x, y) > 0$ for all x, y in X. Define the real-valued function $t(x, y)$ on $X \times X$ by $t(x, y) = \delta(Fx, Gy)/s(x, y)$. Then if $\{(x_n, y_n)\}$ is

an arbitrary sequence in $X \times X$ converging to (x, y), it follows Lemma 1 that the sequence $\{t(x_n, y_n)\}$ converges to $t(x, y)$. The function t is therefore a continuous function defined on the compact metric space $X \times X$ and so achieves its maximum value c. It is easy to see that (4) implies $c < 1$. Choose $\varphi(u, v, w, m, n) = c \cdot \max\{u, v, w, m, n\}$. Then the conditions of Theorem 1 are satisfied. Consequently, f, g, F and G have a unique common fixed point z and $Fz = Gz = \{z\}$. It follows that $s(z, z) = 0$, giving a contradiction. Since $s(x, y) = 0$ for some x, y in X, then $Fx = Gy = \{fx\} = \{gy\}$ is a singleton $\{z\}$ and so $F^2x = Ffx = fFx = \{fz\}$ is also a singleton. Suppose that $fz \neq z$. By (4) we have

$$\begin{aligned} d(fz, z) = \delta(Ffx, Gy) &< \max\{\delta(f^2x, fFx), \delta(gy, Gy), \delta(f^2x, Gy), \delta(gy, Ffx), d(f^2x, gy)\} \\ &= \max\{0, 0, d(fz, z), d(z, fz), d(fz, z)\} = d(fz, z) \end{aligned}$$

which is impossible. Hence $\{z\} = \{fz\} = Fz$. Similarly, we can show that $gy = z$ is also a common fixed point of G and g and $Gz = \{z\}$.

Now suppose that f, g, F and G have a second distinct common fixed point w. Then if $Fw \neq \{w\}$ or $Gw \neq \{w\}$, by (4) we get

$$\delta(Fw, Gw) < \max\{\delta(w, Fw), \delta(w, Gw), \delta(w, Gw), \delta(w, Fw), 0\} \leq \delta(Fw, Gw)$$

since $fw = gw = w$ is in Fw and Gw, giving a contradiction. Thus $Fw = Gw = \{w\}$ and so

$$d(z, w) = \delta(Fz, Gw) < \max\{0, 0, d(z, w), d(z, w), d(z, w)\} = d(z, w)$$

which is a contradiction. The fixed point z must therefore be unique. This completes the proof.

Remark 2 Theorem 2 of Fisher [3] is a special case of our Theorem 2. The following example reveals that our Theorem 2 extends properly Theorem 2 of Fisher [3].

Example Let $X = \{1, 2, 5, 9\}$ with the usual metric. Define a self mapping f of X and a multivalued mapping F of X into $B(X)$ by $f1 = f2 = f5 = 2$, $f9 = 1$, $F1 = \{2, 5\}$, $F2 = \{2\}$, $F5 = \{1, 2\}$, $F9 = \{5\}$. Then Theorem 2 of Fisher [3] is not applicable since

$$\delta(F2, F5) = 1 = \max\{\delta(f2, F2), \delta(f5, F5), \delta(f2, F5), \delta(f5, F2), d(f2, f5)\}.$$

Take $G = F$ and $g = i$ — the identity mapping of X. It is easy to check that f, g, F and G satisfy the conditions of our Theorem 2.

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