Completely Separating Systems of k-sets for $\binom{k-1}{2} \leq n < \binom{k}{2}$ or $11 \leq k \leq 12$

Ian T. Roberts, Sue D'Arcy
Charles Darwin University
Darwin 0909, Australia
ian.roberts@cdu.edu.au
Hans-Dietrich O.F. Gronau
University of Rostock
gronau@uni-rostock.de

Abstract

$R(n,k)$ denotes the minimum possible size of a completely separating system $\mathcal{C}$ on $[n]$ with $|A| = k$ for each $A \in \mathcal{C}$. Values of $R(n,k)$ are determined for $\binom{k-1}{2} \leq n < \binom{k}{2}$ or $11 \leq n \leq 12$. Using the dual interpretation of completely separating systems as antichains, this paper provides corresponding results for dual k-regular antichains.

1 Introduction and Basic Results

This paper extends previous work in [8], [9], [10], [11] and more recent work by Böhm in [1], [2] and [3], to determine the minimum size completely separating systems (CSSs) with a single block size. This simultaneously determines the minimum size ground set for which a k-regular antichain of a given size exists. This section contains some basic results and a summary of relevant results from the papers mentioned above. Subsequent sections determine the unknown values of $R(n,k)$ for $\binom{k-1}{2} \leq n < \binom{k}{2}$ or $11 \leq n \leq 12$.

Let $k < n$ be integers, with $[n] = \{1, 2, \ldots, n\}$, and with $2^{[n]}$ denoting the power set of $[n]$. An **(n)Competely Separating System ((n)CSS)** $\mathcal{C}$ is a collection of blocks of $[n]$ such that for any pair of points $x, y \in [n]$ there exist blocks $A, B \in \mathcal{C}$ with $x \in A \setminus B$ and $y \in B \setminus A$. An **(n,k)Completely Separating System ((n,k)CSS)** $\mathcal{C}$ is an (n)CSS in which each block is of size k.

The size of $\mathcal{C}$ is the number of blocks in $\mathcal{C}$, denoted by $|\mathcal{C}|$. The integers **R(n)** and **R(n,k)** are defined by: $R(n) = \min\{|\mathcal{C}| : \mathcal{C} \text{ is an } (n)CSS\}$ and $R(n,k) = \min\{|\mathcal{C}| : \mathcal{C} \text{ is an } (n,k)CSS\}$. In what follows R is sometimes written instead of $R(n,k)$. An (n)CSS for which $|\mathcal{C}| = R(n)$ is a **minimal** (n)CSS and an (n,k)CSS $\mathcal{C}$ for which $|\mathcal{C}| = R(n,k)$ is a **minimal** (n,k)CSS.

The **volume** of a collection of blocks $\mathcal{C}$ is $V(\mathcal{C}) = \sum_{A \in \mathcal{C}} |A|$. For an (n,k)CSS $\mathcal{C}$, $V(\mathcal{C}) = k|\mathcal{C}|$. A CSS is said to be **fair** if each point occurs in either p or $p+1$ blocks for some integer p. A point is said to **cover** a collection of blocks $\mathcal{C}$ if each block in $\mathcal{C}$ contains the point. Similarly, a point a **covers** another point b if each block containing b also contains a. In any CSS, no point covers another point. A point is called a **p-point** if it occurs in exactly p blocks.

CSSs or partial CSSs are often represented by arrays in this paper, with each row representing a block or partial block. Gaps are sometimes left in the rows to aid in seeing the substructures.

Some values of $R(n)$ from [8] and [11] are restated in the following lemma.

Lemma 1.1. *The following hold for $k < n$:*
1. If $n \leq 4$ then $R(n) = n$.
2. If $n = 5$ or 6 then $R(n) = 4$ and, up to labelling of points, there is a unique way of achieving $R(n)$ in each case.
3. $R(n)$ is a non-decreasing function of n.
4. $R(7) = 5$.
5. For $1 \leq k < n, R(n,k) = R(n, n-k)$.

In [8], [9] and [10] the values of $R(n,k)$ are determined for $n \geq \binom{k}{2}$, or for $k \leq 10$. The main results are summarised here:

Theorem 1.1. *The following hold for $k < n$:*
1. If $n \geq \binom{k+1}{2}$, then $R(n,k) = \lceil 2n/k \rceil$;
2. If $n = \binom{k+1}{2} - 1, k > 3$, then $R(n,k) = k+2 = \lceil 2n/k \rceil + 1$;
3. If $k^2/2 \leq n \leq \binom{k+1}{2} - 2, k \geq 5$, then $R(n,k) = k+1$, with $R(n,k) = \lceil 2n/k \rceil$ except for $n = k^2/2$;
4. If $\binom{k}{2} \leq n < k^2/2, k \geq 5$, then $R(n,k) = k+1 > \lceil 2n/k \rceil$;
5. If $\binom{k}{2} - \frac{k}{3} < n < \binom{k}{2}$ then $R(n,k) \geq k+1$.

The last result can be strengthened to $R(n,k) = k+1$ due to work on regular antichains by Böhm in [1], [2] and [3]. These papers provide various results of interest, including a method to construct (n,k)CSSs in $k+1$ blocks for $k+3 < n \leq \binom{k+1}{2} - 2$. That is,

Lemma 1.2. *An (n,k)CSS in $k+1$ blocks exists for $k+3 < n \leq \binom{k+1}{2} - 2$.*

Let $w = \lfloor \binom{k}{2} - \frac{2}{5}k \rfloor$ for $k \equiv 0, 1, 3, 4 \pmod 5$ and $w = \lfloor \binom{k}{2} - \frac{2}{5}k \rfloor - 1$ for $k \equiv 2 \pmod 5$. The following lemma appears in [1] in the dual context of k-regular antichains.

Lemma 1.3. *Assume that $k \geq 6$. Then $R(n,k) \leq k$ for $k+3 \leq n \leq w$.*

Hence, for all values considered in this paper, it can be assumed that $R(n,k) \leq k+1$. As $R(n,k)$ is known for all cases with $k \leq 10$, it will also be assumed that $k \geq 11$.

A catalogue of non-isomorphic configurations for (n)CSSs for $n \leq 35$ and with at most 7 blocks is developed in [6]. Hence $R(n,k)$ is determined for all $n \leq 35$ provided $R(n,k) \leq 7$. Kündgen et al [7] use an asymptotic approach to determine upper and lower bounds on $R(n,k)$ for $n \geq 2k$. Their Corollary 1 states that $R(\binom{rm}{r}, \binom{rm-1}{r-1}) = rm$ for $r \geq 1$ and $m \geq 2$.

CSSs have a dual formulation as antichains, and this has been useful in the determination of minimum size CSSs. An **antichain** $\mathcal{A}$ on $[m]$ is a collection of subsets of $[m]$ such that $A \not\subseteq B$ for all distinct $A, B \in \mathcal{A}$. Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be a collection of subsets of $[m]$. The **dual** $\mathcal{A}^*$ of $\mathcal{A}$ is the collection $\mathcal{A}^* = \{X_1, \ldots, X_m\}$ of subsets of $[n]$ given by $X_i = \{q : i \in A_q\}$. Antichains are the duals of CSSs: if $\mathcal{A}$ is a CSS then its dual $\mathcal{A}^*$ is an antichain and vice versa. A **flat antichain** is an antichain with $||A| - |B|| \leq 1$ for all $A, B \in \mathcal{A}$. Fair CSSs and flat antichains are dual concepts. An antichain $\mathcal{A}$ in $2^{[m]}$ is m**-native** if the size of the antichain $\mathcal{A}$ exceeds the maximum size antichain on $[m-1]$. That is, $|\mathcal{A}| > \binom{m-1}{\lfloor \frac{m-1}{2} \rfloor}$. This means that all elements of $[m]$ occur in $\mathcal{A}$. If $\mathcal{C}$ is a minimum size (n,k)CSS in m blocks then the dual antichain is m-native.

A k**-regular antichain** $\mathcal{A}$ on $[m]$ has each element of $[m]$ occurring in exactly k sets in $\mathcal{A}$. Thus $V(\mathcal{A}) = km$. The dual of an (n,k)CSS $\mathcal{C}$ in m blocks is a k-regular m-native antichain $\mathcal{A}$ with

$|\mathcal{A}| = n$. Thus, when we determinie $R(n,k) = R$, we are also determining the existence of a dual k-regular R-native antichain of size n.

The determination of values of $R(n,k)$ involves several different approaches. A commonly used method is to establish a lower bound for a particular case and then try to determine whether or not the bound is achievable. The bound is often obtained by showing that if $R(n,k) = R$ then certain partial structures must occur, and that these structures cannot be completed to an (n,k)CSS in R blocks.

For the values of n and k considered here, $2n \le V(\mathcal{C}) \le 3n$ and each point will normally be a p-point with $2 \le p \le 3$. For an (n,k)CSS $\mathcal{C}$ in R blocks with $2 \le V(\mathcal{C}) \le 3n$, $t = 3n - kR$ is the minimum number of 2-points that must occur in $\mathcal{C}$, and $u = n - t$ is the maximum number of p-points in $\mathcal{C}$, for $p \ge 3$.

If $\mathcal{C}$ achieves $R(n,k)$ then it has at least $\frac{2t}{R}$ 2-points in one of its blocks. An upper bound on the number of 2-points in a block is given by the following theorem.

Theorem 1.2. *Let $\mathcal{C}$ be a minimal (n,k)CSS in R blocks. Then each block in $\mathcal{C}$ contains at most p 2-points, where $p = R(n,k) - R(k-p) - 1$, and $R(k-p)$ is the minimum possible size of a $(k-p)$CSS.*

Proof. Assume that a block in $\mathcal{C}$ contains at least p 2-points. Each of these 2-points occur once more, each in different blocks of $\mathcal{C}$. The remaining $k-p$ points must be completely separated in the remaining $R-p-1$ blocks, which is only possible if $R(k-p) \le R(n,k) - p - 1$. □

Noting that $R(n) = n$ for $n \le 4$ and $R(5) = R(6) = 4$, an immediate corollary is

Corollary 1.1. *Let $\mathcal{C}$ be an (n,k)CSS with $|\mathcal{C}| = k$ or $k-1$. Then*
1. Each block in $\mathcal{C}$ contains at most $(k-5)$ or $(k-6)$ 2-points respectively.
2. If $n > \binom{k-1}{2}$ then $R(n,k) \ge k$.

The following related inequality appears in [10].

Lemma 1.4.

$$R(n,k) \ge \left\lceil \frac{5 - 2k + \sqrt{(2k-5)^2 + 24n}}{2} \right\rceil. \qquad (1)$$

2 $R(n,k)$ for $\binom{k-1}{2} \le n < \binom{k}{2}$

The study of values of $R(n,k)$ when n is bounded by (near) quadratic functions of k has produced the results summarised in Theorem 1.1, and this approach is continued in this section to determine $R(n,k)$ for $\binom{k-1}{2} \le n < \binom{k}{2}$.

The main results are summarised in the following theorem. The proof is contained in the subsequent discussion and theorems. Recall that $w = \lfloor \binom{k}{2} - \frac{2}{5}k \rfloor$ for $k \equiv 0, 1, 3, 4 \pmod 5$ and $w = \lfloor \binom{k}{2} - \frac{2}{5}k \rfloor - 1$ for $k \equiv 2 \pmod 5$.

Theorem 2.1. *For $k \ge 6$,*
1.

$$R(\binom{k-1}{2}, k) = \begin{cases} k-1 & k \equiv 1 \pmod 5 \\ k & \textit{otherwise} \end{cases}$$

2. $R(n,k) = k$ *for* $\binom{k-1}{2} < n \leq w$.
3. $R(n,k) = k+1$ *for* $w < n < \binom{k}{2}$.

The proof of Part 2 is straight forward. It follows immediately from Lemma 1.3 and Corollary 1.1. The proof of the other two parts is contained in the following subsections.

2.1 $R(n,k)$ for $w < n < \binom{k}{2}$

By Theorem 1.1 and Lemma 1.2, $R(n,k) = k+1$ for $\binom{k}{2} - k/3 < n < \binom{k}{2}$.

This leaves $R(n,k)$ to be determined in this section for $w < n \leq \binom{k}{2} - k/3 = \lceil\frac{(k-1)^2}{2}\rceil + \frac{k}{6} - 1/2$ for k odd, and $\binom{k}{2} - k/3 = \lceil\frac{(k-1)^2}{2}\rceil + \frac{k}{6} - 1$ for k even. For each $k \geq 11$, the number of values of n in this interval, called the *gap*, is approximately $\frac{k}{15}$. It will be shown that

Theorem 2.2. $R(n,k) = k+1$ *for* $w < n \leq \binom{k}{2} - k/3$.

Proof. Assume that $\mathcal{C}$ is an (n,k)CSS with $|\mathcal{C}| = k$ and $w < n \leq \binom{k}{2} - k/3$. By Corollary 1.1, at most $(k-5)$ 2-points can occur in any block in a minimum size (n,k)CSS in k blocks for $k \geq 6$. A block is said to be *full* if it contains $(k-5)$ 2-points. Within the gap, the minimum number of 2-points that must occur is $t = 3n - k^2$. For fixed k and $|\mathcal{C}|$, t increases with n, so t achieves its minimum in the gap when $n = w+1$.

In the remainder of this section an argument is presented based upon the minimum number of 2-points which occur in an (n,k)CSS $\mathcal{C}$ with $|\mathcal{C}| = k$. For fixed n, k this number is minimised if all other points are 3-points, so the argument assumes that all other points are 3-points. To be precise, cases when there are p-points with $p \geq 4$ follows immediately from the arguments, as the number of 2-points increases by $p-3$ with the inclusion of each p-point, $p > 3$.

For fixed k, the number of 2-points t and the number of 3-points u for each n in the gap is bounded respectively for $k \equiv 0,1,2,3,4 \pmod 5$ by:
$t \geq \frac{5k^2-27k+30}{10}$ and $u \leq \frac{9k-10}{5}$; $t \geq \frac{5k^2-27k+12}{10}$ and $u \leq \frac{9k-4}{5}$; $t \geq \frac{5k^2-27k-6}{10}$ and $u \leq \frac{9k+2}{5}$; $t \geq \frac{5k^2-27k+6}{10}$ and $u \leq \frac{9k-2}{5}$; $t \geq \frac{5k^2-27k+18}{10}$ and $u \leq \frac{9k-6}{5}$.
By calculating $2t - k(k-6)$ it can be seen that there are at least v blocks of $\mathcal{C}$ that contain exactly $(k-5)$ 2-points where
$v = \frac{3k}{5}+6, \frac{3k}{5}+\frac{12}{5}, \frac{3k}{5}-\frac{6}{5}, \frac{3k}{5}+\frac{6}{5}, \frac{3k}{5}+\frac{18}{5}$ respectively for $k \equiv 0,1,2,3,4 \pmod 5$.
This means that in all cases except when $k \equiv 2 \pmod 5$ and $n = w+1$, on average more than three out of every five blocks of $\mathcal{C}$ are full. Note that this average is larger if there is a p-point, $p > 3$. The values listed here are used implicitly throughout the proof when making calculations relating to the number of 2-points or 3-points in various configurations.

Some more structure is needed. Assume that a block $B_1 \in \mathcal{C}$ is full. Let $B_1 = \{1, 2, \ldots, k-5, a, b, c, d, e\}$. Then each of the 2-points $1, 2, \ldots, k-5$ reoccur once more in $\mathcal{C}$, say in blocks $B_6, \ldots, B_k$. This means that the 3-points a, b, c, d, e must be completely separated in blocks in $B = \{B_2, \ldots, B_5\}$. There is one non-isomorphic way to do this, as shown.

$$B = \begin{array}{ccccc} a & b & c & & \\ a & d & e & X & Y \\ b & d & & & \\ c & e & & & \end{array}$$

Here, X represents the subcollection of 3-sets in these blocks, other than a, b, c, d, e, and Y is the subcollection of 2-sets in these blocks. Let X_i, Y_i denote the ith block of X, Y respectively. Call the collection $A = \{B_1, B_2, B_3, B_4, B_5\}$ a *collection of associated blocks*. We proceed by considering the possible size and structure of X and Y. The known partial structure for these associated blocks is used to impose structural constraints within parts of a partitioning of the remaining blocks, without having to deal with each block individually. This will make use of the following process.

Block Partitioning Process (BPP)
Let X_1, X_2 be the sets of 3-points in collections of associated blocks A_1, A_2 respectively in the same form that X is for A above. Assume that A_1, A_2 are disjoint in the sense that $X_1 \cap X_2 = \emptyset$. It is then said that A_1, A_2 are X-disjoint. Then we can partition the blocks of $\mathcal{C}$ into X-disjoint collections of blocks of size five, labelled, $A_1, \ldots, A_s, V, s \leq \lfloor \frac{k}{5} \rfloor$, by recursively choosing a previously unassociated full block, and including it, together with its associated blocks, in the same part. V consists of any remaining blocks called the *excess blocks*.

Let $A = \{B_1, \ldots, B_5\}$ be a collection of associated blocks, let $B = \{B_2, \ldots, B_5\}$, $Z = \{B_6, \ldots, B_k\}$, and let $B \times Z = \{(i, j) : B_i \in B, B_j \in Z\}$. A row in Y and a row in Z are said *to clash* if a row in Z is covered by both a 3-point and a 2-point from the same row in B. This reflects the fact that the 3-point covers the 2-point and so the CSS property is violated.

Case 1: Assume that there is no 2-point which occurs twice in Y.

It is easy to see that $|X| \neq 10, 11$ as follows. If $|X| = 10$ then $|Y| = 4(k-5)$ and each row of Y contains exactly $(k-5)$ 2-points which must be completely separated in rows in Z. We represent this situation with the following notation for the possible configuration of X and Y

$$W = \begin{matrix} 2 & k-5 \\ 2 & k-5 \\ 3 & k-5 \\ 3 & k-5 \end{matrix}$$

where each row represents the number of 3-points in X and 2-points in Y in the corresponding blocks of B. As $|X| = 10$, an element in X must reoccur somewhere in blocks in Z, so rows in Y and Z will clash.

For $|X| = 11$ the possible nonisomorphic configurations are

$$W_1 = \begin{matrix} 3 & k-6 \\ 2 & k-5 \\ 3 & k-5 \\ 3 & k-5 \end{matrix} \quad \text{and} \quad W_2 = \begin{matrix} 2 & k-5 \\ 2 & k-5 \\ 3 & k-5 \\ 4 & k-6 \end{matrix}$$

In either case, it is easily checked that there is a row which contains a 2-point and a 3-point which occur together again in a block in Z, and so a 2-point is covered by a 3-point.

Now consider the case $|X| = 12$. There are five possible configurations, given the possible values in Y:

$$W_1 = \begin{matrix} 2 & k-5 \\ 2 & k-5 \\ 3 & k-5 \\ 5 & k-7 \end{matrix}, \quad W_2 = \begin{matrix} 2 & k-5 \\ 2 & k-5 \\ 4 & k-6 \\ 4 & k-6 \end{matrix}, \quad W_3 = \begin{matrix} 2 & k-5 \\ 3 & k-6 \\ 3 & k-5 \\ 4 & k-6 \end{matrix}, \quad W_4 = \begin{matrix} 2 & k-5 \\ 4 & k-7 \\ 3 & k-5 \\ 3 & k-5 \end{matrix}, \quad W_5 = \begin{matrix} 3 & k-6 \\ 3 & k-6 \\ 3 & k-5 \\ 3 & k-5 \end{matrix}$$

The fact that each 2-point in Y needs to reoccur once in Z means that $|Y \times Z| = 4k - 22$ so $|X \times Z| \leq 2$. As $|X| = 12$, it is easily seen, by considering the possible placement of three points of X in Z, that $|X \times Z| \geq 3$ if more than four distinct points are used in X. So we can assume that only four distinct points occur in X.

For W_1: as there are at least five distinct points in X, $|X \times Z| \geq 3$.
For W_2: assume that $X_4 = fghi$. With only four points allowed, $X_3 = fghi$, and the first two

blocks cannot be filled using only points from $fghi$.

For W_3 or W_4: for W_3 assume that $X_4 = fghi$. Then $X_3 = fgh$ and X_2 cannot be filled by points chosen from $fghi$, given that each point occurs 3 times in $\mathcal{C}$. The configuration W_4 uses the same argument style with $X_2 = fghi$.

For W_5: there is one possibility, namely $X_1 = fgh, X_2 = fgi, X_3 = fhi, X_4 = ghi$. This is the only possible configuration for $|X| = 12$.

It follows that if all associated collections have $|X| = 12$ then there are two possibilities. If $s = \lfloor \frac{k}{5} \rfloor$ then there are insufficient 3-points for the number of associated collections except when $n = w + 1$ and $k \equiv 2 \pmod 5$. Then there are insufficient distinct 3-points left to fill the excess rows. In all other cases there are insufficient full rows as none of the excess rows can be full. This follows from the assumption that $|X| = 12$ for all associated collections and from the fact that 3-points cannot be shared by distinct associated collections.

Assume that $\mathcal{C}$ contains some associated blocks with $|X| = 13$, or equivalently $|Y| = 4k - 23$. Ensuring that no more than three pairs of rows are covered in $B \times Z$, the eight possible configurations are:

$$W_1 = \begin{matrix} 2 & k-5 \\ 2 & k-5 \\ 3 & k-5 \\ 6 & k-8 \end{matrix}, \quad W_2 = \begin{matrix} 2 & k-5 \\ 2 & k-5 \\ 4 & k-6 \\ 5 & k-7 \end{matrix}, \quad W_3 = \begin{matrix} 2 & k-5 \\ 3 & k-6 \\ 3 & k-5 \\ 5 & k-7 \end{matrix}, \quad W_4 = \begin{matrix} 2 & k-5 \\ 3 & k-6 \\ 4 & k-6 \\ 4 & k-6 \end{matrix},$$

$$W_5 = \begin{matrix} 3 & k-6 \\ 3 & k-6 \\ 3 & k-5 \\ 4 & k-6 \end{matrix}, \quad W_6 = \begin{matrix} 3 & k-6 \\ 4 & k-7 \\ 3 & k-5 \\ 3 & k-5 \end{matrix}, \quad W_7 = \begin{matrix} 2 & k-5 \\ 4 & k-7 \\ 3 & k-5 \\ 4 & k-6 \end{matrix}, \quad W_8 = \begin{matrix} 2 & k-5 \\ 5 & k-8 \\ 3 & k-5 \\ 3 & k-5 \end{matrix}$$

Given these configurations, it can be easily checked by considering the possible placement of points from X in Z, that if there are at least six distinct 3-points occuring in X, then $|X \times Z| \geq 4$. So we can assume that there are at most five distinct points in X. Hence the configuration W_1 is not feasible.

Given that there are five distinct 3-points in X, two places in Z include values from X, and there are three possible arrangements of these points in Z: the point x repeated in two different rows of Z; the points x, y occurring in the same row of Z; or the points x, y occurring in different rows of Z. Only the first two cases are feasible as in the third case at least four pairs of rows will be covered by x or y. The following constructions implicitly assume that these are the only two feasible arrangements. Let the rows in Z, which may contain points from X, be labelled Z_1, Z_2.

For W_2: let $X_4 = fghij$ and $X_3 = fghi$. Then one each of f, g, h are in X_1, X_2, Z_1 respectively, and j must be in X_1, X_2, forcing i into Z_2. This is not feasible.

For notational convenience when the points in one row of X are specified, then the sets of points in the remaining rows, including those in Z, are recorded in summary form in numeric order of their row name, as illustrated in the next case with the vertical stroke signifying the divide between X and Z.

For W_3: with $X_4 = fghij$, the only feasible solution is $gi, fij, fgh|hj$ (for rows X_1, X_2, X_3, Z_1 in this order) but then Y_3 clashes with Z_1.

For W_5: with $X_4 = fghi$, these points must occur in pairs in four more rows, giving $fgj, fhj, gij|hi$. Then Y_3 clashes with Z_1. A similar methodology can be used to show that W_6 (without using Z_2), W_7 and W_8 are not feasible.

W_6 is feasible with $fgh, fgij, fhi, ghi|j, j$.

For W_4: with $X_4 = fghi$, X_3 must be $fghj$, forcing $fj, gij|hi$.

Thus W_4, W_6 are the only feasible configurations for $|X| = 13$.

The reason for doing this analysis is the following. Given the feasible configurations above, it will be shown that for any two distinct associated collections A_1, A_2 of five blocks, with A_1, A_2 having their associated blocks with $|X| = 12$ or 13, then A_1, A_2 are X-disjoint. Then BPP can be applied to the blocks in $\mathcal{C}$.

Assume that A_1, A_2 are each collections of associated blocks with blocks $B_{A_1} \in A_1, B_{A_2} \in A_2$ being full, and X is related to A_1. Assume that some 3-points are common to both A_1 and A_2. This cannot happen with the one feasible configuration for $|X| = 12$ so assume that $|X| = 13$. B_{A_1}, B_{A_2} cannot contain any of the same five 3-points, else the 3-points cannot be completely separated appropriately. Assume that the 3-points in B_{A_1}, B_{A_2} are $abcde, fghij$ respectively. Assume that A_1, A_2 share 3-points in a common block B_3 in A_1. Then B_3 contains one of the following non-isomorphic combinations of 3-points $abcfgh, abcfg, abfg$. For $|X| = 13$ there are two feasible configurations for X, namely W_4 and W_6. Neither of these configurations allow for the 3-points $fghij \in B_{A_2}$ to share 3-points with A_1 and have them completely separated in only four blocks.

Hence there are no collections of associated rows with shared blocks, and any excess rows containing 3-points common with an associated collection with $|X| = 13$ cannot be full.

BPP can be applied for the case of associated blocks with $|X| = 12$ or 13. The volume of 2-points in $\mathcal{C}$ is maximised when all but one of the associated collections have $|X| = 12$. As any excess rows are not full, the volume of 2-points in $\mathcal{C}$ is at most $\lfloor \frac{k-5}{5} \rfloor (5k-27) + (5k-28) + i(k-6) = \frac{k-5-i}{5}(5k-27) + (5k-28) + i(k-6) < k^2 - \frac{27k}{5} < 2t$, with $i \equiv 0, 1, 2, 3, 4 \pmod 5$, except possibly when $n = w + 1$ and $k \equiv 2 \pmod 5$. For $k \equiv 2 \pmod 5$ there must be two excess rows which are not full, so the appropriate inequality is $\frac{k-7}{5}(5k-27) + (5k-28) + 2(k-6) =< k^2 - \frac{27k}{5} - \frac{6}{5}$ and again there are insufficient 2-points in $\mathcal{C}$. That is, if all associated collections have $|X| \leq 13$ then the volume of 2-points will be less than $2t$.

Assume that $\mathcal{C}$ contains an associated collection in which $|X| = 14$. If there is more than one collection of associated rows with $|X| = 14$, and which do not share some common 3-points, or where the excess rows are not full, then calculations similar to those above show that the volume of 2-points is less than $2t$.

Assume that all but one collection of associated rows has $|X| = 12$ and one collection A of associated rows has $|X| = 14$. For this latter X it must be that $|X \times Z| \leq 4$. There are several possible arrangements for X, but to show that the volume of 2-points in $\mathcal{C}$ is less than $2t$, it is sufficent in most cases to show that any excess rows associated with A cannot be full.

Assume that A shares 3-points with another full row, say Z_1, which contains the 3-points $fghij$. There is one way to completely separate these in four other blocks. There must be at least two other distinct 3-points in X with one or both reoccurring in an excess row, not Z_1, and contributing at least two to the size of $|X \times Z|$. It does not matter how $fghij$ are arranged in X or excess rows, these contribute at least three additional values to the size of $|X \times Z|$. Hence $|X \times Z| > 4$. This is also the case if any of $fghij$ appear in the other excess row. Hence Z_1 and any other excess rows cannot be full. After using BPP, it is easy to check that for each value of $k \pmod 5$, except for $n = w + 1$ and $k \equiv 2 \pmod 5$, the volume of 2-points in $\mathcal{C}$ is less than $2t$.

When $n = w + 1$ and $k \equiv 2 \pmod 5$ then there are insufficient distinct 3-points to complete the two excess rows.

If there is an associated collection with $|X| \geq 15$ then the volume of 2-points in $\mathcal{C}$ requires that there is one such collection and the remaining ones have $|X| = 12$. Then there are $k \pmod 5$ excess rows, and whether or not these are full, the calculation of the maximum volume of 2-points in $\mathcal{C}$ can be found for each value of $k \pmod 5$, and the volume of 2-points is less than $2t$.

Case 2: Assume that there is a 2-point which occurs twice in Y. Then the following structure occurs with x the repeated 2-point in Y, and rows labelled $B_1, \ldots B_5$.

$$\begin{array}{cccccccc}
1 & \ldots & & k-5 & a & b & c & d & e \\
a & b & c & & & & & & \\
a & d & e & & X & & Y & & \\
b & d & & & & & x & & \\
c & e & & & & & x & &
\end{array}$$

This means that it is necessary that $|X \times Z| \leq 4(k-5) - |Y| + 2$. It also means that all 3-points in the last 2 rows B_4, B_5 are distinct to avoid covering x, so there are at least six distinct points in X in $B_4 \cup B_5$.

It is easy to check that for $|X| = 10$ and $|Y| = 4k - 20$, or for $|X| = 11$ and $|Y| = 4k - 19$, that $|X \times Z| \geq 4 > 4(k-5) - |Y| + 2$, as there must be at least two positions filled in Z by 3-points from each of B_4, B_5. Thus any asssociated collection contains at most 3 full rows.

If $|X| \geq 12$ then X has the following partial structure with 3-points f, g, h, i, j, k

$$\begin{array}{cccccccc}
a & b & c & & & & & \\
a & d & e & & & & & \\
b & d & & i & j & k & & x \\
c & e & & f & g & h & & x
\end{array}$$

with each of B_2, B_3 containing at least two 3-points. Each of the collections fgh, ijk require at least three more rows to completely separate each of them, so each collection has points which occupy at least two places in Z.

For $|X| = 12$ or 13, the possible arrangements of the points $fghijk$ which keep $|X \times Z|$ small enough, require that there are four places in each of B_2, B_3 occupied by pairs of points chosen from each of fgh and ijk. This is not possible as $|X| \leq 13$.

Thus all collections of associated rows have $|X| \geq 14$ and using arguments similar to those used in Case 1, it is easy to verify that the volume of 2-points in $\mathcal{C}$ is less than $2t$.

Thus $R(n, k) \geq k+1$ for all values of n in the gap. The fact that $R(n, k) = k+1$ follows immediately by Lemma 1.2. $\square$

2.2 $R(n, k)$ for $n = \binom{k-1}{2}$

By Lemma 1.4, $R(\binom{k-1}{2}, k) \geq k - 1$.

Lemma 2.1. *If* $k \geq 6$ *and* $R(\binom{k-1}{2}, k) = k - 1$ *then* $k \equiv 1 \pmod 5$.

Proof. A fair minimal CSS on $\binom{k-1}{2}$ points and with $(k-1)$ blocks has each block containing exactly $(k-6)$ 2-points and six 3-points. The 3-points must be distributed as follows:

$$\begin{array}{cccccc}
a & b & c & d & e & f \\
a & b & c & & & \\
a & d & e & & & \\
b & d & f & & & \\
c & e & f & & &
\end{array}$$

Each of the remaining $(k-6)$ blocks contains exactly one 2-point from each of these five blocks. So the remaining 3-points in these blocks occur as shown:

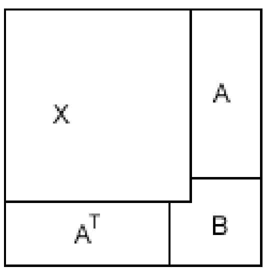
$$\begin{array}{ccccccc} a & b & c & d & e & f & \\ a & b & c & & g & h & i \\ a & d & e & & g & h & j \\ b & d & f & & g & i & j \\ c & e & f & & h & i & j \end{array}$$

Therefore $k-1$ must be divisible by five in order to allow complete separation. $\square$

It follows that if $k \not\equiv 1 \pmod 5$, $R(\binom{k-1}{2}, k) \geq k$, and the following lemma provides a construction for this case.

Lemma 2.2. *For $k \geq 6$, if $R(\binom{k-1}{2}, k) = k$ then $R(\binom{k+4}{2}, k+5) = k+5$*

Proof. Construct a $(k+5) \times (k+5)$ array M with the following structure, where A, B and X are disjoint collections, and X is a minimum size $(\binom{k-1}{2}, k)$CSS.



where $A = \begin{array}{ccccc} 1 & 2 & 3 & \dots & k-2 \\ \vdots & & & & \vdots \\ 4k-7 & & & & 5(k-2) \end{array}$,

which together with A^T completely separates $(5k-10)$ 2-points, and B is the following array which completely separates 15 3-points.

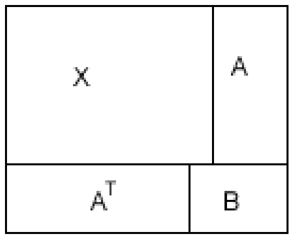
$$B = \begin{array}{ccccccc} a & b & c & d & e & f & g \\ a & b & c & g & h & i & j \\ a & d & e & h & j & m & o \\ b & d & f & h & i & l & n \\ c & e & f & i & k & n & o \\ & & g & j & k & l & m \\ & & k & l & m & n & o \end{array}.$$

Then M is a $(k+5) \times (k+5)$ array which completely separates $\binom{k-1}{2} + (5k-10) + 15 = \binom{k+4}{2}$ points. $\square$

Lemma 2.2 is applicable for $k = 7, \dots, 10$, as shown in Table 1. However, this is not the case for $k = 6$ as $R(10, 6) = 5$.

Lemma 2.3. *For $k \geq 6$, if $R(\binom{k-1}{2}, k) = k-1$ then $R(\binom{k+4}{2}, k+5) = k+4$.*

Proof. Construct a $(k+4) \times (k+5)$ array M with the following structure, where A, B and X are disjoint collections.



where $A = \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \vdots & & & & \vdots \\ 5k-1 & & & & 5(k-1) \end{array}$,

which together with A^T completely separates $5(k-1)$ 2-points. B is a 5×6 array which completely separates ten 3-points and X is a $(k-1) \times k$ array which completely separates $\binom{k-1}{2}$ points.

Then M is a $(k+4) \times (k+5)$ array which completely separates $\binom{k-1}{2} + 5(k-1) + 10 = \binom{k+4}{2}$ points. $\square$

The lemmas give part 1 of Theorem 2.1.

3 Minimal CSSs for $11 \leq k \leq 12$

In Theorem 1.1 the values of $R(n,k)$ for $n \geq k^2/2$ are stated and $R(n,k)$ is known for all n with $k \leq 10$. The remaining unknown values of $R(n,k)$ for $k = 11$ and 12 are determined here. When it is stated in this section, that a construction satisfying certain parameters has been found, then that construction appears in [4]. The construction of Böhm provides an alternative construction technique when $R = k+1$.

3.1 Remaining cases for $R(n, 11)$

$R(n,11)$ can be determined for $n \leq 21$ using results from [10] and applying $R(n,k) = R(n, n-k)$. Roberts [10] gives values of $R(n,11)$ for $n = 23, \ldots, 26$. $R(45,11) = 10$ by Theorem 2.1, $R(50,11) = 11$ by Theorem 2.1, and $R(n,11)$ has been determined for all $n \geq 55$ by Theorem 1.1. The following unknown cases need to be determined.

Roberts [10] gives the lower bound:
(i) $R(22,11) \geq 7$.

By Lemma 1.3 lower bounds for $R(n,k)$ are:
(ii) For $27 \leq n \leq 28$, $R(n,11) \geq 7$
(iii) For $29 \leq n \leq 33$, $R(n,11) \geq 8$
(iv) For $34 \leq n \leq 39$, $R(n,11) \geq 9$
(v) For $40 \leq n \leq 45$, $R(n,11) \geq 10$
(vi) For $46 \leq n \leq 50$, $R(n,11) \geq 10$.

By Theorem 2.2

(vii) For $51 \leq n \leq 54$, $R(n, 11) \geq 12$.

Case (i) $n = 22$
Constructions have been found for $R(22, 11) = 7$ in [4] and [6].

Case (ii) $27 \leq n \leq 28$

Lemma 3.1.
$R(n, 11) = 8$ for $n = 27, 28$.

Proof. For $n = 27, 28$ let $\mathcal{C}$ be an (n, k)CSS in 7 blocks. If $R(27, 11) = 7$, then $V(\mathcal{C}) = 77$, so there have to be at least four 2-points giving a block B with at least two 2-points which is not possible by Theorem 1.2. If $R(28, 11) = 7$ there are at least seven 2-points and a similar argument follows. $\square$

The same non-existence result could be obtained by using the catalogue in [6]. Constructions have been found for $R(27, 11) = 8$ and $R(28, 11) = 8$.

Case (iii) $29 \leq n \leq 33$
For $n = 29, 30, 31$, the lower bound for $R(n, k)$ is achieved and constructions have been found for $R(n, 11) = 8$.

Lemma 3.2.
$R(n, 11) = 9$ for $n = 32, 33$.

Proof. Let $\mathcal{C}$ be a $(32, 11)$CSS in 8 blocks. Then $V(\mathcal{C}) = 88$ and there have to be least eight 2-points giving a block B with at least two 2-points.

There are at least nine 2-points then there is block B with at least three 2-points which is not possible by Theorem 1.2.

Assume that there are exactly eight 2-points in $\mathcal{C}$. By Theorem 1.2 there are at most two 2-points in any block giving the configuration:

$$\begin{array}{ccccc} 1 & 2 & a & \ldots & i \\ 1 & \ldots & & & \\ 2 & \ldots & & & \\ \ldots & & & & \end{array} .$$

The nine 3-points $a, \ldots, i$ need to be completely separated in blocks three to eight. They cover nine of the ten available pairs. This allows at most one pair of 2-points in these blocks, and the 2-points cannot be appropriately included.

Let $\mathcal{C}$ be a $(33, 11)$CSS in 8 blocks. $V(\mathcal{C}) = 88$, and there must be 11 2-points and 22 3-points or, if there is one 4-point there are 12 2-points and 20 3-points. Neither is possible by Theorem 1.2. $\square$

Constructions have been found for $R(32, 11) = R(33, 11) = 9$.

Case (iv) $34 \leq n \leq 39$
Constructions have been found for $R(34, 11) = R(35, 11) = R(36, 11) = 9$.

Lemma 3.3.
$R(n, 11) = 10$ for $37 \leq n \leq 39$.

Proof. Let $\mathcal{C}$ be a $(37, 11)$CSS in nine blocks. $V(\mathcal{C}) = 99$ and if there is a 4-point there must be 13 2-points and 23 3-points as follows:

$$\begin{array}{lllll} X & 1 & 2 & 3 & a-g\ldots \\ X & \ldots & & & \\ X & \ldots & & & \\ X & \ldots & & & \\ 1 & \ldots & & & \\ 2 & \ldots & & & \\ 3 & \ldots & & & \\ \ldots & & & & \\ \ldots & & & & \end{array} \quad \text{or} \quad \begin{array}{lllll} 1 & 2 & 3 & a-h\ldots & \\ 1 & \ldots & & & \\ 2 & \ldots & & & \\ 3 & \ldots & & & \\ X & \ldots & & & \\ X & \ldots & & & \\ X & \ldots & & & \\ X & \ldots & & & \\ & \ldots & & & \end{array} \quad \text{or} \quad \begin{array}{lllll} 1 & 2 & 3 & a-h\ldots & \\ 1 & X & \ldots & & \\ 2 & \ldots & & & \\ 3 & \ldots & & & \\ X & \ldots & & & \\ X & \ldots & & & \\ X & \ldots & & & \\ & \ldots & & & \\ & \ldots & & & \end{array} .$$

By Lemma 1.1, the seven (or eight) points $a, \ldots, g$ (or $a, \ldots, h$) require five rows to be completely separated and this will allow at most three pairs of 2-points in these rows, so this is not possible.

If there are no 4-points then there are 25 3-points and 12 2-points, giving the following configuration:

$$\begin{array}{llll} 1 & 2 & 3 & a\ldots h \\ 1 & \ldots & & \\ 2 & \ldots & & \\ 3 & \ldots & & \\ \ldots & & & \\ \ldots & & & \end{array} .$$

The eight points $a, \ldots, h$ require five rows to completely separate and this will not allow for any pair of 2-points in these rows, so this is not possible.

Let $\mathcal{C}$ be a $(38, 11)$CSS with $R = 9$. $V(\mathcal{C}) = 99$ so there must be at least 15 2-points with at least four 2-points in at least one block. Let $B_1 = \{1, 2, 3, 4, a, \ldots, g\}$ then $a, \ldots, g$ cannot be completely separated in the last four blocks.

Let $\mathcal{C}$ be a $(39, 11)$CSS with $R = 9$. $V(\mathcal{C}) = 99$ so there must be at least 17 2-points with at least four 2-points in at least one block. The proof involves the same argument as used for $n = 38$. $\square$

Constructions have been found for $R(37, 11) = R(38, 11) = R(39, 11) = 10$.

Case (v) $40 \leq n \leq 44$
Constructions have been found for $R(n, 11) = 10$ for $40 \leq n \leq 44$.

Case (vi) $46 \leq n \leq 49$
By Corollary 1.1, $R(n, 11) \geq 11$ for $n > 45$, and constructions have been found for $R(n, 11) = 11$ for $46 \leq n \leq 49$.

Case (vii) $51 \leq n \leq 54$
$R(51, 11), \ldots, R(54, 11) \geq 12$ by Theorem 2.2, and constructions have been found for $R(n, 11) = 12$ for $51 \leq n \leq 54$.

3.2 Remaining cases for $R(n, 12)$

$R(n, 12)$ can be determined for $n \leq 24$ using results from [10] and applying $R(n, k) = R(n, n-k)$, and in [10] there is a construction for $R(25, 12) = 7$.

$R(55, 12) = 11$ by Theorem 2.1. Theorem 2.1 gives $R(61, 12) = 13$. All values of $R(n, k)$ have been determined for $n \geq 66$ in Theorem 1.1.

The remaining unknown cases are for $n = 24$, $26 \leq n \leq 30$, $36 \leq n \leq 54$, $56 \leq n \leq 60$ and $62 \leq n \leq 65$.

Roberts [10] gives the lower bound:

(i) $R(24,12) \geq 7$.

By Lemma 1.4 lower bounds for $R(n,k)$ are:
(ii) For $26 \leq n \leq 30$, $R(n,12) \geq 7$
(iii) For $31 \leq n \leq 36$, $R(n,12) \geq 8$
(iv) For $37 \leq n \leq 42$, $R(n,12) \geq 9$
(v) For $43 \leq n \leq 48$, $R(n,12) \geq 10$
(vi) For $49 \leq n \leq 54$, $R(n,12) \geq 11$
(vii) For $56 \leq n \leq 60$, $R(n,12) \geq 12$.

By Theorem 2.2
(viii) For $62 \leq n < 65$, $R(n,12) \geq 13$.

By Theorem 1.2, if $\mathcal{C}$ is an $(n,12)$CSS with $|\mathcal{C}| = 7$, then each block in $\mathcal{C}$ contains at most $|\mathcal{C}| - 6 = 1$ 2-point.

Lemma 3.4. *For $k = 12$ and $R(n,12) = 7$ there cannot be a 2-point in any block.*

Proof. Let $\mathcal{C}$ be an $(n,12)$CSS with $R = 7$. Assume there is a block containing a 2-point. This gives the following partial structure:

$$\begin{array}{cccccccccccc}
1 & a & b & c & d & e & f & g & h & i & j & k \\
1 & \ldots & & & & & & & & & & \\
\vdots & & & & & & & & & & &
\end{array}.$$

The 11 points $a, \ldots, k$ must be completely separated in five blocks which is not possible since $R(11) = 6$. □

Case (i) $n = 24$
A construction has been found for $R(24,12) = 7$.

Case (ii) $26 \leq n \leq 30$
By the catalogue in [6], $R(n,12) \neq 7$ for $n = 26, 27, 29, 30$ and a construction is given there for $R(28,12) = 7$. Constructions have been found for $R(26,12) = R(27,12) = R(29,12) = R(30,12) = 8$.

Case (iii) $31 \leq n \leq 36$
Constructions has been found for $R(n,12) = 8$ for $31 \leq n \leq 33$.

Lemma 3.5. $R(n,12) = 9$ *for* $34 \leq n \leq 36$.

Proof. Assume $\mathcal{C}$ is a $(34,12)$CSS in eight blocks. By Theorem 1.2 there must be eight 2-points and 26 3-points, with exactly two 2-points per block giving the following configuration:

$$\begin{array}{cccccccccccc}
1 & 2 & a & b & c & d & e & f & g & h & i & j \\
1 & - & \ldots & & & & & & & & & \\
2 & - & \ldots & & & & & & & & & \\
\vdots & & & & & & & & & & &
\end{array}.$$

The ten 3-points $a, \ldots, j$ must be completely separated in the last five blocks covering exactly ten pairs of these blocks (see [10]). Hence the remaining 2-points cannot be completely separated. Adaptations of this proof show that $R(35,12), R(36,12) > 8$. Constructions have been found for $R(n,12) = 9$ for $34 \leq n \leq 36$. □

Case (iv) $37 \le n \le 42$
Constructions have been found for $R(n, 12) = 9$ for $37 \le n \le 39$.

Lemma 3.6. *$R(n, 12) = 10$ for $40 \le n \le 42$.*

Proof. Assume $\mathcal{C}$ is an $(n, 12)$CSS ($40 \le n \le 42$) in 9 blocks. $V(\mathcal{C}) = 108$ so there are at least 14 2-points, but this would require some blocks to contain at least four 2-points. This is not possible by Theorem 1.2. If there were any p-points, $p \ge 4$, the number of 2-points would be greater. Constructions have been found for $R(n, 12) = 10$ for $40 \le n \le 42$. $\square$

Case (v) $43 \le n \le 48$
Constructions have been found for $R(n, 12) = 10$ for $43 \le n \le 45$.

Lemma 3.7. *$R(46, 12) = 11$.*

Proof. Assume that $\mathcal{C}$ is a $(46, 12)$CSS in ten blocks. There are at most four 2-points per block by Lemma 1.1. It can be seen that there are no p-points, $p \ge 4$ as follows. Assume that the point A occurs in at least four blocks. Then there are at least 19 2-points in $\mathcal{C}$ and the partial structure below must occur to completely separate the points of row 1. This forces all of the 2-points $1, \dots, 19$ to occur in rows five to eight and this is not possible.

$$\begin{array}{llllllllllll}
A & 1 & 2 & 3 & 4 & a & b & c & d & e & f & g \\
A & b & e & \dots \\
A & c & f & \dots \\
A & d & g & \dots \\
1 & \dots \\
2 & \dots \\
3 & \dots \\
4 & \dots \\
a & b & c & d & \dots \\
a & e & f & g & \dots
\end{array}$$

The following arguments can be made without loss of generality. $V(\mathcal{C}) = 120$ so there are 18 2-points and 28 3-points, and there is a block which contains exactly four 2-points. This gives the following partial structure:

$$\begin{array}{llllllllllll}
1 & 2 & 3 & 4 & a & b & c & d & e & f & g & h \\
1 & \dots \\
2 & \dots \\
3 & \dots \\
4 & \dots \\
\vdots
\end{array} .$$

The eight 3-points $a, \dots, h$ must be completely separated in the last five blocks, and by [10] there are two ways of doing this with volume 16. This leads to two cases based upon the partial forms of rows six to ten:

$$\text{Case 1:} \quad \begin{array}{llll}
a & b & c & d \\
a & e & f & g \\
b & e & h \\
c & f & h \\
d & g
\end{array} , \quad \text{Case 2:} \quad \begin{array}{llll}
a & b & c & d \\
a & e & f \\
b & e & g \\
c & f & h \\
d & g & h
\end{array} .$$

These points cover eight of the 10 pairs in rows six to ten, and so at most two 2-points can occur here twice. This means that the first five rows must have the following form:

1	2	3	4	a	b	c	d	e	f	g	h
1	5	6	$7\ldots$								
2	8	9	$10\ldots$								
3	11	12	$13\ldots$								
4	14	15	$16\ldots$								

with each of $5,\ldots,16$ occurring exactly once more in rows six to ten, and with 17 in rows eight and ten, and 18 in rows nine and ten. It follows that at least one of the rows $6,\ldots,10$ contains exactly four 2-points.

Case 1:
Assume that $5,8$ are in row ten, along with the six 3-points $A,\ldots F$. These 3-points must be completely separated in the four rows $4,\ldots,7$ with three of each of the 3-points in each row. This means that at least one of $11,12,13$ cannot be completely separated from them.
Assume that $5,8,11$ are in row nine, along with the five 3-points $A,\ldots,E$. Each of the points $A,\ldots,E$ must occur twice in rows $5,\ldots,8$ and they cover five of the six pairs of rows. This means that at least one of $14,15,16$ cannot be completely separated from them.
Assume that $5,8,11,14$ are in row six, along with the four 3-points $A,\ldots,D$. Each of the points $A,\ldots,D$ must occur twice in rows $7,\ldots,10$ as follows, to avoid covering 17 or 18. The placement of $9,10,11,E,F$ is then forced as shown.

A	B	C	9	10	11	E	F
A	D	E					
B	D	F					
C							

Similarly to the above arguments, it can be seen that $14,15,16$ cannot be completely separated from some of the other points. Hence Case 1 does not lead to a $(46,12)$CSS.

Case 2 follows by similar reasoning to Case 1 to show that a $(46,12)$CSS cannot be created, so only some details are included.
Assume that $5,8,11,14$ are in row six, along with the four 3-points $A,\ldots D$. Then rows $7,\ldots,10$ have the following partial form and there are two subcases to check based upon the possible ways to complete row seven: with $6,9,12,E,F,G$ or $6,9,E,F,G,H$.

a	e	f	18	A	B
b	e	g	17	A	C
c	f	h	17	B	D
d	g	h	18	C	D

This means that one of rows $7,\ldots,10$ contain four 2-points, say row seven,
containing $5,8,11,18,I,J,K,L,M$. Then $I,\ldots,M$ must be completely separated in rows 5, 6, 8 and 9, with I,J,K in row five, and one each of these in rows 6, 8, 9. This means that $14,15,16$ cannot be completely separated from them. □

Lemma 3.8. $R(n,12)=11$ *for* $n=47,48$.

Proof. Assume $\mathcal{C}$ is an $(n,12)$CSS in ten blocks. $V(\mathcal{C})=120$ and there are 21 or 24 2-points respectively, but this would require some blocks to contain at least five 2-points which is not possible by Theorem 1.2. If there was a 4-point there would be more 2-points and a similar argument would follow. □

Constructions have been found for $R(n,12)=11$ for $46\le n\le 48$.

Cases (vi) $49\le n\le 54$
Constructions have been found for $R(n,12)=11$ for $49\le n\le 54$.

Case (vii) $54\le n\le 60$
Constructions have been found for $R(n,12)=13$ for $54\le n\le 60$.

$R(61, 12) = 13$ by Theorem 2.2. It is worth noting that a computer search (see [5]) had previously determined that $R(61, 12) \neq 12$, and a long analytic proof of this is outlined in [6]. The new method using BPP is a significant improvement to these previously used methods which dealt with this one case only. A construction for $R(61, 12) = 13$ is shown in Section 4.

Case (viii) $62 \leq n \leq 65$
Constructions have been found for $R(n, 12) = 13$ for $62 \leq n \leq 65$.

Table 1 provides a complete set of values of $R(n, k)$ for $2 \leq n \leq 63$ and $k \leq 12$.

4 Comments

Quadratic Bands

It is interesting to note that in the case of $n = \lceil \frac{(k-1)^2}{2} \rceil$, $R(n, k) = k + 1$ for $6 \leq k \leq 8$ or $k = 12$, and $R(n, k) = k$ for $k > 8$, $k \neq 12$. For $n = \binom{k-1}{2}$, $R(n, k) = k - 1$ if and only if $k \equiv 1 \pmod 5$, and $R(n, k) = k$ otherwise. This is the first time that the values of $R(n, k)$ have not been found to be the same function of k for each n of the form $n = \lceil \frac{(k+a)^2}{2} \rceil$ or $n = \binom{k+a}{2}, a \geq -1, k \geq 10$.

A $(61, 12)$CSS in 13 blocks is shown below, partly because of the property just mentioned, but also because it was very difficult to show that $R(61, 12) \neq 12$ before BPP was developed.

1	2	3	4	5	a	b	c	d		u	v	w		
6	7	8	9	10	a	e	f	g		u	v	x		
11	12	13	14	15	b	e	h	i		u	w	y		
16	17	18	19	20	c	f	h	j		v	x	y		
21	22	23	24	25	d	g	i	j		w	x	y		
26	27		a	b	c	d	e	f	g	h	i	j		
26		k	l	m	n	o		p	q	r	s	t		H
27		k	l	m	n	o			z	A	B	C	D	E
1	6	11	16	21		k		p	q		C	F	G	H
2	7	12	17	22		l		p	r		B	E	F	G
3	8	13	18	23		m		q	s		A	D	G	E
4	9	14	19	24		n		r	t		z	D	H	F
5	10	15	20	25		o		s	t		z	A	B	C

Monotonicity in n
Lemma 1.1 implies that $R(n)$ is a non-decreasing function of n. One of the questions posed in [9] was whether or not the value of $R(n, k)$ is monotonic in n, for fixed $k \neq 4, 5$ and $n \geq 2k$. In [8] it was shown that the lower bound of $\lceil \frac{2n}{k} \rceil = k + 1$ cannot be achieved for $n = \binom{k+1}{2} - 1$, whilst it is achieved for $\binom{k+1}{2} - 2$, and for $n = \binom{k+1}{2}$. Thus $R(n, k)$ is not a non-decreasing function of n at least at one point for $k \geq 4$. This paper shows that $R(26, 12) = R(27, 12) = 8$ but $R(28, 12) = 7$ and this is the first case found of another value of n and k where $R(n, k)$ is not monotonic as a function of n.

Regular antichains
The dual of an (n, k)CSS in $R(n, k)$ blocks is an R-native k-regular AC of size n. Hence Table 1 provides the smallest size ground set for the existence of a k-regular AC of size n.

Flat antichains
All of the constructions mentioned in Section 3 are fair CSSs. Hence their duals are flat ACs. This continues support for the conjecture that $R(n, k)$ can always be achieved by a fair CSS. Equivalently, the results support the conjecture that whenever there is a k-regular AC of size n on $[R]$, then there is also a k-regular flat AC of size n on $[R]$.

	k											
n	1	2	3	4	5	6	7	8	9	10	11	12
2	2											
3	3											
4	4	4										
5	5	5										
6	6	6	4									
7	7	7	5									
8	8	8	6	5								
9	9	9	6	6								
10	10	10	7	5	6							
11	11	11	8	6	6							
12	12	12	8	6	6	6						
13	13	13	9	7	6	7						
14	14	14	10	7	7	7	6					
15	15	15	10	8	6	7	7					
16	16	16	11	8	7	7	7	6				
17	17	17	12	9	7	7	7	7				
18	18	18	12	9	8	7	8	7	6			
19	19	19	13	10	8	7	8	7	7			
20	20	20	14	10	8	8	8	8	7	6		
21	21	21	14	11	9	7	8	8	7	7		
22	22	22	15	11	9	8	8	8	8	7	7	
23	23	23	16	12	10	8	8	8	8	7	7	
24	24	24	16	12	10	8	8	8	8	8	7	7
25	25	25	17	13	10	9	8	9	8	8	7	7
26	26	26	18	13	11	9	8	9	8	8	7	8
27	27	27	18	14	11	9	9	9	9	8	7	8
28	28	28	19	14	12	10	8	9	9	8	8	7
29	29	29	20	15	12	10	9	9	9	9	8	8
30	30	30	20	15	12	10	9	9	9	9	8	8
31	31	31	21	16	13	11	9	9	9	9	8	8
32	32	32	22	16	13	11	10	9	9	9	8	8
33	33	33	22	17	14	11	10	9	10	9	9	8
34	34	34	23	17	14	12	10	9	10	9	9	9
35	35	35	24	18	14	12	10	10	10	10	9	9
36	36	36	24	18	15	12	11	9	10	10	9	9
37	37	37	25	19	15	13	11	10	10	10	10	9
38	38	38	26	19	16	13	11	10	10	10	10	9
39	39	39	26	20	16	13	12	10	10	10	10	9
40	40	40	27	20	16	14	12	10	10	10	10	10
41	41	41	28	21	17	14	12	11	10	10	10	10
42	42	42	28	21	17	14	12	11	10	11	10	10
43	43	43	29	22	18	15	13	11	10	11	10	10
44	44	44	30	22	18	15	13	11	11	11	10	10
45	45	45	30	23	18	15	13	12	10	11	10	10
46	46	46	31	23	19	16	14	12	11	11	11	11
47	47	47	32	24	19	16	14	12	11	11	11	11
48	48	48	32	24	20	16	14	12	11	11	11	11
49	49	49	33	25	20	17	14	13	11	11	11	11
50	50	50	34	25	20	17	15	13	12	11	11	11
51	51	51	34	26	21	17	15	13	12	11	12	11
52	52	52	35	26	21	18	15	13	12	11	12	11
53	53	53	36	27	22	18	16	14	12	11	12	11
54	54	54	36	27	22	18	16	14	12	12	12	11
55	55	55	37	28	22	19	16	14	13	11	12	12
56	56	56	38	28	23	19	16	14	13	12	12	12
57	57	57	38	29	23	19	17	15	13	12	12	12
58	58	58	39	29	24	20	17	15	13	12	12	12
59	59	59	40	30	24	20	17	15	13	12	12	12
60	60	60	40	30	24	20	18	15	14	12	12	12
61	61	61	41	31	25	21	18	16	14	13	12	13
62	62	62	42	31	25	21	18	16	14	13	12	13
63	63	63	42	32	26	21	18	16	14	13	12	13

Table 1: Values of $R(n,k)$ for $2 \leq n \leq 63$ and $k \leq 12$.

Addendum: The value of $R(34, 10)$ has been amended in Table 1 from 10 to 9. This corrects a typographic error in [10].

References

[1] M. Böhm, *Reguläre Antiketten*. Diploma Thesis Universität Rostock, 2008.

[2] M. Böhm, *Regular Antichains*. Submitted to Graphs and Combinatorics, 2008.

[3] M. Böhm, $(m-1)$*-Regular Antichains on* $[m]$. Journal of Combinatorial Mathematics and Combinatorial Computing. To appear.

[4] S. D'Arcy *Some New Completely Separating Systems Designs*, Honours Thesis, Charles Darwin University, 2006.

[5] D. Dau, *Vollständig separierende Systeme und Schatten*, Diplomarbeit, Universität Rostock, 2003.

[6] M. Grüttmüller, I.T. Roberts, L. Rylands *Completely Separating Systems - a catalogue and applications*. Submitted to Discrete Mathematics, 2010.

[7] A. Kündgen, D. Mubayi, P. Tetali, *Minimal Completely Separating Systems of* k*-sets*, J. Combin. Theory A **93**, 192-198, 2001.

[8] C. Ramsay, I.T. Roberts, *Minimal Completely Separating Systems of Sets*, Australasian Journal of Combinatorics, **13**, 129-150, 1996.

[9] C.Ramsay, I.T. Roberts, F. Ruskey *Completely Separating Systems of* k*-sets*, Discrete Mathematics, **183**, 265-275, 1998.

[10] I.T. Roberts, *Completely Separating Systems of* k*-sets for* $7 \leq k \leq 10$, Australasian Journal of Combinatorics, **33**, 87-98, 2005.

[11] I.T. Roberts, L. Rylands *Minimal* (n) *and* (n, h, k) *Completely Separating Systems*, Australasian Journal of Combinatorics, **33**, 57-67, 2005.