An abstract algebraic–topological approach to the notions of a first and a second dual space I

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1 Introduction

By $I\!R$ and $\mathbb{C}$ we denote the reals and the complex numbers respectively and by $I\!K$ we mean $I\!R$ or $\mathbb{C}$. Now let $(X, ||\cdot||)$ be a normed $I\!K$–vector space and $X' := \{f : X \to I\!K | \ f$ linear and continuous$\}$ the (first) dual space (dual) of X. Hence X' consists of functions which at the same time are an algebraic homomorphism (a linear map) and a topological homomorphism (a continuous map). By the usual operatornorm, $(X', ||\cdot||)$ again becomes a normed space (even a Banach–space) meaning that X and X' belong to the same class of spaces. Hence at once we can construct the second dual space (bidual) X'' by: $X'' := ((X, ||\cdot||)', ||\cdot||)'$.
Then the canonical map $J : X \to X''$ is defined via the evaluation map[1] $\omega : X \times I\!K^X \to I\!K$ by $J(x) := \omega(x, \cdot) \in X''$ with $\forall x \in X : \omega(x, \cdot) : X' \to I\!K$, $\forall x' \in X' : \omega(x, \cdot)(x') = \omega(x, x') = x'(x)$.

1.1 Proposition
$\omega(x, \cdot) : (I\!K^X, \tau_p) \to I\!K$ is continuous for all $x \in X$.

Proof: For any net $(f_i)_{i \in I}$ from $I\!K^X$, $f \in I\!K^X$, $f_i \stackrel{\tau_p}{\longrightarrow} f$ implies especially $f_i(x) \to f(x)$ and thus $\omega(x, f_i) = f_i(x) \to f(x) = \omega(x, f)$. ■

1.2 Remark
So the map J is well–defined since we have $\forall x \in X : \omega(x, \cdot) \in X''$, i.e. $\omega(x, \cdot) : (X', ||\cdot||) \to I\!K$ is linear and continuous (w.r.t. the pointwise topology τ_p).

Proof: For $x', y' \in X', \alpha, \beta \in I\!K$ we find $\omega(x, \alpha x' + \beta y') = (\alpha x' + \beta y')(x) = \alpha x'(x) + \beta y'(x) = \alpha\omega(x, x') + \beta\omega(x, y')$ showing that $\omega(x, \cdot)$ is linear. Continuity is ensured by the even more general proposition 1.1. ■

1.3 Corollary
(a) *$\forall x \in X : \omega(x, \cdot) : (X', \tau_p) \to I\!K$ is continuous.*

[1] Given sets X, Y, ω is defined to map $\omega : X \times Y^X \to Y$ with $\omega(x, f) := f(x)$

(b) $\forall x \in X : \omega(x,\cdot) : (X', ||\cdot||) \to I\!K$ *is continuous.*

(c) *If σ is an arbitrary topology for X', then $\tau_p \leq \sigma$ implies $\omega(x,\cdot) : (X',\sigma) \to I\!K$ is continuous.*

Now we consider another quite different example. Let X, Y be rings and for simplicity let us assume that X, Y are commutative rings with units. Then we can consider $X^d := \{h \in Y^X |\ h \text{ is a ring homomorphism } \}$ as the first dual space of X (w.r.t. Y); in the usual pointwise manner we can in Y^X define an addition and a multiplication but then in general X^d is absolutely not closed under this operations, i.e. $f+g$, $f \cdot g$ may not be homomorphisms if f, g are.

Of course, we can ask now:

Question 1: How to define the second dual space X^{dd}?
We come back to our standard example: a normed space $(X, ||.||)$ and to the canonical map $J : Y \to X''$. There exists still a homomorphic map $X \to C(A, I\!K)$ to a space of continuous functions, where A is a suitable topological space (see for instance [2]).

Question 2: Is this embedding completely different from $J : X \to X''$?
Starting from these examples and generalizing the situation we will establish an abstract scheme for the construction of a first and a second dual space for a suitable space X. We will consider spaces X with algebraic or with algebraic–topological structures. We give some general assertions within this context and we will finally subsume some important well-known possibilities to embed X into X^{dd} by the map J (which may be regarded as representation of X by X^{dd}) using our scheme. Our approach will in the general context be based on topological arguments. And within this context we will answer the questions.

2 Basic topological and algebraic Notions

What is an algebraic operation on a set X?
An unitary operation is simply a function from the set into itself, a binary operation is a function from the set $X \times X$ of all ordered pairs into X, and so on. But, of course, we may think of any n–ary operation on X as a function from $X^{\boldsymbol{N}}$ into X by identifying a n–ary operation $o : X^n \to X$ with the map $o' : X^{\boldsymbol{N}} \to X : o'\left((x_i)_{i \in \boldsymbol{N}}\right) := o(x_1, ..., x_n)$. To choose $I\!N$ above is not necessary, it can be replaced by any other set – even by finite ones (with at least n elements), if we restrict our observation to operations with few arguments. So we have generalized the notion of an algebraic operation here in topological looking.
Now, we first define an algebraic structure in a quite rough manner:

2.4 Definition
Given sets X, B and $O \subseteq X^{(X^B)}$, we call the pair (X, O) a B–algebraic structure *on X.*

Second, we have to observe, that, whenever we are interested in homomorphic mappings between algebraic structures with more than one operation, we have to respect a (mostly "natural given") map between the *sets of operations* for this structures. If we have, for instance, two rings $(R_1, +_1, *_1)$ and $(R_2, +_2, *_2)$, we will require $\forall x, y \in R_1 : f(x *_1 y) = f(x) *_2 f(y)$ and $f(x +_1 y) = f(x) +_2 f(y)$ for a map f to be a homomorphism, but not $f(x +_1 y) = f(x) *_2 f(y)$. So our requirement respects the map $\{(+_1, +_2), (*_1, *_2)\}$. This is natural, because of our wish to get some more than trivial homomorphisms - which will fail in general if the operations are switched. But we will keep in mind the existence of this mapping between the sets of operations – in order to preserve the possibility to change it, if a trivially given one is "unprofitable", sometime. The remarks above lead to

2.5 Definition
If (X_1, O_1) and (X_2, O_2) are B–algebraic structures, we call a pair of mappings (φ, Ω) with $\varphi : X_1 \to X_2$ and $\Omega : O_1 \to O_2$ to be a free generalized homomorphism *iff*

$$\forall o \in O_1, \underline{x} \in X_1^B : \ \varphi\,(\ o(\underline{x})\ ) \ = \ \Omega(o)\,(\ \varphi(\underline{x})\ ) \ .$$

It's easy to check, that, for instance, the only "unusual" free generalized homomorphism between rings as above, which we possibly get more by this definition, is (φ_0, Ω_0) with the constant map $\varphi_0(x) = 1;\ \forall x \in R_1$ and $\Omega_0 = \{(+_1, *_2), (*_1, *_2)\}$, if R_2 has an multiplicative identity 1.

The "roughly defined" notions above hopefully illustrates the idea to look at algebraic operations as a kind of functionals into a set X from special function spaces over X. But they don't respect carefully the arity of the algebraic operations, because of the automatic B–arity of all operations in a B–algebraic structure. So we are not able to require for a generalized homomorphism to map n–ary operations onto n–ary ones, for example, which we may wish to do sometimes. This motivates the following little more sophisticated explanations.

2.6 Definition
Given sets X, O and a family $(B_o)_{o \in O}$ with $o \in X^{(X^{B_o})}$, $\forall o \in O$, we call the triple $(X, O, (B_o)_{o \in O})$ a $(B_o)_{o \in O}$–algebraic structure. We assume that the elements of the family $(B_o)_{o \in O}$ are cardinals, where we for simplicity denote at the same time by B_o also a set representing the cardinal B_o, such that X^{B_o} is the set of all maps from B_o to X. (If B_o is finite, $B_o = n \in I\!N$, then we write X^n instead of $X^{\{1,\ldots,n\}}$ for instance.)

Here we may choose all B_o equal, so we get back the case of definition 2.4. For abbreviation we will say "algebraic structure" if no trouble seems to be possible.

If $\mathbf{X} = (X, O, (B_o)_{o \in O})$ is an algebraic structure and $A \subseteq X$, then we call the set $opc_O(A) := \bigcap\{M | \ A \subseteq M \subseteq X, \forall o \in O, \underline{m} \in M^{B_o} : o(m) \in M\}$ the *operational closure* of A w.r.t. O.

2.7 Definition
If $(X_1, O_1, (B_o)_{o \in O_1})$ and $(X_2, O_2, (C_p)_{p \in O_2})$ are (B_o)– and (C_p)–algebraic structures, we call a pair (φ, Ω) of mappings $\varphi : X_1 \to X_2$ and $\Omega : O_1 \to O_2$ a generalized homomorphism iff

(a) $(B_o)_{o \in O_1}$ *is a subfamily of* $(C_p)_{p \in O_2}$ *such that* $\forall o \in O_1 : B_o = C_{\Omega(o)}$

(b) $\forall o \in O_1, \underline{x} \in X_1^{B_o} : \ \varphi\left(o(\underline{x})\right) \ = \ \Omega(o)\left(\varphi(\underline{x})\right)$,
where $\varphi(\underline{x}) = \varphi((x_i)_{i \in B_o}) := (\varphi(x_i))_{i \in B_o} \in X_2^{C_{\Omega(o)}} = X_2^{B_o}$.

The family of all generalized homomorphisms from X to Y we denote by $GHom(X, Y)$.

2.8 Remark
If $(X_1, O_1, (B_o)_{o \in O_1})$, $(X_2, O_2, (C_p)_{p \in O_2})$ and $(X_3, O_3, (D_q)_{q \in O_2})$ are algebraic structures, $(\varphi, \Omega) \in GHom(X_1, X_2)$ and $(\psi, \Sigma) \in GHom(X_2, X_3)$ then $(\psi \circ \varphi, \Sigma \circ \Omega) \in GHom(X_1, X_3)$.

3 Definition of an abstract (first) dual space

At first we want to explain, how to define algebraic operations in Y^X if Y carries algebraic operations:

3.9 Definition
For an arbitrary set X and an algebraic structure $\mathbf{Y} = (Y, P, (C_p)_{p \in P})$ we can define operations for Y^X using these in Y pointwise: for $p \in P$ define $p' \in (Y^X)^{\left[(Y^X)^{C_p}\right]}$ by $\forall \underline{f} \in (Y^X)^{C_p}, x \in X : p'(\underline{f})(x) := p(\underline{f}(x))$, where $\underline{f}$ is a vector $\underline{f} = (f_i)_{i \in C_p}$ from $(Y^X)^{C_p}$ and $(\underline{f}(x)) = (f_i(x))_{i \in C_p}$.

3.10 Assumption
From now on, Y has always a topology σ; if Y has no "natural" topology, then let σ be the discrete topology on Y.

3.11 Remark
If Y has a natural topology σ, then we have the advantage that then the pointwise topology τ_p on Y^X is defined. In our considerations τ_p will play an important role. One reason for that is, that for a normed space X the weak topology in X', τ_{w^*}, also called weak star topology coincides with the pointwise topology: $\tau_{w^*} = \tau_p$

3.12 Definition
Let $\mathbf{X} = (X, O, (B_o)_{o\in O})$ *and* $\mathbf{Y} = (Y, P, (C_p)_{p\in P})$ *be algebraic structures and* σ *a topology on* Y *such that the assumptions of definition 2.7 are fulfilled; especially let* $\Omega : O \to P$ *be a fixed map and we identify* (φ, Ω) *with* φ.
Then we define:

$$X^d := \{\varphi \in Y^X \mid (\varphi, \Omega) \in GHom(X, Y)\} \;;$$

if X *has a topology, too, we assume in addition that each* $\varphi \in X^d$ *is continuous. We also provide* X^d *with a topology* ρ *such that* $\tau_p \le \rho$ *holds. The space* (X^d, ρ) *is called* the Y-dual of X or the (generalized) dual space of X with respect to $\Omega, (Y, \sigma)$ and ρ.

3.13 Remark
In general the attempt to define algebraic operations in X^d*, corresponding to them in* X *or* Y*, in a natural way will fail. Here we mean, the "main" natural way would be the pointwise, mentioned in definition 3.9. But in general we can not ensure, that any function we get in this manner as the result of pointwise defined operations from homomorphisms is a homomorphism again. (The pointwise sum of two ring–homomorphisms, for example, is not a ring–homomorphism in most cases.)*

3.14 Examples

(1) Let $(X, ||.||)$ be a normed space; we have $X^d = X' = (X', ||.||), \rho = \tau_{||.||}$; $\tau_p \le \tau_{||.||}$ since τ_p is the weak topology in $\tau_{||.||}$.

(2) Let X, Y be our rings (see the introduction); we assume that Y has no natural topology and hence let σ be the discrete topology for Y. Then τ_p is defined in X^d and we define $\rho := \tau_p$ and get (X^d, τ_p) as a generalized dual space.

(3) Let X, Y be $I\!K$-normed spaces, then $X^d = L(X, Y) = \{h : X \to Y \mid h$ is linear and continuous $\}$ is the "natural "Y-dual of X, $\rho = \tau_{||.||}$, where $||.||$ denotes the operator-norm.

4 The notion of a second dual space of X w.r.t. a space Y

In order to define the second dual space Xdd of X with respect to Y, we must consider a bifurcation: We come back to the definition of $X^d \subseteq Y^X$.
Let be $\mathbf{Y} = (Y, O, (B_o)_{o\in O})$. We know (see definition 3.9): to each $o \in O$ we can assign an operation in Y^X by pointwise definition on X; for instance: $o \equiv +$, $f_1, f_2 \in Y^X$, $f_1 + f_2 : \forall x \in X : (f_1 + f_2)(x) = f_1(x) + f_2(x)$.

4.15 Definition
We say that X^d *has the* defect D *iff* $\exists (o,n,\underline{f}) \in O \times (I\!N \setminus \{0\}) \times (X^d)^n, B_o = n, \underline{f} = (f_1,...,f_k) \in (X^d)^n : o(\underline{f}) \notin X^d$. *We abbreviate the two possible cases:*

- non D*:* X^d *has not the defect D;*
- D*:* X^d *has the defect D.*

4.16 Definition

$$X^{dd} := \begin{cases} \left((X^d,\rho)^d,\mu\right) \subseteq Y^{X^d} & : \text{ if non } D \\ \left(C((X^d,\tau_p),(Y,\sigma)),\mu\right) \subseteq Y^{X^d} & : \text{ if } D \end{cases} ,$$

with $\tau_p \leq \mu$ *in* Y^{X^d}, *is called the* second dual space (bidual) of X w.r.t. Y,σ,ρ,μ.

What do we need?
We must prove several important properties of the dual system

$$(X,Y,X^d,X^{dd},J : X \to X^{dd})$$

in order that our approach becomes workable.

4.1 Introduction of pointwise defined algebraic operations in X^{dd}

4.17 Definition
Now we use again our standard notations: $\mathbf{X} = (X,O,(B_o)_{o\in O})$, $\mathbf{Y} = (Y,P,(C_p)_{p\in P})$. *We have* $X^{dd} \subseteq Y^{X^d}$*; for all* $p \in P$ *we denote the map* $p \to p'$ *as defined in definition 3.9, where*

$$p' : \left(Y^{X^d}\right)^{C_p} \to Y^{X^d} ,$$

by Δ, $\Delta : p \to p'$. *We assume that* P *has no redundant elements and that* Δ *is injective. Let* $Q := \Delta(P), q = \Delta(p), D_q = D_{\Delta(p)} := C_p$. *Hence* $(Y^{X^d},Q,(D_q)_{q\in Q})$ *is an algebraic structure for* Y^{X^d}, *and we can restrict this structure to* X^{dd}.

4.18 Remark
Concerning our maps we get: $\Omega : O \to P$ *and* $\Delta : P \to Q$, *hence* $\Delta \circ \Omega : O \to Q$.

4.2 When does $J(X) \subseteq X^{dd}$ hold?

We have: $J : X \to Y^{X^d} : \forall x \in X : J(x) = \omega(x,.)$; $\omega(x,.) : X^d \to Y : \forall f \in X^d : \omega(x,f) = f(x) \in Y$.
In order to show that $J(X) \subseteq X^{dd}$ holds, we provide a lemma.

4.19 Lemma
We consider the assumptions of definition 3.12 and remark 3.13. According to definition 4.16 we endow X^d in case D *with pointwise topology and in case* non D *with a topology $\rho \geq \tau_p$.*
Then holds for all $x \in X$:

(1) *In both cases $\omega(x, .) : (X^d) \to Y$ is continuous.*

(2) *In case* non D, *$\omega(x, .)$ is a homomorphism from X^d to Y.*

Proof: (1) Let $(f_i)_{i \in I}$ a net in X^d, $f \in X^d$ and $f_i \xrightarrow{\tau_p} f$, then $f_i(x) \to f(x)$ in Y and we have $\omega(x, f_i) = f_i(x) \to f(x) = \omega(x, f)$, hence $\omega(x, f_i) \to f$, so $\omega(x, .) : (X^d, \tau_p) \to Y$ is continuous. Because of $\tau_p \leq \rho$, this holds for $\omega(x, .) : (X^d, \rho) \to Y$, too.
(2) Let $p \in P$ be given, with arity $C_p = k \in I\!N, k \geq 1$, and let $f_1, ..., f_k \in X^d$; since we have case *non D*, p' as defined in 3.9 yields $p'(f_1, ..., f_k) \in X^d$. So we find $\forall x \in X$: $\omega(x, p'(f_1, ..., f_k)) = p'(f_1, ..., f_k)(x) = p(f_1(x), ..., f_k(x)) = p(\omega(x, f_1), ..., p(x, f_k))$. ■

4.20 Corollary
In both cases $J(X) \subseteq X^{dd}$ holds.

4.3 Redefinition of X^d

It would be very useful if (X^d, ρ) and especially (X^d, τ_p) have nice topological properties, for instance, that (X^d, τ_p) is compact. This would enforce, that for all $f \in C\big((X^d, \tau_p), (Y, \sigma)\big)$ the image $f(X^d)$ is compact in Y and hence is bounded if (Y, σ) is metrizable.

But to get τ_p-compactness for $X^d \subseteq Y^X$ often is a difficult problem. Therefore we will allow in some cases to substitute X^d by a suitable subspace $A \subseteq X^d$. But we assume always that $A \neq \emptyset$ and if a zero-element $\underline{0}$ of X^d belongs to A, then even $A \neq \{\underline{0}\}$ holds.

Now using such a subspace, we define $X^d := A$ to be the new Y-dual of X. By the same arguments as above, we find that for $A \subseteq Y^A$ holds either D or *non D*, too.

A simple example: in some cases we must work with $A := X^d \setminus \{\underline{0}\}$ instead of X^d, where $\underline{0}$ is the zero-homomorphism.

4.4 The homomorphy theorem for the map J

We consider the algebraic structures

$$\begin{aligned} \mathbf{X} &= (X, O, (B_o)_{o\in O}) \ , \\ \mathbf{Y} &= (Y, P, (C_p)_{p\in P}) \ , \\ \mathbf{Y}^{\mathbf{X}^{\mathbf{d}}} &= (Y^{X^d}, Q, (D_q)_{q\in Q}) \ , \\ \mathbf{X}^{\mathbf{dd}} &= (X^{dd}, Q, (D_q)_{q\in Q}) \ , \end{aligned}$$

we use fixed mappings $\Omega : O \to P$, $\Delta : P \to Q$, and $\forall o \in O : D_{(\Delta\circ\Omega)(o)} = C_{\Omega(o)} = B_o$.

By corollary 4.20 we have $J(X) \subseteq X^{dd}$. We will still remember, that we use the common notion of equality of functions: let $x_1, x_2 \in X$, then $J(x_1) = J(x_2) \Leftrightarrow \omega(x_1, .) = \omega(x_2, .) \Leftrightarrow \forall h \in X^d : \omega(x_1, .)(h) = \omega(x_2, .)(h)$.

If $\underline{x}$ is a vector from X^{B_o}, we write for brevity $(\omega(x_i, .))_{i\in B_o} = \omega(\underline{x}, .)$ and $(J(x_i))_{i\in B_o} = J(\underline{x})$.

Now we come to our homomorphy theorem for J.

4.21 Theorem

(1) $J : X \to X^{dd}$ is a $(J, \Delta \circ \Omega)$-homomorphism.

(2) If X has a topology τ, then $J : (X, \tau) \to (X^{dd}, \tau_p)$ is continuous.

Proof: (1) Let $o \in O$, $\underline{x} \in X^{B_o}$; we want to show that $J(o(\underline{x})) = (\Delta \circ \Omega(o))(J(\underline{x}))$ holds:
$\forall h \in X^d : J(o(\underline{x}))(h) = \omega(o(\underline{x}), .)(h) = h(o(\underline{x})) = \Omega(o)(h(\underline{x}))$, since by definition 3.12 each $h \in X^d$ is a (h, Ω)-homomorphism. And we have $C_{\Omega(o)} = B_o$. Now we find,

$$\begin{aligned} \Omega(o)(h(\underline{x})) &= \Omega(o)((h(x_i))_{i\in B_o}) = \Omega(o)((\omega(x_i, h))_{i\in B_o}) \\ &= (\Delta(\Omega(o))(\omega(x_i, .)_{i\in D_{\Delta\circ\Omega(o)}}))(h) \\ &= (\Delta(\Omega(o))(\omega(\underline{x}, .)))(h) = ((\Delta \circ \Omega)(o)(J(\underline{x}))(h) \\ &= (((\Delta \circ \Omega)(o))(J(\underline{x})))(h) \ , \end{aligned}$$

implying $J(o(\underline{x}))(h) = ((\Delta \circ \Omega)(o)J(\underline{x}))(h)$, yielding $J(o(\underline{x})) = (\Delta \circ \Omega(o))(J(\underline{x}))$. This holds in Y^{X^d}, but $o(\underline{x}) \in X$ implies $J(o(\underline{x})) \in X^{dd}$, thus $((\Delta \circ \Omega)(o))(J(\underline{x})) \in X^{dd}$, and hence $J(o(\underline{x})) = (\Delta \circ \Omega(o))(J(\underline{x}))$ in X^{dd}.

(2) If X has a topology, too, by definition 3.12 we find $X^d \subseteq C((X, \tau), (Y, \sigma))$. Now, let $(x_i)_{i\in I}$ be a net in X, $x \in X$ and $x_i \xrightarrow{\tau} x$. Since $J(X) \subseteq X^{dd}$ holds, we have $J(x) \in X^{dd}$ and $\forall i \in I : J(x_i) \in X^{dd}$; $J(x_i) = \omega(x_i, .)$, $J(x) = \omega(x, .)$. We have $\forall h \in X^d : \omega(x_i, .)(h) = \omega(x_i, h) = h(x_i) \to h(x) = \omega(x, h) = \omega(x, .)(h)$ in Y because of the continuity of each h here. Thus, $(J(x_i))_{i\in I}$ converges pointwise to $J(x)$.■

4.5 Algebraic properties of spaces of continuous functions which can serve as second dual spaces

At first we consider rings X, X, where Y is a topological ring. We easily find:

4.22 Proposition
Let X be a topological space and Y a topological ring; if we define addition and multiplication in $C(X,Y)$ pointwise, then $C(X,Y)$ becomes a ring, too. If Y is commutative, then so is $C(X,Y)$. If Y has a unit 1, then the constant function $\underline{1}(x) \equiv 1$ is a unit in $C(X,Y)$.

4.23 Corollary
Let (Y,σ) be a topological commutative ring with unit. Then $C((X^d,\tau_p),(Y,\sigma))$ with pointwise topology and operations is a commutative ring with unit.

4.24 Proposition
Let X be a completely regular topological space; let Y be

(1) an algebra

(2) a normed algebra

(3) a Banach algebra

(4) a C^*-algebra;

let in cases (2), (3), (4) $(C^*(X,Y), ||.||_{sup})$ denote the set of bounded continous functions with the supremum-norm (yielding the Tchebycheff-metric). For cases (1), (2), (3) we consider the scalar field $I\!K$, for case (4) we use $\mathbb{C}$. Then with pointwise defined algebraic operations in Y^X and $C^*(X,Y)$, respectively, we get:

(1) Y^X is an algebra.

(2) $(C^*(X,Y), ||.||_{sup})$ is a normed algebra.

(3) $(C^*(X,Y), ||.||_{sup})$ is a Banach-algebra.

(4) $(C^*(X,Y), ||.||_{sup})$ is a C^*-algebra.

Proof: (1) Clearly Y^X is a $I\!K$-vector space and Y^X is a ring, where the additive group is the same for the vector space as for the ring. We still show: $\forall f,g \in Y^X, \alpha \in I\!K : \alpha(fg) = (\alpha f)g = f(\alpha g)$, by simply calculating $\forall x \in X : \alpha(fg)(x) = \alpha(f(x)g(x)) = (\alpha f(x))g(x) = ((\alpha f)g)(x)$ and similar $\forall x \in X : \alpha(fg)(x) = \alpha(f(x)g(x)) = f(x)(\alpha g(x)) = f(\alpha g)(x)$.

(2) Since the algebraic operations in Y are continuous, we find: $f,g \in C^*(X,Y) \Rightarrow f+g, \alpha f, fg \in C(X,Y)$ and we still must show, that these functions are bounded,

too. $f,g \in C^*(X,Y) \Rightarrow \exists c_1,c_2 \in I\!R, c_1 > 0, c_2 > 0 : \forall x \in X : ||f(x)|| \leq c_1, ||g(x)|| \leq c_2$, immediately implying $||f(x)+g(x)|| \leq ||f(x)|| + ||g(x)|| \leq c_1 + c_2$, thus $f+g \in C^*(X,Y)$, and analogously for $\alpha \in I\!K$, $\alpha f(x) \leq |\alpha| \cdot ||f(x)|| \leq |\alpha| c_1$ as well as $||f(x)g(x)|| \leq ||f(x)|| \cdot ||g(x)|| \leq c_1 c_2$.
As is well known, $||.||_{sup} : ||f||_{sup} := \sup_{x\in X} ||f(x)||$ is a norm in $C^*(X,Y)$; finally: $\forall f,g \in C^*(X,Y) : ||fg||_{sup} = \sup_{x\in X} ||f(x)g(x)|| \leq \sup_{x\in X}(||f(x)|| \cdot ||g(x)||) \leq (\sup_{x\in X} ||f(x)||) \cdot (\sup_{x\in X} ||g(x)||) = ||f||_{sup} \cdot ||.g||_{sup}$, since $\forall z \in X : 0 \leq ||f(z)|| \leq \sup_{x\in X} ||f(x)||, 0 \leq ||g(z)|| \leq \sup_{x\in X} ||g(x)|| \Rightarrow 0 \leq ||f(z)|| \cdot ||g(z)|| \leq (\sup_{x\in X} ||f(x)||) \cdot (\sup_{x\in X} ||g(x)||) \Rightarrow \sup_{z\in X}(||f(z)|| \cdot ||g(z)||) \leq (\sup_{x\in X} ||f(x)||) \cdot (\sup_{x\in X} ||g(x)||)$.

(3) Having proven (2), we only must show, that $(C^*(X,Y), ||.||_{sup})$ is complete, whenever Y is: Let (f_n) be a Cauchy sequence in $C^*(X,Y)$; $\forall x \in X, k,l \in I\!N : ||f_k(x) - f_l(x)|| \leq ||f_k - f_l||_{sup} \Rightarrow (f_n(x))$ is Cauchy in Y, thus $(f_n(x))$ converges to an uniquely determined element $y_x \in Y$, yielding a function $f : X \to Y : f(x) := y_x$ with $(f_n(x)) \to f(x)$ for all $x \in X$. Now, $\forall \varepsilon > 0 : \exists n_\varepsilon \in I\!N : \forall k,l \geq n_\varepsilon : ||f_k - f_l||_{sup} < \varepsilon$ $\Rightarrow \forall x \in X : \forall m \geq n_\varepsilon : ||f_m(x) - f_{n_\varepsilon}|| < \varepsilon$, so from $f_m(x) \to f(x)$ just follows $f_m(x) - f_{n_\varepsilon}(x) \to f(x) - f_{n_\varepsilon}(x)$, implying $||f_m(x) - f_{n_\varepsilon}(x)|| \to ||f(x) - f_{n_\varepsilon}(x)||$ and hence $||f(x) - f_{n_\varepsilon}(x)|| \leq \varepsilon$; now $||f(x)|| - ||f_{n_\varepsilon}(x)|| \leq \varepsilon$, thus $||f(x)|| \leq \varepsilon + ||f_{n_\varepsilon}(x)|| \leq \varepsilon + ||f_{n_\varepsilon}||_{sup}$, showing that f is bounded.
Furthermore, we have $\forall x \in X : \forall n \geq n_\varepsilon : ||f_n(x) - f(x)|| \leq ||f_n(x) - f_{n_\varepsilon}(x)|| + ||f_{n_\varepsilon}(x) - f(x)|| < 2\varepsilon$; hence the sequence (f_n) converges uniformly to f in the space of all bounded functions from X to Y, showing, that f is continuous. Thus $(C^*(X,Y), ||.||_{sup})$ is complete.

(4) Let $y \to \overline{y}$ denote the involution in Y; then we define: $\forall f \in C^*(X,Y) : \overline{f} : X \to Y : \overline{f}(x) := \overline{f(x)}$; of course $f \to \overline{f}$ fulfills all general properties of an involution, because $y \to \overline{y}$ does. Finally: $||\overline{f}f||_{sup} = \sup_{x\in X} ||\overline{f}f(x)|| = \sup_{x\in X} ||\overline{f}(x)f(x)|| = \sup_{x\in X}(||f(x)||^2) = (\sup_{x\in X} ||f(x)||)^2 = ||f||^2_{sup}$. ■

4.6 Properties of the canonical map J for normed spaces

4.25 Proposition
Let $(X, ||.||)$ be a normed space, $X^d = (X', ||.||)$ the dual space; let $\emptyset \neq A \subseteq X^d, A \neq \{\underline{0}\}$, when $\underline{0}$ is the zero-map. Let hold for A:

(a) $\forall h \in A : ||h|| \leq 1$ *and*

(b) $J(X) \subseteq (C^*((A,\tau_p), I\!K), ||.||_{sup})$.

Then

(1) $\forall x \in X : ||J(x)||_{sup} \leq ||x||$.

(2) *J is uniformly continuous and hence continuous.*

(3) *If in addition for A holds $\forall x \in X, x \neq 0 : \exists h \in A : ||x|| \leq |h(x)|$ then $||x|| \leq ||J(x)||_{sup}$ holds, too.*

(4) *$J(X)$ separates the points of A.*

Proof: (1) $||J(x)||_{sup} = \sup_{h \in A} |(J(x))(h)| = \sup_{h \in A} |\omega(x,h)| = \sup_{h \in A} h(x)| \leq \sup_{h \in A}(||h|| \cdot ||x||) \leq \sup_{h \in A} ||x||$, since $||h|| \leq 1$.

(2) By (1) we find $\forall x, y \in X : ||J(x) - J(y)|| = ||J(x-y)|| \leq ||x-y||$.

(3) By assumption, $\forall x \in X, x \neq 0 : \exists h \in A : ||x|| \leq |h(x)|$, hence $||x|| \leq |h(x)| \leq \sup_{g \in A} |g(x)| = ||J(x)||_{sup}$.

(4) Let $f, g \in A$, $f \neq g \Rightarrow \exists x \in X : f(x) \neq g(x)$; $\omega(x,.) \in J(X)$ and $\omega(x,.)(f) \neq \omega(x,.)(g)$. ∎

4.7 Features of the zero-homomorphism $\underline{0} \in X^d$

In some cases when a zero-homomorphism $\underline{0} \in X^d$ is defined, this homomorphism can have special properties which we must observe. We formulate these facts in two lemmas.

(1) We consider the dual system $(X, Y, X^d, X^{dd}, J : X \to X^{dd})$ as defined in 3.12, 4.16 and 4.17. We assume that within the algebraic structures of X, Y and X^{dd} respectively are included ring structures $(+, *)$ and the ring structure for X corresponds to the ring structure for X^{dd} via the ring structure for Y. Let these ring structures be commutative with units. If 0_Y is the $+$-unit in Y, the zero-homomorphism $\underline{0} \in X^d$ is defined: $\forall x \in X : \underline{0}(x) = 0_Y$.; by 1_Y we mean the $*$-unit and we assume $0_Y \neq 1_Y$.

4.26 Lemma
Let $e, 1$ be the units in X and X^{dd} respectively. By the homomorphy theorem 4.21 holds $J(e) = 1$. Then this equality does not hold on X^d, but only on $X^d \setminus \{\underline{0}\}$.

Proof: $J(e) = 1$ means $\forall h \in X^d : (J(e))(h) = 1(h)$, hence $\omega(e,h) = h(e) = 1_Y$, but for $h = \underline{0}$ we have $(J(e))(\underline{0}) = \omega(e, \underline{0}) = \underline{0}(e) = 0_Y$ by definition of the zero-homomorphism. ∎

4.27 Remark
In such situations we (have to) consider $X^d \setminus \{\underline{0}\}$ instead of X^d as the first dual space of X w.r.t. Y.

(2) We consider now an important property of the pointwise topology τ_p.

4.28 Lemma
(On τ_p-isolated points)
Let X be a set containing at least two distinct elements a, b; let (Y, σ) be a Hausdorff topological space containing at least the elements y_0, y_1 with $y_0 \neq y_1$; let $H \subseteq Y^X$ with cardinality at least 2; let h_0 denote the constant map $h_0(x) \equiv y_0$. Now let H have the properties

(a) $h_0 \in H$,

(b) $h \in H$ *and* $h \neq h_0$ *implies* $h(b) = y_1$.

Then h_0 is an isolated point of H in (Y^X, τ_p).

Proof: In Y we find open sets U, V such that $y_0 \in U$, $y_1 \in V$ and $U \cap V = \emptyset$; now, we have $(\{b\}, U) \in \tau_p$ and $h_0 \in (\{b\}, U)$; we find $(\{b\}, U) \cap H = \{h_0\}$. So, h_0 is a τ_p-isolated point of H. ■

We still need a simple topological fact.

4.29 Lemma
Let (X, τ) be a Hausdorff topological space; B a compact subset of X, $B \neq X$ and $a \in B$. Then are equivalent:

(a) $B \setminus \{a\}$ *is compact.*

(b) a *is an isolated point of* B.

Now, we want to give a simple example for an application of the lemma on isolated points.

4.30 Proposition
Let $(X, +, *, e)$ be a non-trivial ring with unit e, $F_2 = \{0, 1\}$ the two-element ring (field) and we provide F_2 with the discrete topology; we have $X^d = \{h : X \to F_2 \mid h \text{ is a ring homomorphism } \}$; let $H \subseteq X^d$ contain at least two elemets; h_0 denotes the zero-homomorphism, i.e. $\forall x \in X : h_0(x) = 0 \in F_2$; now let H fulfill the properties

(a) $h_0 \in H$

(b) $\forall h \in H, h \neq h_0 : h(e) = 1 \in F_2$.

Then h_0 is an isolated point of H in (F_2^X, τ_p).

4.8 When is the map J injective?

We remind of the definition of the evaluation map for sets X, Y $\omega : X \times Y^X \to Y$: $\omega(x, f) := f(x)$; let $A \subseteq Y^X, A \neq \emptyset$ and we restrict ω to $X \times A : X \times A \to Y, J : X \to Y$; then holds:

4.31 Proposition
J is injective if and only if A separates the points of X.

Proof: Let J be injective. $\forall x, y \in X, x \neq y \Leftrightarrow J(x) = \omega(x, .) \neq \omega(y, .) = J(y) \Leftrightarrow \exists f \in A : \omega(x, f) \neq \omega(y, f) \Leftrightarrow f(x) \neq f(y)$, thus A separates the points of X. ■

4.9 Algebraic closedness of X^d in (Y^X, τ_p)

We consider X^d as is defined in 3.12; since by assumtion 3.10 Y has a topology σ, we find with $(X^d)_{alg} := \{\varphi \in Y^X | \ (\varphi, \Omega) \in GHom(X, Y)\}$ w.r.t. a fixed Ω: if X has no topology, then $X^d = (X^d)_{alg}$; otherwise by definition 3.12 $X^d = (X^d)_{alg} \cap C(X, Y)$.

4.32 Proposition
We assume (Y, σ) to be Hausdorff and all algebraic operations in Y being continuous w.r.t. σ. Then $(X^d)_{alg}$ is τ_p-closed in Y^X.

Remark: Compare proposition 3.1 of the paper [4], where the (uncomplicated) proof can be found.

5 Some examples and applications

At first we want to answer our questions.

(1) Let X, Y be commutative rings, either both with units or both without. We assume that X^d has D in the sense of definition 4.15. In most concrete cases this will happen. We set $\rho = \tau_p$, σ discrete topology on Y, $\mu = \tau_p$; then by definition 4.16: $X^{dd} = \big(C((X^d, \tau_p), (Y, \sigma)), \ \tau_p \big)$. If X and Y have units and $\underline{0} \in X^d$ is the zero-homomorphism, we use as (new) dual space $X^d \setminus \{\underline{0}\}$ (see 4.26 and 4.27); then

$$X^{dd} = \Big(\ C((X^d \setminus \{\underline{0}\}, \tau_p), (Y, \sigma)), \ \tau_p \Big) \ .$$

And this is the answer to question 1.
By the homomorphy theorem 4.21 we get

(a) $J : X \to X^{dd}$ is a homomorphism.

(b) If X has a topology τ, $J : (X, \tau) \to (X^{dd}, \tau_p)$ is continuous.

(2) Let $(X, ||.||)$ be a (non-trivial) normed space, $B' \subseteq X'$ the (norm-)closed unit-ball in X'; we use X^d $(= A)$ $= B'$ as dual space of X (see 4.3, redefinition of X^d); of course, since $(X, ||.||)$ is non-trivial, $\emptyset \neq B' \neq \{\underline{0}\}$; B' is no vector-subspace of X'; hence by definition 4.16 $X^{dd} = C((B', \tau_p), (I\!\!K, \tau_{|.|}))$ $= \big(C_b((B', \tau_p), (I\!\!K, \tau_{|.|})),\ \tau_{||.||_{sup}} \big)$, since by Alaoglu's theorem (B', τ_p) is compact. By the homomorphy theorem and by proposition 4.25 we get:

(a) $J : X \to X^{dd}$ is a homomorphism, i.e. a linear map.

(b) $\forall x \in X : ||J(x)||_{sup} = ||x||$; finally: J is a linear isometry from X to the subspace $J(X) \subseteq \big(C_b((B', \tau_p), (I\!\!K, \tau_{|.|})),\ \tau_{||.||_{sup}} \big)$. If X is a Banach space, then $J(X)$ is closed in $\big(C_b((B', \tau_p), (I\!\!K, \tau_{|.|})),\ \tau_{||.||_{sup}} \big)$.

Hence the answer to question 2 is simple: the representation of $(X, ||.||)$ by a space of continuous functions follows informally from our general duality approach.

(3) Within our approach we want to prove the theorem of M.H. Stone on representation of Boolean rings. At first we will recall some definitions and facts.

5.33 Definition
A ring $(X, +, \cdot)$ is called a Boolean *ring, iff $\forall x \in X : x^2 = x$ holds.*

In a Boolean ring $(X, +, \cdot)$ we have

- $\forall x \in X : x + x = 0$
- X is commutative.

In proposition 4.30 we mentioned the smallest (non-degenerated, commutative) ring with unit $F_2 = \{0, 1\}$, which is just a field - and a Boolean ring, too. Conversely, we find:

5.34 Proposition
Each Boolean ring with unit, which is a field, is isomorphic to F_2.

We still need the following facts:

5.35 Proposition
Let X be a Boolean ring with unit e; let I be an ideal of X; then hold:

(a) The factor ring X/I is a Boolean ring.

(b) If M is a maximal ideal of X, then X/M is isomorphic to F_2.

The first dual space X^d of a Boolean ring:

We set $Y = F_2$ and hence by definition 3.12 $X^d = \{h : X \to F_2 |\ h$ is a ring-homomorphism $\}$. If X has a unit e, by simple properties of homomorphisms we get $X^d \setminus \{\underline{0}\} = \{h \in X^d |\ h(e) = 1\}$, where $\underline{0}$ denotes the zero-homomorphism.

Now, according to 3.10, we consider for F_2 the discrete topology and since $X^d \subseteq F_2^X$, for X^d the pointwise topology is defined. But then proposition 4.30 yield at once:

5.36 Proposition
The zero-homomorphism $\underline{0} \in X^d$ is an isolated point of X^d in (F_2^X, τ_p).

Now we can prove the essential properties of (X^d, τ_p) and $(X^d \setminus \{\underline{0}\}, \tau_p)$, respectively.

5.37 Theorem
Let $(X, +, \cdot)$ be a Boolean ring with unit e; then both (X^d, τ_p) and $(X^d \setminus \{\underline{0}\}, \tau_p))$ are Hausdorff, compact and totally disconnected.

Proof: Hausdorffness, compactness and totally disconnectedness are stable under formation of arbitrary products, F_2 with discrete topology has all these properties, and so (F_2^X, τ_p) does. Hausdorffness and totally disconnectedness are stable under formation of subspaces, so X^d and $X^d \setminus \{\underline{0}\}$ are Hausdorff and totally disconnected. X^d is compact as a closed subspace of the compact space F_2^X (4.32), and $X^d \setminus \{\underline{0}\}$ is compact because $\underline{0}$ is an isolated point of X^d.∎

By the following we show, that X admits non-trivial homomorphisms.

5.38 Lemma
Let $(X, +, \cdot)$ be a Boolean ring with unit e, $X \neq \{0\}$; then X^d separates the points of X.

Proof: Since X is a ring, it is enough to show that $\forall 0 \neq x \in X : \exists h \in X^d : h(x) = 1$. So, let an arbitrary $0 \neq x \in X$ be given. Then $I := \{(x + e) \cdot z \mid z \in X\}$ is an ideal in X; we have $(x + e) = e \cdot (x + e) \in I$ and hence $\emptyset \neq I$. If $e \in I$ we find $z \in X$ such that $e = z(x + e)$, implying $e = z(x + e) = z(x + e)(x + e) = e(x + e) = x + e$, thus $x = 0$, a contradiction. $e \notin I$ implies $I \neq X$, and hence I is contained in a maximal Ideal M; let $h : X \to X/M = F_2$ be the canonical homomorphism, then M is the kernel of h; now, $e(x + e) = x + e \in I$ implies $x + e \in M$ and hence $h(e) + h(x) = h(x + e) = 0$, thus $h(x) = -h(e) = h(e) = 1$. ∎

Finally we want to state some special properties of the pointwise topology τ_p for X^d. These properties are induced by the fact, that F_2 is considered with discrete topology. Let X be a set, (Y, σ) a topological space and Y^X (as usual) the space of all maps from X to Y; for $A \subseteq X$, $B \subseteq Y$, we define $(A, B) := \{f \in Y^X | \ f(A) \subseteq B\}$; then $\{ \ (\{x\}, H) | \ x \in X, H \in \sigma\}$ is an open subbase for τ_p on Y^X. Now let us come back to the situation that X is a Boolean ring with unit e and $Y = F_2$ with discrete topology. Then nontrivial open sets of F_2 are $\{0\}$, $\{1\}$ and hence τ_p has as a subbase the family of all $(\{x\}, \{0\})$ and $(\{x\}, \{1\})$ and we write for brevity $(x, 0)$ and $(x, 1)$, respectively, for them.

5.39 Lemma

(a) $X^d \setminus (x, 1) = (x, 0)$ *in* F_2^X

(b) $(x, 0) = (x + e, 1)$ *in* X^d

(c) $(x, 1) \cap (y, 1) = (xy, 1)$ *in* X^d

5.40 Corollary

The family $\{(x, 1) | \ x \in X\}$ *is a base for* τ_p *in* X^d.

Proof: Let $G \subseteq X^d$ be τ_p-open and let $h \in G$; we find subbase elements $S_1, ..., S_n$ such that $h \in \bigcap_{i=1}^n S_i \subseteq G$, where $S_i = (x_i, 1)$ or $S_j = (y_j, 0)$; but by the foregoing lemma we have $(y_j, 0) = (y_j + e, 1)$, so we denote these $y_j + e$ by x_j and finally we get from the third part of the lemma $\bigcap_{i=1}^n S_i = (x_1 x_2 ... x_n, 1)$.■

5.41 Lemma

Let A *be open and closed in* X^d $(= \{h : X \to F_2 | \ h$ *is a homomorphism and* $h \neq \underline{0}\})$*; then there exists a point* $x_A \in X$ *such that* $A = (x_A, 1)$.

Proof: $X^d \setminus A$ is τ_p-open in X^d and hence $\forall h \in X^d \setminus A : \exists x_h \in X : h \in (x_h, 1) \subseteq X^d \setminus A$ by 5.40, implying $X^d \setminus A = \bigcup_{h \in X^d \setminus A} (x_h, 1)$; $X^d \setminus A$ is closed, too, and hence it is compact, thus $X^d \setminus A = \bigcup_{i=1}^n (x_{h_i}, 1)$ with suitable x_{h_i}; now we have $A = \bigcap_{i=1}^n X^d \setminus (x_{h_i}, 1) = \bigcap_{i=1}^n (x_{h_i}, 0) = \bigcap_{i=1}^n (x_{h_i} + e, 1) = (x_A, 1)$ where $x_A := (x_{h_1} + e) \cdots (x_{h_n} + e)$. ■

The second dual space X^{dd} *of a Boolean ring:*

If X is a (non-trivial) Boolean ring and if $f, g \in X^d$ are given, if we define addition and multiplication in X^d pointwise, then in general $f + g$ is not a multiplicative homomorphism between the rings X and F_2. Therefore by definition 4.16 we let be $X^{dd} = C((X^d, \tau_p), F_2)$, and it is not hard to see that $C((X^d, \tau_p), F_2)$ is a Boolean ring with unit, too. Now, we can prove the representation theorem of M.H. Stone for Boolean rings.

5.42 Theorem
Let X be a (non-trivial) Boolean ring with unit and let be $X^d = \{h : X \to F_2 |\ h$ is a homomorphism and $h \neq \underline{0}\}$. $J : X \to X^{dd} = C((X^d, \tau_p), F_2)$ is an isomorphic map from X onto X^{dd}, and (X^d, τ_p) is a Hausdorff, totally disconnected and compact space.

Proof: The properties of (X^d, τ_p) we proved in 5.37. By 4.20 $J(X) \subseteq X^{dd}$ holds and then by theorem 4.21 J is a homomorphism from X into X^{dd}. By lemma 5.38 and proposition 4.31 we easily find that J is injective, thus J is an isomorphism. It remains only to show, that J is surjective: it is evident, that the space of continuous maps $C((X^d, \tau_p), F_2)$ consists of all characteristic functions $\chi_A : X^d \to F_2$ where $\emptyset \neq A \subseteq X^d$ is open and closed (w.r.t. τ_p); now let be $\chi_A \in X^{dd}$; by lemma 5.41 we find a $x_A \in X$ such that $A = (x_A, 1)$, implying $J(x_A) = \omega(x_A, .) = \chi_A : \forall h \in A : \omega(x_A, h) = h(x_A) = 1 = \chi_A(h)$; $\forall h \in X^d \setminus A : \omega(x_A, h) = h(x_A) = 0 = \chi_A(h)$. ■

(4) Finally we consider commutative Banach-algebras (with units).

We assume here, that the elementary theory of commutative Banach-algebras is known.

Let X be a commutative $\mathbb{C}$-Banach algebra with unit e. By definition 3.12 we get $X^d = \{h : X \to \mathbb{C} \mid h$ is linear, continuous, multiplicative $\}$; $X^d \subseteq L(X, \mathbb{C})$ and for X^d we consider the operator norm. By the Banach algebra theory we find:

5.43 Proposition
(a) *X^d has a zero-homomorphism $\underline{0}$ and $||\underline{0}|| = 0$.*

(b) *$\{h \in X^d |\ ||h|| \leq 1\} = X^d$, since $||\underline{0}|| = 0$ and $h \neq \underline{0}$ implies $||h|| = 1$.*

(c) *$\{h \in X^d |\ h(e) = 1\} = \{h \in X^d |\ h \neq \underline{0}\} = X^d \setminus \{\underline{0}\}$.*

(d) *X^d is no vector subspace of $\mathbb{C}^X$, since, for instance, if $h \in X^d$, $h \neq \underline{0}$, then $||2h|| = 2||h|| = 2 \neq 1$. Using definition 4.15 we find that X^d has defect D. Hence by definition 4.16 we get for the second dual space of the Banach algebra X: $X^{dd} = \big(C((X^d, \tau_p), (\mathbb{C}, \tau_{|.|})), \mu \big)$.*

5.44 Proposition
(a) *(X^d, τ_p) is a Hausdorff compact topological space.*

(b) *The zero-homomorphism $\underline{0} \in X^d$ is an isolated point of X^d in $(\mathbb{C}^X, \tau_p)$.*

(c) *$X^d \setminus \{\underline{0}\}$ is a Hausdorff compact space, too.*

Proof:

(a) Let $X' := \{h \in \mathbb{C}^X | \; h$ is linear and continuous$\}$ the dual space of the Banach space $(X, ||.||)$ and $B_{X'}$ the (norm-)closed unit ball in X'. Then by the Alaoglu theorem $B_{X'}$ is τ_p-compact in X' and hence $B_{X'}$ is compact in $(\mathbb{C}^X, \tau_p)$, too; $(\mathbb{C}^X, \tau_p)$ is Hausdorff since $\mathbb{C}$ with Euclidian $\tau_{|.|}$ is; thus $B_{X'}$ is closed in $(\mathbb{C}^X, \tau_p)$. Let $A_{rm} = \{h \in \mathbb{C}^X | \; h$ is ring multiplicative$\}$. Then $X^d = B_{X'} \cap A_{rm}$.($X^d \subseteq B_{X'}$ and $X^d \subseteq A_{rm}$; otherwise let $h \in B_{X'} \cap A_{rm}$, then $h \in \mathbb{C}^X$ and $||h|| \leq 1$, h is linear, continuous, multiplicative, hence $h \in X^d$.) Now, A_{rm} is closed in $(\mathbb{C}^X, \tau_p)$ by proposition 4.32 since of course the multiplication in $\mathbb{C}$ is continuous. Thus $B_{X'} \cap A_{rm} = X^d$ is closed in $(\mathbb{C}^X, \tau_p)$; X^d is also τ_p-closed in $B_{X'}$ yielding that (X^d, τ_p) is compact; but (X^d, τ_p) is Hausdorff, too.

(b) $\mathbb{C}$ has the units 0, 1 and $0 \neq 1$; hence we can apply the lemma 4.28 on τ_p-isolated points.

(c) This follows from lemma 4.29. ■

5.45 Remark
We now use as (new) dual space of X the space $X^d \setminus \{\underline{0}\}$ and denote it again by X^d. Hence $X^d = \{h \in \mathbb{C}^X | \; h \neq \underline{0}, h$ linear, continuous and multiplicative$\}$. And (X^d, τ_p) is Hausdorff and compact. Here, (X^d, τ_p) is also called Gelfand-space of X.

Since (X^d, τ_p) is compact and by proposition 4.24 we obtain

5.46 Corollary

(a) $\big(C((X^d, \tau_p), (\mathbb{C}, \tau_{|.|})), \mu \big) = \big(C^*((X^d, \tau_p), (\mathbb{C}, \tau_{|.|})), \tau_{||.||_{sup}} \big)$

(b) $\big(C^*((X^d, \tau_p), (\mathbb{C}, \tau_{|.|})), \tau_{||.||_{sup}} \big)$ *is a commutative Banach algebra with unit.*

By 4.20, the homomorphy theorem 4.21, propositions 4.25, 5.44 and corollary 5.46 we get the representation theorem of Gelfand.

5.47 Theorem
Let X be a (nontrivial) commutative Banach algebra with unit. Then hold

(a) (X^d, τ_p) *is a compact Hausdorff space.*

(b)

$$J : X \to X^{dd} = \big(C^*((X^d, \tau_p), (\mathbb{C}, \tau_{|.|}), \tau_{||.||_{sup}}\big)$$

is an algebra homomorphism.

(c) $\forall x \in X : ||J(x)||_{sup} \leq ||x||$.

(d) *J is uniformly continuous and hence continuous.*

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