Representation of non-commutative C^*-algebras by spaces of continuous mappings

H. Poppe, R. Bartsch

1 Introduction

By the theorems of Gelfand and Gelfand-Naimark we know that we can represent commutative Banach algebras and commutative C^*-algebras respectively by spaces of continuous functions. For non-commutative C^*-algebras we have the GNS-construction.

In our paper we will show that under mild assumptions a non-commutative C^*-algebra can be represented by a space of continuous mappings. This can be done just in the same way as in the commutative case.

For the presentation and proofs of the results we will use definitions and results from our paper [4].

In part 2 of the present paper we consider the C^*-algebra $M_n(\mathbb{C})$ of complex $n \times n$-matrices. Finally, in part 3 we treat general C^*-algebras.

Without proof we state two important results concerning homomorphisms between C^*-algebras (see [2]). The assertion holds for arbitrary C^*-algebras and can be proved without commutativity arguments.
If X, Y are C^*-algebras with units, $h : X \to Y$ is called an algebra homomorphism iff h is linear, multiplicative and involutary; h is also called a $*$-homomorphism since often the involution is denoted by $x \to x^*$.

1.1 Proposition
Let X, Y be C^*-algebras with units; let $h : X \to Y$ be an algebra-homomorphism. Then holds:

(a) $\forall x \in X : ||h(x)|| \leq ||x||$.
Hence h is continuous and $||h|| := \sup_{||x|| \leq 1} ||h(x)|| \leq 1$.

(b) *The range $h(X) := \{h(x) |\ x \in X\}$ of h is a C^*-subalgebra of Y; that means especially that $h(X)$ is closed in Y w.r.t. the norm-topology $\tau_{||.||}$ in Y.*

1.2 Corollary
With the assumptions of proposition 1.1 we get the equivalence of the following:

(a) *h is injective.*

(b) *$ker(h) = \{0\}$.*

(c) *$\forall x \in X : ||h(x)|| = ||x||$, hence h is an isometric isomorphism from X onto $h(X) \subseteq Y$.*

Proof: (a)$\Leftrightarrow$(b) and (c)$\Rightarrow$(a) are obvious; we show (a)$\Rightarrow$(c): by 1.1(b) we know that $h(X)$ is a C^*-subalgebra of Y; since h is injective, h^{-1} exists uniquely and $\forall x \in X : h^{-1}(h(x)) = x$, hence h^{-1} maps $h(X)$ onto X; h^{-1} is an algebra homomorphism. (Linearity and multiplicativity are trivial by injectivity and the homomorphy of h, so we show only that $h^{-1}(h(x)^*) = (h^{-1}(h(x)))^*$ holds: $h^{-1}(h(x)^*) = h^{-1}(h(x^*)) = x^* = (h^{-1}(h(x)))^*$). Now by 1.1(a) we find $\forall x \in X : ||x|| = ||h^{-1}(h(x))|| \leq ||h(x)|| \leq ||x|| \Rightarrow ||x|| \leq ||h(x)|| \leq ||x||$. ■

Now we generalize proposition 4.3. of [4] assuming that for the space Y not only $Y \in \{\mathbb{R}, \mathbb{C}\}$ is allowed, but that Y is an arbitrary normed space.
Let X, Y be $\mathbb{K}$-normed spaces and let in addition exist finitely many algebraic operations in X and in Y respectively, such that X, Y belong to the same class of such spaces. We assume that we can assign to each algebraic operation in X an algebraic operation in Y (in a natural manner). According to definition 2.1. of [5] we define the dual space X^d of X w.r.t. Y:

1.3 Definition
$X^d := \{h : X \to Y |$ *h is linear, continuous and a homomorphism w.r.t. each pair of corresponding algebraic operations in* $X, Y\}$.

Remark: In the paper [4] one finds precise definitions of an abstract dual space X^d (definition 3.2.) and an abstract second dual space X^{dd} (definition 4.2.), and also the properties of the canonical map $J : X \to X^{dd}$, where $\forall x \in X : Jx = \omega(x, .)$, $\omega(x, .) : X^d \to Y : \forall h \in X^d : \omega(x, .)(h) = \omega(x, h) = h(x)$, where ω is just the evaluation map.

1.4 Proposition
Let X, Y be $\mathbb{C}$-Banach algebras with units. Let $L(X, Y)$ be the set of all linear and continuous maps from X to Y; then $X^d \subseteq L(X, Y)$ and for $L(X, Y)$ we consider the operator norm: $||h|| := \sup_{||x|| \leq 1} ||h(x)||$ and we restrict this norm to X^d. Let $A \subseteq X^d$ with $A \neq \emptyset$ and $A \neq \{\underline{0}\}$ be given, where $\underline{0}$ here denotes the zero-homomorphism in X^d. Now let hold:

(a) $\forall h \in A : ||h|| \leq 1$

(b) $J(X) \subseteq (C_b((A,\tau_p),Y),||.||_{\sup})$.
Here are: τ_p the pointwise topology, $C_b((A,\tau_p),Y)$ the space of bounded functions from A to the metric space Y; $||.||_{\sup}$ is the supremum-norm (Tchebycheff-norm).

Then hold:

(c) $\forall x \in X : ||Jx||_{\sup} \leq ||x||$,

(d) J is uniformly continuous and hence continuous, and

(e) J separates the points of A.

Proof: (c) $||Jx||_{\sup} = \sup_{h\in A} ||(Jx)(h)|| = \sup_{h\in A} ||\omega x, h|| = \sup_{h\in A} ||h(x)|| \leq \sup_{h\in A}(||h|| \cdot ||x||) \leq \sup_{h\in A} ||x|| = ||x||$.

(d) From theorem 4.1. of [4] we know that J is linear; hence by (a) we get $\forall x, y \in X : ||Jx - Jy|| = ||J(x-y)|| \leq ||x-y||$.

(e) Let $f, g \in A$ with $f \neq g$, i.e. $\exists x_0 \in X : f(x_0) \neq g(x_0)$. So, $Jx_0 = \omega(x_0, .)$ separates f, g. ■

2 The C^*-algebra $M_n(\mathbb{C})$

We want to test the possibility to represent $M_n(\mathbb{C})$ by a space of continuous mappings. By $M_n(\mathbb{C})$, we mean the family of all $n \times n$-matrices with complex entries and with n strictly greater than 1. This becomes a $\mathbb{C}$-vector space by the usual addition of matrices and scalar multiplication with elements of $\mathbb{C}$. By the usual multiplication of matrices it becomes an algebra with unit.

Now, for $\mathbb{C}^n$ we can define the Euclidian norm $||(z_1,...,z_n)|| := \sqrt{\sum_{i=1}^n |z_i|^2}$. Hence, we can consider $L(\mathbb{C}^n, \mathbb{C}^n)$. But we see at once, that the vector spaces $L(\mathbb{C}^n, \mathbb{C}^n)$ and $M_n(\mathbb{C})$ are isomorphic: For $h \in L(\mathbb{C}^n, \mathbb{C}^n)$ there exists a unique matrix $A_h \in M_n(\mathbb{C}) : \forall x \in \mathbb{C}^n : h(x) = A_h \cdot x^T$; but for each $B \in M_n(\mathbb{C})$ the function $g_B : \mathbb{C}^n \to \mathbb{C}^n : g_B(x) := B \cdot x^T$ is clearly linear and - since $\mathbb{C}^n$ with euclidian norm is a finite dimensional normed space - g_B is continuous, too; thus $g_B \in L(\mathbb{C}^n, \mathbb{C}^n)$.

For $L(\mathbb{C}^n, \mathbb{C}^n)$ we can define the operator norm: $\forall h \in L(\mathbb{C}^n, \mathbb{C}^n) : ||h|| := \sup_{||x||\leq 1} ||h(x)||$; but now we can carry over this norm to $M_n(\mathbb{C})$: $\forall A \in M_n(\mathbb{C}) : ||A|| := \sup_{||x||\leq 1} ||A \cdot x^T||$, where $x \in \mathbb{C}^n$.

As is well known, to compute $||A||$ for $A \in M_n(\mathbb{C})$, we can use the fact, that this norm coincide with the spectral norm: let A^* the conjugate transpose of A (if

$A = (a_{ij})$ then $A^* = (\overline{a_{ji}})$), then the spectral norm of A is defined as the square root of the largest eigenvalue of the positive semidefinit matrix A^*A.
Remark: With these norm, $M_n(\mathbb{C})$ is a Banach algebra, as we know. It is a C^*-algebra, too, where $A \to A^*$ is the involution.

First dual spaces of $M_n(\mathbb{C})$:

(1) Using definition 1.3 we set $X = M_n(\mathbb{C})$ and $Y = \mathbb{C}$; hence $M_n(\mathbb{C})^d = \{h : M_n(\mathbb{C}) \to \mathbb{C} \mid h$ is linear, multiplicative and involutory$\} = \{h : M_n(\mathbb{C}) \mid h$ is an algebra (ring) homomorphism, $h(A^*) = \overline{h(A)}$, and h is continuous $\}$.

2.1 Lemma
Let $\underline{0} \in M_n(\mathbb{C})^d$ be the zero map. Then $M_n(\mathbb{C})^d = \{\underline{0}\}$.

Proof: For $n = 2$ we have $E := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. Now assume $\exists h \in M_n(\mathbb{C})^d : h \neq \underline{0}$. Since h is a ring homomorphism with $h \neq \underline{0}$, we find $h(E) = 1$ and $1 = h(E) = h \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} h \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$, thus $h \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \neq 0$.
Otherwise, $h \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = h \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} h \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} - h \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} h \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = 0$ because the multiplication is commutative in the range space $\mathbb{C}$. So, we get a contradiction, yielding $M_n(\mathbb{C}) = \{\underline{0}\}$ here.

For $n = 3$ observe that the matrix $A := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ yields an invertible $B := AA^T - A^TA = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ leading to the same contradiction as above, when used at the place of $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$.

Now, by building block matrices, the assumption will follow for all $n > 1$. ■

(2) As we have seen, the problem in the foregoing case was the commutativity of the multiplication in $\mathbb{C}$. Now, the multiplication in $Y = M_n(\mathbb{C})$ is non-commutative, hence we chose here $Y = X = M_n(\mathbb{C})$.

Thus, $M_n(\mathbb{C})^d = \{h : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \mid h$ is linear, multiplicative and involutory $\} = \{h : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \mid h$ is linear, multiplicative, involutory and continuous }. (The continuity follows from the fact, that $X = M_n(\mathbb{C})$ is finite-dimensional as $\mathbb{C}$-algebra.)

Here we have the advantage, that the identity map $\mathbb{1} : M_n(\mathbb{C}) \to M_n(\mathbb{C}) : \mathbb{1}(A) := A$ belongs to $M_n(\mathbb{C})^d$. But we find still more elements of $M_n(\mathbb{C})^d$.

2.2 Proposition

(a) Let $U \in M_n(\mathbb{C})$ be invertible; for the map $h_U : M_n(\mathbb{C}) \to M_n(\mathbb{C}) : h_U(A) := UAU^{-1}$ hold:

- *(i) h_U is linear,*
- *(ii) h_U is multiplicative,*
- *(iii) h_U is bijective,*
- *(iv) $h_U(E_n) = E_n$ where E_n means the unit matrix in $X = Y = M_n(\mathbb{C})$.*
- *(v) h_U is continuous.*

(b) If U is an unitary matrix, then additionally holds

- *(vi) $\forall A \in M_n(\mathbb{C}) : h_U(A^*) = (h_U(A))^*$.*

Proof: (i), (ii), (iii), (iv) and (vi) are straightforward matrix calculations, (v) again follows from the fact, that $M_n(\mathbb{C})$ is a finite dimensional normed space.■

By definition of $M_n(\mathbb{C})^d$ we get $M_n(\mathbb{C})^d \subseteq L((M_n(\mathbb{C}), ||.||), (M_n(\mathbb{C}), ||.||)) = L(M_n(\mathbb{C}), M_n(\mathbb{C}))$, and for $L(M_n(\mathbb{C}), M_n(\mathbb{C}))$ we consider the operator norm $||.|| : \forall h \in L(M_n(\mathbb{C}), M_n(\mathbb{C})) : ||h|| = \sup_{normA\leq 1} ||h(A)|| = \sup_{||A||=1} ||h(A)||$ and we restrict $||.||$ to $M_n(\mathbb{C})^d$. For instance, $\mathbb{1} \in M_n(\mathbb{C})^d$: $||\mathbb{1}|| = \sup_{||A||=1} ||\mathbb{1}(A)|| = 1$.

2.3 Lemma

The algebraic operations on $Y = M_n(\mathbb{C})$ (addition and multiplication of matrices, scalar multiplication, involution) are continuous w.r.t. the operator norm.

Proof: For the finite dimensional Banach space $(M_n(\mathbb{C}), ||.||)$ the vector space operations are contiuous, in arbitrary normed algebra, hence in $M_n(\mathbb{C})$ too, the multiplication is continuous. (And is easy to compute by the obvious submultiplicativity of the norm.) Now, we have $\forall A \in M_n(\mathbb{C}) : ||A|| = ||A^*||$. Let a net (A_n) in $M_n(\mathbb{C})$ be given with $A_n \to 0$ w.r.t. the norm topology $\tau_{||.||}$, just meaning $||A_n|| \to 0$, hence $||A_n^* - 0|| = ||A_n^*|| = ||A_n|| \overset{\tau_{||.||}}{\to} 0$; thus the involution is continuous on $M_n(\mathbb{C})$, too. ■

2.4 Proposition

(a) $M_n(\mathbb{C})^d$ *has a zero-homomorphism*
$h_0 : \forall A \in X = M_n(\mathbb{C}) : h_0(A) = Z_n \in M_n(\mathbb{C}) = Y$, *where* Z_n *means the zero-matrix.*

(b) $\forall h \in M_n(\mathbb{C})^d : ||h|| \leq 1$.

(c) Since $M_n(\mathbb{C})^d \subseteq M_n(\mathbb{C})^{M_n(\mathbb{C})}$, *using the algebraic operations in* $Y = M_n(\mathbb{C})$ *by pointwise definition, we can carry over these operations to* $M_n(\mathbb{C})^d$. *Endowed with these pointwise operations,* $M_n(\mathbb{C})^d$ *is not an algebra.*

Proof: (a) is evident, since Z_n is a zero-element of $Y = M_n(\mathbb{C})$.
(b) follows from proposition 1.1(a).
(c) $M_n(\mathbb{C})^d$ is no vector space: $\mathbb{1} \in M_n(\mathbb{C})^d$ and $||\mathbb{1}|| = 1$; hence $||2 \cdot \mathbb{1}|| = 2 > 1$, implying $2 \cdot \mathbb{1} \notin M_n(\mathbb{C})^d$ by (a). ∎

By (c) and the definitions of an abstract second dual space in [4], definition 4.2., and by [4], corollary 4.1. we obtain:

2.5 Corollary
The second dual space of $X = M_n(\mathbb{C})$ *w.r.t.* $Y = M_n(\mathbb{C})$ *is*

$$M_n(\mathbb{C})^{dd} = C\left(\left(M_n(\mathbb{C})^d, \tau_p\right), \left(M_n(\mathbb{C}), ||.||\right)\right)$$

and for the canonical map J *holds* $J(M_n(\mathbb{C})) \subseteq M_n(\mathbb{C})^{dd}$.

2.6 Proposition

(a) $(M_n(\mathbb{C})^d, \tau_p)$ *is a Hausdorff and compact topological space.*

(b) The zero-homomorphism $\underline{0} \in M_n(\mathbb{C})^d$ *is an isolated point of* $M_n(\mathbb{C})^d$ *in* $\left(M_n(\mathbb{C})^{M_n(\mathbb{C})}, \tau_p\right)$.

(c) $M_n(\mathbb{C})^d \setminus \{\underline{0}\}$ *is a Hausdorff compact space, too.*

Proof: (a) $Y = M_n(\mathbb{C})$ is a finite-dimensional normed space and by lemma 2.3 the algebraic operations in Y are $\tau_{||.||}$-continuous; especially, Y is Hausdorff. Then by the generalized Alaoglu theorem in [5], corollary 3.3., $(M_n(\mathbb{C})^d, \tau_p) = \left(\{h \in M_n(\mathbb{C})^{M_n(\mathbb{C})} | \; h \text{ is a continuous algebra homomorphism and } ||h|| \leq 1\}, \tau_p\right)$ is a compact Hausdorff topological space.
(b) By proposition 2.6 there exists a $h \in M_n(\mathbb{C})^d$ with $h \neq \underline{0}$ and $h(E_n) = E_n$, where E_n is the unit-matrix in $M_n(\mathbb{C})$. Then [4], lemma 4.3., yields the assertion.
. (c) follows from (a) and (b). ∎

Notation: Following the arguments of [4], lemma 4.2 and section 4.3 ("Redefinition of X^d") we consider $M_n(\mathbb{C})^d \setminus \{\underline{0}\}$ as new dual space of $X = M_n(\mathbb{C})$ and denote this space again by the symbol $M_n(\mathbb{C})^d$, from now on.

2.7 Theorem
Let

$$C_b\left((M_n(\mathbb{C}), \tau_p), (M_n(\mathbb{C}), ||.||)\right)$$

be the space of all bounded continuous functions from $(M_n(\mathbb{C}), \tau_p)$ into $(M_n(\mathbb{C}), ||.||)$ and let this space be endowed with the supremum-norm. Then hold:

(a) *The redefined $(M_n(\mathbb{C})^d, \tau_p)$ is a compact Hausdorff topological space.*

(b) *$M_n(\mathbb{C})^{dd} = C_b\left((M_n(\mathbb{C}), \tau_p), (M_n(\mathbb{C}), ||.||)\right)$ and*

$$\left(C_b\left((M_n(\mathbb{C}), \tau_p), (M_n(\mathbb{C}), ||.||)\right), ||.||_{\sup}\right)$$

is a C^*-algebra with unit $\underline{1}$: $\forall h \in M_n(\mathbb{C})^d : \underline{1}(h) := E_n \in Y = M_n(\mathbb{C})$.

(c) *$J : (M_n(\mathbb{C}), ||.||) \to M_n(\mathbb{C})^{dd}$ is an isomorphism, i.e. an injective algebra homomorphism from $M_n(\mathbb{C})$ onto the C^*-subalgebra $J(M_n(\mathbb{C})) \subseteq \left(C_b\left((M_n(\mathbb{C}), \tau_p), (M_n(\mathbb{C}), ||.||)\right), ||.||_{\sup}\right)$.*

(d) *J is is an isometric map, i.e. $\forall A \in M_n(\mathbb{C}) : ||J(A)||_{\sup} = ||A||$.*

Proof: (a) is just 2.6(c).
(b) Since $(M_n(\mathbb{C}), \tau_p)$ is compact and Hausdorff, we get
$M_n(\mathbb{C})^{dd} = C\left((M_n(\mathbb{C}), \tau_p), (M_n(\mathbb{C}), ||.||)\right) = C_b\left((M_n(\mathbb{C}), \tau_p), (M_n(\mathbb{C}), ||.||)\right)$
and in [4], proposition 4.2., was proved that the space
$C_b\left((M_n(\mathbb{C}), \tau_p), (M_n(\mathbb{C}), ||.||)\right)$ is a C^*-algebra and a space of continuous functions, which has a natural unit.
(c) By the homomorphy theorem in [4], theorem 4.1., we get that $J : (M_n(\mathbb{C}), ||.||) \to M_n(\mathbb{C})^{dd}$ is an algebra homomorphism; but then by 1.1(b) $J(M_n(\mathbb{C}))$ is a C^*-subalgebra of $(M_n(\mathbb{C})^{dd}, ||.||_{\sup})$.
Now, $\mathbb{1} \in M_n(\mathbb{C})^d$ implies, that $M_n(\mathbb{C})^d$ separates the points of $X = M_n(\mathbb{C})$ and hence by [4], proposition 4.5., J is injective, yielding by corollary 1.2, that hold: $\forall A \in M_n(\mathbb{C}) : ||J(A)||_{\sup} = ||A||$. So, (d) is proved. ∎

3 Representing arbitrary non-commutative C^*-algebras

Let X be a (nontrivial) non-commutative C^*-algebra with unit e; we want to define the dual space X^d (in the sense of our approach), the second dual space X^{dd} and to prove a representation theorem.

To define X^d we set $Y = (M_n(\mathbb{C}), ||.||)$.

3.1 Definition
$X^d := \{h : X \to M_n(\mathbb{C})|\ h$ is a continuous, linear ring-homomorphism with $h(x^*) = h(x)^*\}$ is called the first dual space of X w.r.t. $Y = M_n(\mathbb{C})$.

Remark: By 1.1(a) we know, that it is enough to say, that each h is an algebra homomorphism onto it's image. Now we can follow the procedure of the case $X = M_n(\mathbb{C})$ and $Y = M_n(\mathbb{C})$. Again we have here $X^d \subseteq L(X,Y) = L(X, M_n(\mathbb{C}))$; hence we can define the operator norm for X^d.
Then by 1.1(a) we get: $\forall h \in X^d : ||h|| \leq 1$; thus $X^d = \{h : X \to M_n(\mathbb{C})|\ h$ is continuous algebra homomorphism with $||h|| \leq 1\}$.

X^d has a zero-element $\underline{0}$ and we assume, that $X^d \neq \{\underline{0}\}$.

3.2 Proposition

(1) (X^d, τ_p) is a compact Hausdorff topological space.

(2) The zero homomorphism $\underline{0} \in X^d$ is an isolated point of X^d in $(M_n(\mathbb{C})^X, \tau_p)$.

(3) $X^d \setminus \{\underline{0}\}$ is a compact Hausdorff topological space, too.

Proof: (a) We know: $Y = (M_n(\mathbb{C}), ||.||)$ is Hausdorff, finite dimensional and all concerned algebraic operations on $M_n(\mathbb{C})$ are $\tau_{||.||}$-continuous. Thus using again the generalized Alaoglu theorem [5], corollary 3.3., we get (X^d, τ_p) being compact and Hausdorff.
(b) Sinve $X^d \neq \{\underline{0}\}$ by assumtion we can prove (b) quite analogously to part (b) of proposition 2.6 - and (c) will follow immediately. ■

Hence again we define $X^d \setminus \{\underline{0}\}$ as the new dual space and **redefine** from this point on the symbol X^d to denote this new dual.

Thus $X^d = \{h : X \to M_n(\mathbb{C})|\ h$ is a continuous algebra homomorphism and $h \neq \underline{0}\}$.

3.1 Definition of the second dual space

Again we have $X^d \subseteq M_n(\mathbb{C})^X$ and by using the algebraic operations on $Y = M_n(\mathbb{C})$, we can pointwise define similar those operations on X^d, too. In general X^d then will not be a vector space, as we know from the case $X = M_n(\mathbb{C})$. In fact, whenever there exists an $h \in X^d$ with $h \neq \underline{0}$, then follows $\exists x \in X : h(x) \neq 0$, implying $x \neq 0$, so $||\frac{1}{||x||}h(x)|| > 0$, yielding $\exists n \in \mathbb{N} : \frac{1}{n} < ||\frac{1}{||x||}h(x)||$. From the assumption, that X^d is a vector space, we get $nh \in X^d$, but we have $1 < ||\frac{n}{||x||}h(x)|| = ||(nh)(\frac{x}{||x||})|| \leq ||nh|| \leq 1$ - a contradiction. So, X^d is not an algebra if it is nontrivial.

By the definition of an abstract second dual space in [4], definition 4.2., and by [4], corollary 4.1., we obtain:

3.3 Proposition
The second dual space of X w.r.t. $Y = M_n(\mathbb{C})$ is

$$X^{dd} := C\left((X^d, \tau_p), (M_n(\mathbb{C}), ||.||)\right)$$

endowed with supremum-norm, and for the canonical map J holds $J(X) \subseteq X^{dd}$.

3.4 Theorem
Let X be a non-commutative C^*-algebra with unit; we assume $\exists h \in X^d : h \neq \underline{0}$. Then hold:

(1) *(The redefined) (X^d, τ_p) is a Hausdorff and compact topological space.*

(2)
$$\left(C_b\left((X^d, \tau_p), (M_n(\mathbb{C}), ||.||)\right), ||.||_{\sup}\right)$$
is a (non-commutative) C^*-algebra with unit, the canonical map
$$J : X \to C\left((X^d, \tau_p), (M_n(\mathbb{C}), ||.||)\right)$$
is an algebra homomorphism and $J(X)$ is a C^*-subalgebra of X^{dd}.

(3) $\forall x \in X : ||Jx||_{\sup} \leq ||x||$.

(4) *J is uniformly continuous and hence continuous.*

(5) *$J(X)$ separates the points of X^d.*

Proof: (a) comes from 3.2(a),(c).
(b) Since (X^d, τ_p) is compact and Hausdorff, X^{dd} equals the space of bounded continuous mappings, and by [4], proposition 4.2., this space is a C^*-algebra. The homomorphy theorem [4], theorem 4.1., shows that $J : X \to X^{dd}$ is an algebra-homomorphism. By 1.1(b) we know, that $J(X)$ is a C^*-subalgebra of X^{dd}.
Assertions (c), (d) and (e) we obtain from proposition 1.4. ■

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Harry Poppe
Dept. of Math., University of Rostock
Ulmenstrasse 69
18057 Rostock (Germany)
email: harry.poppe@uni-rostock.de

René Bartsch
University of Rostock, Dept. of Math
Ulmenstrasse 69
18057 Rostock (Germany)
email: math@marvinius.net