NOTE ON A CONJECTURE CONCERNING SPANNING SUBGRAPHS WITH DEGREE CONSTRAINTS

CHRISTIAN REIHER

Abstract. In this brief note, we give a positive answer to a question that has recently been asked by L. Addario–Berry et al. in this journal, namely: Given a graph $G = (V, E)$ and for every vertex $v \in V$ a list $D_v \subseteq \{0, 1, \dots, d(v)\}$ satisfying $|D_v| \geqq \frac{d(v)}{2} + 1$, does there necessarily exist a spanning subgraph H of G such that $d_H(v) \in D_v$ holds for every $v \in V$?

In a recent paper (see section 4.2 of [2]), the following conjecture has been proposed.

Conjecture 1. Given a graph $G = (V, E)$ and for every vertex $v \in V$ a list $D_v \subseteq \{0, 1, \dots, d(v)\}$ satisfying $|D_v| \geqq \frac{d(v)}{2} + 1$, there is a spanning subgraph H of G such that $d_H(v) \in D_v$ holds for every $v \in V$.

The optimality of this conjecture is discussed in [2]. Partial results replacing the lower bound on $|D_v|$ at first by $\frac{11}{12}d(v) + 1$ and then by $\frac{7}{8}d(v) + 1$ have appeared in [2] and [3]. The purpose of this note is to provide a complete proof of the above conjecture. Our argument uses a certain observation concerning partitions of graphs into stars, namely

Lemma 2. For every graph $G = (V, E)$, there exists a partition $\{B_v \mid v \in V\}$ of its edge–set such that the following conditions are satisfied.

(a) Whenever $v \in V$ and $e \in B_v$, the edge e is incident to v.
(b) If $v \in V$, then $|B_v| \geqq \frac{d(v)-1}{2}$.

Proof. We argue by induction on $|E|$. If $E = \emptyset$, we may take $B_v = \emptyset$ for every $v \in V$ and are done. For the induction step, we distinguish two cases. First, if some vertex $x \in V$ has a unique neighbour y, we apply the induction hypothesis to $G' = (V, E - \{xy\})$, thus getting a family $\{B'_v \mid v \in V\}$ satisfying all of the above conditions with respect to G' instead of G. Now let

$$B_v = \begin{cases} \emptyset & \text{if } v = x, \\ B'_y \cup \{xy\} & \text{if } v = y, \\ B'_v & \text{if } v \in V - \{x, y\} \end{cases}$$

2000 *Mathematics Subject Classification.* Primary: 05C07, Secondary: 05C15.

Key words and phrases. Louigi's Conjecture, Degree–constrained subgraphs, Combinatorial Nullstellensatz.

and observe that the family $\{B_v \,|\, v \in V\}$ is as desired. So it remains to consider the case where G has at least one edge and every vertex of G is either isolated or has at least degree two. For known reasons, G then has to contain some cycle $x_1x_2\cdots x_n$ with $n \geqq 3$. Now apply the induction hypothesis to $G^* = (V, E - \{x_1x_2, x_2x_3, \ldots, x_nx_1\})$, thereby obtaining a partition $\{B^*_v \,|\, v \in V\}$ of the edge–set of G^* satisfying

$$|B^*_v| \geqq \begin{cases} \frac{d(v)-3}{2} & \text{if } v \in \{x_1, x_2, \ldots, x_n\} \\ \frac{d(v)-1}{2} & \text{if } v \in V - \{x_1, x_2, \ldots, x_n\}. \end{cases}$$

Setting

$$B_v = \begin{cases} B^*_v \cup \{x_ix_{i+1}\} & \text{if } v = x_i, i \in \{1, 2, \ldots, n\}, \\ B^*_v & \text{if } v \in V - \{x_1, x_2, \ldots, x_n\}, \end{cases}$$

where, for convenience, $x_{n+1} = x_1$, the set $\{B_v \,|\, v \in V\}$ is as demanded.

■

Let us now turn to the proof of Conjecture 1. Suppose that $G = (V, E)$ and $\{D_v \,|\, v \in V\}$ are given as above. By our lemma, there is a family $\{B_v \,|\, v \in V\}$ of mutually disjoint subsets of E such that

(a) If $v \in V$, $e \in B_v$, then v is an endvertex of e.
(b) If $v \in V$, then $|B_v| = d(v) + 1 - |D_v|$.

With every $e \in E$, we associate a variable z_e and then we consider the product

$$P(z_e \,|\, e \in E) = \prod_{v \in V, m \in \{0,1,\ldots,d(v)\} - D_v} \left(\sum_{w \in N(v)} z_{vw} + m \right)$$

as a polynomial over the field of rational numbers. The degree of P is (at most)

$$\sum_{v \in V} (d(v) + 1 - |D_v|) = \sum_{v \in V} |B_v|$$

and when expanding the above expression into a sum of monomials only nonnegative coefficients appear throughout the computation. Now, for every $v \in V$, the monomial $\prod_{e \in B_v} z_e$ has positive coefficient in

$$\prod_{m \in \{0,1,\ldots,d(v)\} - D_v} \left(\sum_{w \in N(v)} z_{vw} + m \right),$$

namely $|B_v|!$, and hence $\prod_{v \in V, e \in B_v} z_e$ occurs with a positive coefficient in P. Recalling that the various sets B_v have been chosen to be mutually disjoint, we may invoke the Combinatorial Nullstellensatz (see [1]) to obtain values $z_e \in \{-1, 0\}$ for which $P(z_e \,|\, e \in E) \neq 0$. Now let H be the spanning subgraph of G with edge set $\bar{E} = \{e \in E \,|\, z_e = -1\}$. If H were not as desired, then there existed some $v \in V$ and $m \in \{0, 1, \ldots, d(v)\} - D_v$ such that $d_H(v) = m$. But then $\sum_{w \in N(v)} z_{vw} + m = 0$, whence $P(z_e | e \in E) = 0$, contrary to our construction. Thereby we have proved Conjecture 4.1 of [2]. ■

References

[1] N. Alon: Combinatorial Nullstellensatz, *Combinatorics, Probability and Computing* **8** (1999), 7–29.

[2] L. Addario–Berry, K. Dalal, C. McDiarmid, B. Reed and A. Thomason: Vertex–colouring Edge–weightings, *Combinatorica* **27** (1) (2007), 1–12.

[3] L. Addario–Berry, K. Dalal and B. Reed: Degree constrained subgraphs, *Discrete Applied Mathematics* **156** (7) (2008), 1168–1174.

Institut für Mathematik, Universität Rostock, 18051 Rostock, Germany

E-mail address: Christian.Reiher@uni-rostock.de