Criteria of Chaos for Discrete Dynamical Systems and Application of them to the Anti-control of Chaos

Ju Hyon Hui*

Abstract

In this paper we study criteria of chaos for discrete dynamical systems governed by noninvertible maps and apply them to the anti-control of chaos.

We get some criteria of chaos for discrete dynamical systems governed by noninvertible maps using the subshift in the space of one-side symbolic sequences, and discuss the structural stability of chaotic invariant subsets on a Riemann manifold, and generic chaos. And using these criteria of chaos, we get more generalized conditions to design a controller for the anti-control of chaos.

Key words: chaos, discrete dynamical system, symbolic dynamical system, anti-control of chaos

1 Introduction

In this paper we discuss criteria of chaos for discrete dynamical systems governed by noninvertible maps and we apply these results to the anti-control of chaos.

This paper is concerned with chaos in the following discrete dynamical system

$$x_{n+1} = f(x_n), \quad n \geq 0, \tag{1}$$

where $f : X \to X$ is a map and X is a metric space.

It is well known that the research on the criteria of chaos for dynamical systems is one of three main research branches of chaos theory ([3]).

The criteria of chaos for the system (1) in which f is a diffeomorphism in $\mathbb{R}^n$ have been fully researched ([3]).

The research on chaos for the system (1) in which f is a noninvertible map began some years after Poincare. It's development has been accelerated particularly since computer revolution, and today it is a young and active field of study ([8]). The earliest development of this theory came in the context of $X = \mathbb{R}$ and two-dimensional context has been the current research frontier ([8]).

*Department of Mathematics and Mechanics, Kim Il Sung University, Pyongyang D.P.R of Korea.

In the pursuit of criteria of chaos for the system (1) in which f is not a diffeomorphism in $\mathbb{R}^n$, it has been discussed only for the case f has a snap-back repeller fixed point ([2],[12]) except [10]. In [10], a criterion of chaos was discussed which is concerned with the shift in the space of one-side symbolic sequences.

We study more generalized criteria of chaos for the system (1) in which X is a finite-dimensional manifold, where we use the subshift in the space of one-side symbolic sequences, and apply them to the problem of anti-control of chaos.

The research on the anti-control of chaos (also is called chaotification) has attracted interest from many scientists recently. For a long time, a system was not considered useful and healthy if it behaves chaotically. So the traditional control engineering design always tries to stabilize a chaotic system. The process of making a chaotic system stable is called control of chaos. Over the last decade, research on control of chaos has rapidly developed.

However, in recent years, it has been found that chaos can actually be very useful under some circumstances, for example, in human brain analysis, heartbeat regulation, encryption, digital communications, etc. So sometimes it is useful and even important to make an originally non-chaotic system chaotic in such engineering applications. This process is called chaotification or the anti-control of chaos.

In the pursuit of chaotifying discrete dynamical systems, a mathematically rigorous chaotification method was first developed by Chen and Lai in 1996 ([9]) and ever since, this topic has been extensively studied by many scientists. Here is a very important remark on this problem. It is that a practical desine of a controller should be simpler than the given system. As for the mathematical view, there is great necessity for the anti-control of chaos to generalize so that it can be applied more implement and cheap systems and we can design more simple and cheap controller. In the previous papers was considered the case when the maps, corresponding to the original system ,were continuously differentiable, and at least had one fixed point (see the Introduction of [10]). In [10], this topic was considered under more relaxed conditons than the previous studies.

We get some more reduced conditions for the design of a controller for the chaotification by using our criteria obtained in this paper.

The rest of this paper is organized as follows. In Section 2, we discuss the criteria of chaos in the sense of Devaney for the system (1). Since Li and Yorke ([1], 1975) first introduced a precise mathmatical definition of chaos, there appeared several different definitions of chaos, some are stronger or not, depending on the requirments in different problems. The definition of chaos in the sense of Devaney([4], 1987) is known as well as the definition of chaos in the sense of Li-Yorke. Recently, in [11](2002), was shown that chaos in the sense of Devaney implies chaos in the sense of Li-Yorke. Here we consider the criteria of chaos in the sense of Devaney using the topological conjugacy to the subshift in the space of one-side symbolic sequences, which is a noninvertible map. In Subsection 2.1, we first recall the concepts of chaos in sense of Devaney and symbolic dynamical system, and mention the relation between them. Then we establish more generalized criterion of chaos for noninvertible maps than the one of [10] and discuss the structural stability of chaotic invariant subsets on Riemann manifold. In Subsection 2.2, we discuss the generic chaos for the discrete dynamical system. It is known that generic chaotic map is chaotic in the sense of Li-Yorke([5]). We prove that subshift in the space of one-side symbolic sequences is generic chaotic. This result obviousely implies that the criterion of chaos obtained in Subsection 2.1 is also the criterion of generic chaos. In Section 3, we apply a criterion obtained in this paper to the problem of anti-control of chaos.

2 Criteria of chaos for discrete dynamical system

2.1 Criteria of chaos for discrete dynamical systems governed by noninvertible maps and structural stability of chaotic invariant set

Since Li and Yorke[1] first introduced a precise mathematical definition of chaos, there appeared several different definitions of chaos depending on the requirements in different problems. In 1987, Devaney [4] presented an explicit definition of chaos for following system:

$$x_{n+1} = f(x_n), \quad n \geq 0$$

where $f : X \to X$ is a map and (X, d) is a metric space. A continuous map f is said to be chaotic on X (in the sense of Devaney) if

(1) f is transitive;

(2) the periodic points of f are dense in X;

(3) f has sensitive dependence on initial conditions.

In [13] was given that Properties (1) and (2) together imply Property (3). So Property (3) is redundant in the above definition.

Since chaos for discrete dynamical systems governed by noninvertible maps is related to a symbolic dynamical system in the space of one-side symbolic sequences, now we recall its concept.

Let N be a positive integer, Σ^N be a space of one-side symbolic sequences as follows:

$$\Sigma^N = \{S = (s_0 s_1 s_2 \cdots) \big| \quad i = 0, 1, 2, \ldots\},$$

and define the distance between two points $s = (s_0 s_1 s_2 \cdots)$ and $t = (t_0 t_1 t_2 \cdots)$ of Σ^N by $\rho(s,t) = \sum\limits_{i=1}^{\infty} \frac{|s_i - t_i|}{2^i}$. Then we know that (Σ^N, ρ) is a compact metric space.

Let A be an irreducible $N \times N$ matrix (see [10]) and Σ_A^N be as follows:

$$\Sigma_A^N = \{S = (s_0 s_1 s_2 \cdots) \in \Sigma^N \big| \quad (A)_{s_i s_{i+1}} = 1, i = 0, 1, 2, \ldots\}$$

where $(A)_{ij}$ denotes i, j element of matrix A , Then Σ_A^N is also considered as above. One-side symbolic dynamical system $\sigma : \Sigma^N \to \Sigma^N$ defined by $\sigma(s_0 s_1 s_2 \cdots) = (s_1 s_2 s_3 \cdots)$, $(s_0 s_1 s_2 \cdots) \in \Sigma^N$, is called shift, and the restriction of σ within Σ_A^N, that is, $\sigma\big|_{\Sigma_A^N}$ is called subshift. (See [3])

Symbolic dynamical system $\sigma : \Sigma^N \to \Sigma^N$ is continuous and has the following properties ([3]):

(1) Card $Per_n(\sigma) = N^n$;

(2)$Per(\sigma)$ is dense in Σ^N;

(3) there exists a dense orbit of σ in Σ^N;

where Card $Per_n(\sigma)$ denotes the number of periodic points of period n for σ. These Properties also are satisfied for subshift $\sigma\big|_{\Sigma_A^N}$ ([3]).

It is clear that Property (3) implies that the symbolic dynamical system is transitive. Therefore, this symbolic dynamical system $\sigma\big|_{\Sigma_A^N}$ is chaotic in the sense of Devaney. In this paper, we assume that A is an irreducible matrix. Our main results are as follows.

Theorem 1 *Let X be Hausdorff space and $f : X \to X$ be a continuous map. There is an f-invariant subset $\Lambda \subset X$ such that $f|_\Lambda$ is topologically conjugate to the subshift $\sigma\big|_{\Sigma_A^N}$ if and only if there are distinct nonempty compact subsets $\Lambda_1, \Lambda_2, \cdots, \Lambda_N \subset X$ satisfying the following conditions:*

$$1)\quad f(\Lambda_i) \supset \bigcup_{\substack{j \\ (A)_{ij}=1}} \Lambda_j, \quad (1 \le i \le N),$$

$$2)\quad card \bigcap_{s=0}^{\infty} f^{-s}(\Lambda_{i_s}) \le 1, \quad (i_0 i_1 \cdots) \in \Sigma_A^N$$

where card$(\cdot)$ denotes its cardinal. Consequently f is chaotic on Λ in the sense of Devaney.

Proof.

($\Rightarrow$): From the above conditions, there exist an f-invariant subset $\Lambda \subset X$ and homeomorphism $h : \Lambda \to \Sigma_A^N$ such that $h \circ f\big|_\Lambda = \sigma \circ h$ (where $\sigma = \sigma\big|_{\Sigma_A^N}$).

Let B_i be as following:

$$B_i := \{(b_0 b_1 \cdots) \in \Sigma_A^N \big| b_0 = i\}.$$

Then it follows that $B_i \neq \emptyset$ for any $i \in \{1, \cdots, N\}$ because matrix A is irreducible. Let

$$\Lambda_i := h^{-1}(B_i), (i = 1, \cdots, N),$$

then the set Λ_i is compact. Indeed, if $\{s^i\}$ is a sequence in B_i converging to $\overline{s}$ and $\overline{s} \notin B_i$, then there is an integer M such that $(A)_{\overline{s}_M \overline{s}_{M+1}} = 0$ where $\overline{s} = (\overline{s}_0 \overline{s}_1 \cdots)$.

Since $\{s^k\}$ converges to $\overline{s}$, there is a positive integer $\overline{M}$ such that $d(s^k, \overline{s}) \le \frac{1}{2^{M+2}}$ for any $i \ge \overline{M}$. Therefore, from lemma 2.2.2 of [10], we have

$$k \ge \overline{M}, j \le M + 1 \Rightarrow s_j^k = \overline{s}_j$$

where $s^k = (s_0^k s_1^k \cdots)$. But since $s^k \in B_i$, it follows that

$$(A)_{s_M^k s_{M+1}^k} = (A)_{\overline{s}_M \overline{s}_{M+1}} = 1.$$

This contradicts to above fact. Therefore $\overline{s} \in B_i$ and B_i is a closed subset in Σ_A^N. Since Σ_A^N is compact, B_i is also compact.

Now we are going to show that sets $\Lambda_1, \Lambda_2, \cdots, \Lambda_N$ satisfy the conditions 1) and 2). Since $B_i \cap B_j = \emptyset$ $(i \neq j)$ and h is a homeomorphism, we have $\Lambda_i \cap \Lambda_j = \emptyset$

$(i \neq j)$. If $(c_0c_1c_2\cdots) \in B_j$, then $c_0 = j$ and if $(A)_{ij} = 1$, then $(ijc_2c_3\cdots) \in B_i$ and $\sigma(B_i) \supset B_j$, therefore

$$f(\Lambda_i) = f(h^{-1}(B_i)) = h^{-1} \circ \sigma(B_i) \supset h^{-1}(B_j) = \Lambda_j$$

and

$$f(\Lambda_i) \supset \bigcup_{\substack{j \\ (A)_{ij}=1}} \Lambda_j.$$

And it follows that

$$f^{-s}(\Lambda_{i_s}) = f^{-s}(h^{-1}(B_{i_s}) = (h^{-1} \circ \sigma^{-s} \circ h)(h^{-1}(B_{i_s})) = h^{-1} \circ \sigma^{-s}(B_{i_s}).$$

Therefore

$$\bigcap_{s=0}^{\infty} f^{-s}(\Lambda_{i_s}) = \bigcap_{s=0}^{\infty} h^{-1} \circ \sigma^{-s}(B_{i_s}) = h^{-1}(\bigcap_{s=0}^{\infty} \sigma^{-s}(B_{i_s}))$$

and since

$$card \bigcap_{s=0}^{\infty} \sigma^{-s}(B_{i_s}) \leq 1,$$

the condition 2) is true.

($\Leftarrow$): First, we are going to prove that

$$card \bigcap_{s=0}^{\infty} f^{-s}(\Lambda_{i_s}) = 1, \quad (i_0i_1\cdots) \in \Sigma_A^N.$$

For any nonnegetive integer k and an admissible finite sequence $(i_0i_1\cdots i_k) \in \prod_{s=o}^{k} E_s$ (where $E_s = \{1, 2, \ldots, N\}$ for any s), it follows that

$$f^k(\bigcap_{s=0}^{k} f^{-s}(\Lambda_{i_s})) \supset \Lambda_{i_k}.$$

Indeed, for $k = 0$, it is trivial. Assume that it is true for $k - 1$, then

$$\begin{aligned}
f^k(\bigcap_{s=0}^{k} f^{-s}(\Lambda_{i_s})) &= f \circ f^{k-1}(\bigcap_{s=0}^{k-1} f^{-s}(\Lambda_{i_s}) \cap f^{-(k-1)}(f^{-1}(\Lambda_{i_k}))) \\
&= f(f^{k-1}(\bigcap_{s=0}^{k-1} f^{-s}(\Lambda_{i_s})) \cap f^{-1}(\Lambda_{i_k})) \\
&\supset f(\Lambda_{i_{k-1}} \cap f^{-1}(\Lambda_{i_k})) \\
&\supset \Lambda i_k.
\end{aligned}$$

Therefore

$$\bigcap_{s=0}^{k} f^{-s}(\Lambda_{i_s}) \neq \emptyset.$$

Since Λ_{i_s} is compact and f is continuous, it follows that

$$\bigcap_{s=0}^{\infty} f^{-s}(\Lambda_{i_s}) \neq \emptyset.$$

Therefore, from the assumption, for any $(i_0 i_1 \cdots) \in \Sigma_A^N$,

$$card \bigcap_{s=0}^{\infty} f^{-s}(\Lambda_{i_s}) = 1.$$

Now, let the set Λ and the map h be defined as follows:

$$\Lambda := \bigcup_{(i_0 i_1 \cdots) \in \Sigma_A^N} \bigcap_{s=0}^{\infty} f^{-s}(\Lambda_{i_s})$$

$$h : \Lambda \to \Sigma_A^N$$

$$h(x) = (i_0 i_1 \cdots), \quad x \in \bigcap_{s=0}^{\infty} f^{-s}(\Lambda_{i_s})$$

Then h is a homeomorphism, and it follows that

$$h \circ f\big|_{\Lambda} = \sigma \circ h.$$

Indeed, if

$$x \in \bigcap_{s=0}^{\infty} f^{-s}(\Lambda_{i_s}) \subset \Lambda$$

then

$$f(x) \in f[\bigcap_{s=0}^{\infty} f^{-s}(\Lambda_{i_s})] = \bigcap_{s=0}^{\infty} f^{-(s-1)}(\Lambda_{i_s}).$$

Therefore $h[f(x)] = (i_1 i_2 \cdots)$, and since $h(x) = (i_0 i_1 \cdots)$, it follows that

$$\sigma(h(x)) = (i_1 i_2 \cdots).$$

$\square$

Corollary 1 *Let X be a metric space and $f : X \to X$ be a map. If there exist distinct nonempty compact subsets $\Lambda_1, \Lambda_2, \ldots, \Lambda_N \subset X$ satisfying following conditions:*

1) for every $i (1 \le i \le N)$, $\quad f\big|_{\Lambda_i}$ is continuous.

2) $f(\Lambda_i) \supset \bigcup_{\substack{j \\ (A)_{ij}=1}} \Lambda_j, \quad 1 \le i \le N.$

3) $card \bigcap_{s=0}^{\infty} f^{-s}(\Lambda_{i_s}) \le 1, \quad (i_0 i_1 \cdots) \in \Sigma_A^N.$

Then there exists a f -invariant subset $\Lambda \subset X$ such that $f\big|_{\Lambda}$ is conjugate topologically to $\sigma\big|_{\Sigma_A^N}$, consequently f is chaotic in the sense of Devaney.

Proof.

It follows directly from the process of the proof of Theorem 1. $\square$

Example 1 *Let $X = [-1,3] \times [-1,3] \subset \mathbb{R}^2$ and A_1, A_2, A_3, A_4 be subsets of X as follows:*

$$\begin{aligned}
A_1 &= [0, \frac{1}{2}) \times [\frac{1}{2}, 1) \\
A_2 &= [0, \frac{1}{2}) \times [0, \frac{1}{2}) \\
A_3 &= [\frac{1}{2}, 1) \times [0, \frac{1}{2}) \\
A_4 &= [\frac{1}{2}, 1) \times [\frac{1}{2}, 1)
\end{aligned}$$

Let $f : X \to X$ be as follows:

$$\begin{aligned}
A_1 &\ni (x,y) \longmapsto (3x - \frac{1}{3}, y - \frac{1}{2}) \\
A_2 &\ni (x,y) \longmapsto (x + \frac{1}{2}, 3y - \frac{1}{3}) \\
A_3 &\ni (x,y) \longmapsto (3(x - \frac{1}{2}) - \frac{1}{3}, y + \frac{1}{2}) \\
A_4 &\ni (x,y) \longmapsto (x - \frac{1}{2}, 3(y - \frac{1}{2}) - \frac{1}{3}) \\
X \setminus \bigcup_{i=1}^{4} A_i &\ni (x,y) \longmapsto (|x| - [|x|], |y| - [|y|])
\end{aligned}$$

Then it follows that

$$\begin{aligned}
f(A_1) &\supset A_2 \cup A_3 \\
f(A_2) &\supset A_3 \cup A_4 \\
f(A_3) &\supset A_1 \cup A_4 \\
f(A_4) &\supset A_1 \cup A_2.
\end{aligned}$$

Now define $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ as follows:

$$\begin{aligned}
\Lambda_1 &:= [\frac{1}{12}, \frac{5}{12}] \times [\frac{7}{12}, \frac{11}{12}] \subset A_1 \\
\Lambda_2 &:= [\frac{1}{12}, \frac{5}{12}] \times [\frac{1}{12}, \frac{5}{12}] \subset A_2 \\
\Lambda_3 &:= [\frac{7}{12}, \frac{11}{12}] \times [\frac{1}{12}, \frac{5}{12}] \subset A_3 \\
\Lambda_4 &:= [\frac{7}{12}, \frac{11}{12}] \times [\frac{7}{12}, \frac{11}{12}] \subset A_4 \\
&.
\end{aligned}$$

Then the sets $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ are compact sets satisfying the conditions of Corollary 1 and f also satisfys the conditions of Corollary 1 for the irreducible matrix A:

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Therefore there exists an invariant set $\Lambda \subset X$ such that $f\big|_\Lambda$ is topologically conjugate to $\sigma\big|_{\Sigma_A^N}$, consequantly f is chaotic in the sense of Devaney.(See Figure 1.)

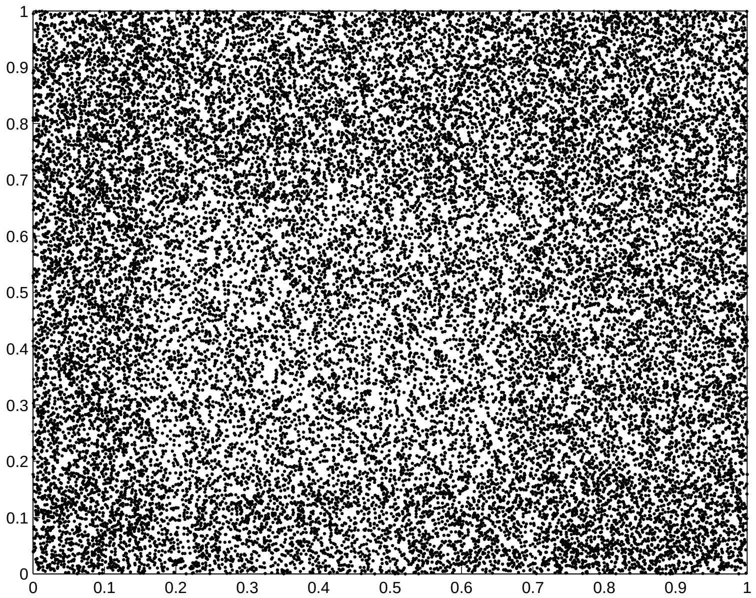

Figure 1: This shows the sequence $(x_n, y_n) = f^n(x_0, y_0)$ plotted in the rectangle $[0,1] \times [0,1]$ in the case of $x_0 = 0.7$, $y_0 = 0.5$ and $n = 1, 2, \ldots, 40000$.

Theorem 2 *Let M be a differentiable finite-dimensional Riemann manifold satisfying the second countable axiom, and $D \subset M$ be an open subset with a compact closure.*

Let $K_1, K_2, \ldots, K_N \subset D (N \geq 2)$ be distinct nonempty compact subsets and $int(K_i)$ $(i = 1.2. \ldots, N)$ be a connected set, where $int(\cdot)$ denotes interior of the set.

If $f : D \to D$ is a map of C^1 -class satisfying following conditions:

1) $f(intK_i) \supset \bigcup_{\substack{j \\ (A)_{ij}=1}} K_j$, $\quad f\big|_{K_i} (i = 1, \ldots, N)$ is injection

2) there exist $n_0 \in N$ and $\lambda > 1$ such that

$$\|Df^{n_0}(\xi)\| \geq \lambda \|\xi\|, \quad \xi \in T_{K_i}M, \quad i = 1, 2, \ldots, N,$$

then there exists a f-invariant subset $\Lambda \subset D$ such that $f\big|_\Lambda$ is conjugate topologically to $\sigma\big|_{\sum_A^N}$, consequently f is chaotic in the sense of Devaney. Moreover the invariant set Λ is C^1-structural stable.

Proof.

Firstly, we are going to prove that

$$card \bigcap_{s=0}^{\infty} f^{-s}(K_{i_s}) \leq 1, \quad (i_0 i_1 \cdots) \in \Sigma_A^N$$

Let M_i and W_i be defined as follows:

$$\begin{aligned} M_i : &= K_i \cap f^{-1}(\bigcup_{\substack{j \\ (A)_{ij}=1}} K_j) \\ W_i : &= intK_i, \quad (i = 1, \ldots, N). \end{aligned}$$

From condition 1), it follows that $M_i \subset W_i, (i = 1, \ldots, N)$. Indeed, since

$$f(intK_i) \supset \bigcup_{\substack{j \\ (A)_{ij}=1}} K_j,$$

it follows that

$$intK_i = f^{-1}f(intK_i) \supset f^{-1}(\bigcup_{\substack{j \\ (A)_{ij}=1}} K_j).$$

Therefore

$$M_i = K_i \cup f^{-1}(\bigcup_{\substack{i \\ (A)_{ij}=1}} K_j) \subset f^{-1}(\bigcup_{\substack{j \\ (A)_{ij}=1}} K_j) \subset intK_i = W_i.$$

By lemma from [7], we can reset a Riemann metric deriving a norm $\| \bullet \|_1$ equivalent to original norm $\| \bullet \|$ of M so that there exists $\lambda_1 > 1$ such that

$$\|Df(\xi)\|_1 \geq \lambda_1 \|\xi\|_1, \quad \xi \in T_\Lambda M.$$

Let $\rho_i = \rho_{w_i} (i = 1, \ldots, N)$ be a distance derived by this Riemann metric in W_i, then $\rho\big|_{M_i \times M_i}$ is bounded because W_i is connected and M_i is compact.

Therefore, there exists $L > 0$ such that

$$\rho_i(a, b) \leq L, a, b \in M_i, i = 1, \ldots, N.$$

Let $(i_0 i_1 \cdots) \in \Sigma_A^N$. Then for any k, it follows that

$x, y \in \bigcap\limits_{s=0}^{\infty} f^{-s}(k_{i_s}) \Rightarrow f^k(x), f^k(y) \in K_{i_k} \cap f^{-1}(K_{i_{k+1}}) \subset intK_{i_k} = W_{i_k}$.

Indeed,

$x, y \in \bigcap\limits_{s=0}^{\infty} f^{-s}(K_{i_s}) \Rightarrow$

$$x, y \in K_{i_0} \cap f^{-1}(K_{i_1}) \cap \ldots \cap f^{-k}(K_{i_k}) \cap f^{-k-1}(K_{i_{k+1}}) \cap \ldots$$

$$f^k(x), f^k(y) \in f^k(K_{i_0}) \cap f^{k-1}(K_{i_1}) \cap \ldots \cap K_{i_k} \cap f^{-1}(K_{i_{k+1}}) \cap \ldots$$

and $K_{i_{k+1}} \in f(intK_{i_k})$ because $(A)_{i_k i_{k+1}} = 1$, therefore

$$K_{i_k} \cap f^{-1}(K_{i_{k+1}}) \subset intK_{i_k} = W_{i_k}$$

and

$$f^k(x), f^k(y) \in M_{i_k}.$$

For $\varepsilon > 0$, take a C^1-class curve γ connecting between $f^k(x)$ and $f^k(y)$ in W_{i_k} such that

$$\int_o^1 \|\dot{\gamma}(t)\|_1 dt \leq \rho_{i_k}(f^k(x), f^k(y)) + \varepsilon.$$

The mapping $f^k\big|_{K_{i_0} \cap f^{-1}(K_{i_1}) \cap \ldots \cap f^{-k}(K_{i_k})}$ is a diffeomorphism its image is K_{i_k}.

So there is a C^1-class curve β in $K_{i_0} \cap f^{-1}(K_{i_1}) \cap \ldots \cap f^{-k}(K_{i_k})$ such that $f^k \circ \beta = \gamma$.

Therefore

$$\begin{aligned} L + \varepsilon &\geq \rho_{i_k}(f^k(x), f^k(y)) + \varepsilon \\ &> \int_0^1 \|\dot{\gamma}(t)\|_1 dt \\ &= \int_0^1 \|Df^k(\dot{\beta}(t))\|_1 dt \\ &\geq \lambda_1^k d(x, y) \end{aligned}$$

Then since $\lambda_1 > 1$ and k is any integer, $d(x, y) = 0$. Therefore

$$card \bigcap_{s=0}^{\infty} f^{-s}(K_{i_s}) \leq 1, \quad (i_0 i_1 \cdots) \in \Sigma_A^N.$$

Consequantly, by Theorem 1, there exists a f-invariant set $\Lambda \subset X$ such that $f\big|_\Lambda$ is conjugated topologically to $\sigma\big|_{\Sigma_A^N}$. Otherwise from the proof of Theorem 1, the f-invariant set Λ is as follows:

$$\Lambda = \bigcup_{(i_0 i_1 \cdots) \in \Sigma_A^N} \bigcap_{s=0}^{\infty} f^{-s}(K_{i_s})$$

And since

$$\bigcap_{s=0}^{\infty} f^{-s}(K_{i_s}) \subset K_{i_0}, \quad (i_0 i_1 \cdots) \leq \Sigma_A^N,$$

it follows that $\Lambda \subset \bigcup_{i=1}^{N} K_i$. Therefore we can prove that Λ is C^1-structurally stable.

□

2.2 Generic chaos for the descrete dynamical system

Following concept of the generic chaos has been accepted in [5]. We proved the subshift $\sigma\big|_{\Sigma_A^N}$ is generic chaotic.

Definition 1 *([5])*

Let (X, d) be a metric space and $f : X \to X$ is a map. f is called generic chaotic if following set

$$G := \{(x, y) \in X \times X \big| \liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0, \quad \limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0\}$$

is second category in $X \times X$.

It is known that a generic chaotic map is chaotic in the sense of Li-Yorke.

Theorem 3 *The subshift $\sigma\big|_{\Sigma_A^N}$ in the space of one-side symbolic sequences is generic chaotic.*

Proof.

First of all, we are going to prove that the following sets L_n and U_n

$$\begin{aligned} L_n &= \{(\alpha, \beta) \in \Sigma_A^N \times \Sigma_A^N \big| \quad \inf_{K\geq n} \rho(\sigma^K(\alpha), \sigma^K(\beta)) < \frac{1}{n}\} \\ U_n &= \{(\alpha, \beta) \in \Sigma_A^N \times \Sigma_A^N \big| \quad \sup_{K\geq n} \rho(\sigma^K(\alpha), \sigma^K(\beta) > \lambda\} \end{aligned}$$

are open and dence sets in $\Sigma_A^N \times \Sigma_A^N$ (here $n \in N, n \neq 0, \lambda > 0$).

If $(\alpha^\circ, \beta^\circ) \in L_n$, then for any $\varepsilon > 0$, there exist $h \in (0, \varepsilon)$ and $K(\geq n)$ such that

$$\rho(\sigma^K(\alpha^\circ), \sigma^K(\beta^\circ)) < \frac{1}{n} - h.$$

Since σ^K is continuous, there are a neighbourhood O_{α° of α° and a neighbourhood O_{β° of β° such that

$$\begin{aligned} \rho(\sigma^K(\alpha), \sigma^K(\alpha^\circ)) &< \frac{h}{2}, \quad \alpha \in O_{\alpha^\circ} \\ \rho(\sigma^K(\beta), \sigma^K(\beta^\circ)) &< \frac{h}{2}, \quad \beta \in O_{\beta^\circ}. \end{aligned}$$

Therefore

$$(\alpha, \beta) \in O_{\alpha^\circ} \times O_{\beta^\circ} \Rightarrow \rho(\sigma^K(\alpha), \sigma^K(\beta)) < \frac{1}{n}$$

and consequently L_n is open.

And then , we can prove that U_n are open as above mentioned method.

Next, U_n is dense in $\Sigma_A^N \times \Sigma_A^N$. Indeed, if $(\alpha^\circ, \beta^\circ) \in \Sigma_A^N \times \Sigma_A^N, \varepsilon > 0$, then there exists $\overline{M} \in N$ such that,

$$\alpha = (\alpha_i) \in \Sigma_A^N, \alpha_i = \alpha_i^\circ(i \leq \overline{M}) \Rightarrow \rho(\alpha^\circ, \alpha) < \varepsilon.$$

(here $\alpha^\circ = (\alpha_i^\circ)$.)

Since $U_n \neq \emptyset$, we can take $(\overline{\alpha}, \overline{\beta}) \in U_n$. Then there exists $K \in N(K \geq n)$ such that

$$\rho(\sigma^K(\overline{\alpha}), \sigma^K\overline{\beta})) > \lambda.$$

Let α_i, β_i be as following:

$$\alpha_i := \begin{cases} \alpha_i^\circ & : \quad i \leq \overline{M} \\ \overline{\alpha_i} & : \quad n < i \\ v_i & : \quad \overline{M} < i \leq n(\text{ here } \alpha_{\overline{M}} v_{\overline{M}+1} \cdots v_n \alpha_{n+1} \text{ is admissible.}) \end{cases}$$

$$\beta_i := \begin{cases} \beta_i^\circ & : \quad i \leq \overline{M} \\ \overline{\beta_i} & : \quad n < i \\ w_i & : \quad \overline{M} < i \leq n(\text{ here } \beta_{\overline{M}} w_{\overline{M}+1} \cdots w_n \beta_{n+1} \text{ is admissible.}) \end{cases}$$

Then, it follows that

$$\rho(\alpha, \alpha^\circ) < \varepsilon, \qquad \rho(\beta, \beta^\circ) < \varepsilon$$

and

$$\rho(\sigma^K(\alpha), \sigma^K(\beta)) = \rho(\sigma^K(\overline{\alpha}), \qquad \sigma^K(\overline{\beta})) > \lambda.$$

Therefore $(\alpha, \beta) \leq U_n$, consequently, U_n is dense.

The fact that L_n is dense in $\Sigma_A^N \times \Sigma_A^N$, is proved by using lemma 2.2.8 in [3].

Now let G be the following set:

$$G := \{(\alpha, \beta) \in \Sigma_A^N \times \Sigma_A^N \big| \quad \begin{array}{l} \liminf_{n\to\infty} \rho[\sigma^n(\alpha), \sigma^n(\beta)] = 0 \\ \limsup_{n\to\infty} \rho[\sigma^n(\alpha), \sigma^n(\beta)] > 0\}. \end{array}$$

σ is chaotic in the sense of Li-Yorke, therefore there exists $\lambda > 0$ such that $U_n \neq \emptyset$, otherwise it follows that

$$G = (\bigcap_{n \in N} L_n) \cap (\bigcap_{n > \overline{M}} U_n).$$

Conseqently G is second category in $\Sigma_A^N \times \Sigma_A^N$. $\square$

Remark 1 *Obviously this theorem implies that a map in a metric space satisfying the criterion of Theorem 1 is generic chaotic.*

3 Application of the criteria of chaos to the anti-control of chaos

Concider a discrete dynamical system, chaotic or not, in the form

$$x_{n+1} = f(x_n), \quad n \geq 0, \tag{2}$$

where $f : D \to \mathbb{R}$ is a map and $D \subset \mathbb{R}$. The objective is to design a (simple) control input sequence, $\{u_n\}$, such that the output of the controled system

$$x_{n+1} = f(x_n) + u_n, \quad n \geq 0 \tag{3}$$

will be chaotic. In [10], they put the controller to be designed in the form

$$u_n = \mu g(x_n),$$

where μ is a parameter and $g : D \to \mathbb{R}$. For convenience, let us introduce the notation $F_\mu(x) := f(x) + \mu g(x)$.

In the previous research was concidered the case that f and g are continuously differentiable and f is bounded uniformly and has at least one fixed point. In [10] improved conditions were studied, that is, f, g are only continuous and monotonic in $\mathbb{R}$.

Now we get a result in which the condition on the controller g is reduced in comparison with the Theorem 3.1 of [10] by using our criterion of chaos obtained above.

Theorem 4 *Let $f, g : I \to \mathbb{R}$ be maps, where I is an interval in $\mathbb{R}$, and let a, p, q and b be constants with $a < p < q < b$ and, $K_1 := [a, p]$ and $K_2 :_= [q, b]$ be intervals in I. Assume that f and g satisfy the following conditions:*

(1) f and g are continuous on $K_1 \cup K_2$.

(2) There exists an irreducible 2×2matrix A and $\mu_0 \in \mathbb{R}$ such that

$$(f + \mu_0 g)(K_i) \supset \bigcup_{\substack{j \\ (A)_{ij}=1}} K_j, \quad (i, j = 1, 2).$$

Then there exists a neighbourhood $O(\mu_0) \subset \mathbb{R}$ of μ_0 and an invariant subset Λ_μ (for $\mu \in O(\mu_0)$) such that $F_\mu = f + \mu g$ is chaotic on Λ_μ in the sense of Devaney.

Proof.

Applying the Corollary 1 in this paper, this result follows from the above assumptions. □

Remark 2 *From the proof of the Theorem 3.1 of [10] concerning the chaotification, we can prove that f and g in the expression (3) satisfy the conditions of above Theorem 5 for the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ if f and g satisfy the conditions of the Theorem 3.1 of [10]. In other words, Theorem 5 is a generalization of the Theorem 3.1 of [10].*

Example 2 *Consider the discrete dynamical system (2), with*

$$f(x) = \begin{cases} x \sin \frac{1}{x} & : \quad x \in [-1, 0) \cup (0, 1] \\ 0 & : \quad x = 0 \end{cases}$$

This system is the same as the one of the Example 3.1 in [10]. Now the map g in the controller can be designed as follows:

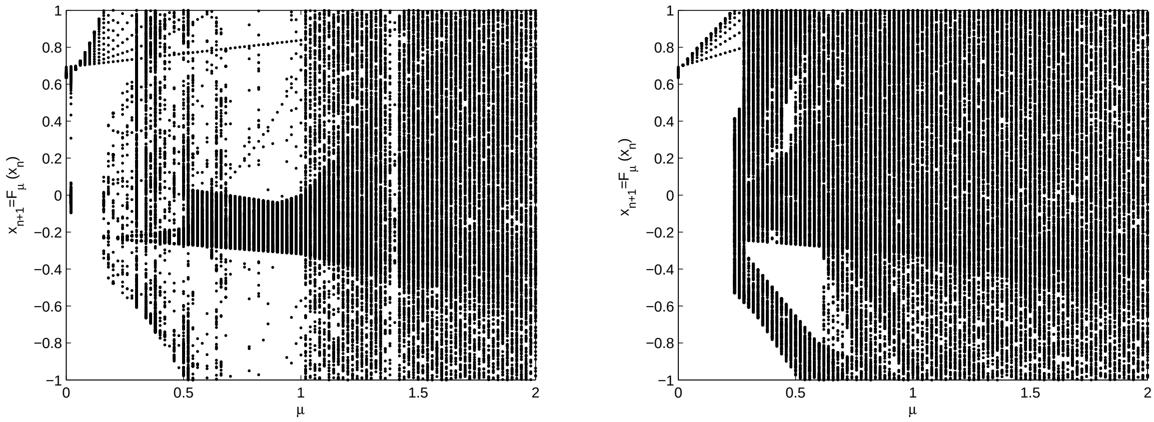

Figure 2: This shows the anti-control of chaos using the controller of $g = g_1(x)$ (left figure) and $g = g_2(x)$ (right figure) where $x_0 = 0.7$, $\Delta\mu$=0.02, n=1000 .

$$g_1(x) = \begin{cases} -\frac{1}{\pi} - x & : \quad x \in [-1, -\frac{1}{\pi}] \\ 3x - 1 - \frac{3}{\pi} & : \quad x \in [\frac{1}{\pi}, 1] \end{cases}$$

$$g_2(x) = \begin{cases} -3x - 1 - \frac{3}{\pi} & : \quad x \in [-1, -\frac{1}{\pi}] \\ -\frac{1}{\pi} + x & : \quad x \in [\frac{1}{\pi}, 1] \end{cases}$$

where g_1 and g_2 can be defined as any function on (-$\frac{1}{\pi}$, $\frac{1}{\pi}$).

Then map g_1 and g_2 obviousely do not satisfy the condition (3) of Theorem 3.1 in [18] discussed on the design of controllers. Now let $a = -1, p = -\frac{1}{\pi}$, q=$\frac{1}{\pi}$ and b=1, and let $K_1 = [-1, -\frac{1}{\pi}]$ and $K_2 = [\frac{1}{\pi}, 1]$. Then the maps f and g_1 satisfy the conditions of Theorem 5 with the irreducible matrix $A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, and also f and g satisfy them with the irreducible matrix $A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ for K_1, K_2 and $\mu = 1$. Therefore we can see that, there exist a neighbourhood O(1) in $\mathbb{R}$ and an invariant subset $\Lambda^i_\mu \subset [-1, -\frac{1}{\pi}] \cup [\frac{1}{\pi}, 1]$ (for $\mu \in O(1)$, i=1,2) so that $F^i_\mu = f + \mu g_i$ is chaotic in the sense of Devaney in Λ^i_μ (See figure 2).

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