The minimum number of blocks in Pairwise Balanced Designs with maximum block size 4 - the final cases

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November 3, 2005

Abstract

The minimum number of blocks having maximum size precisely four that are required to cover, exactly λ times, all pairs of elements from a set of cardinality v is denoted by $g_\lambda^{(4)}(v)$ (or $g^{(4)}(v)$ when $\lambda = 1$). All values of $g_\lambda^{(4)}(v)$ are known except for $\lambda = 1$ and $v = 17$ or 18. It is known that $30 \leq g^{(4)}(17) \leq 31$ and $32 \leq g^{(4)}(18) \leq 33$. In this paper we show that $g^{(4)}(17) \neq 30$ and $g^{(4)}(18) \neq 32$, thus finalising the determination of $g_\lambda^{(4)}(v)$ for all λ and v.

Introduction

Let K be a set of positive integers. A *pairwise balanced design* PBD(v, K) (denoted by $\boldsymbol{P}$) of order v with block sizes from K is a pair $\boldsymbol{P} = (V, \mathcal{B})$, where V is a finite set (the *point set*) of cardinality v and $\mathcal{B}$ is a family of subsets (called *blocks*) of V which satisfy the following properties:

(i) every pair of distinct elements of V occurs in exactly one block of $\mathcal{B}$;

(ii) if $B \in \mathcal{B}$, then $|B| \in K$.

A *partial* PBD(v, K) is defined similarly, with the difference that $(V, \mathcal{B})$ satisfies instead of property (i) the property:

(i') every pair of distinct elements of V occurs in at most one block of $\mathcal{B}$.

The value $g_\lambda^{(4)}(v)$ (or $g^{(4)}(v)$ if $\lambda = 1$) has been investigated in a number of papers including those in the reference list. The papers [1, 2, 3, 5] provide new results or a survey of known results. From [8] it is known that $30 \leq g^{(4)}(17) \leq 31$. In [7] an upper bound for $g^{(4)}(18)$ is established by constructing a PBD on 18 points with 33 blocks. Independently, [4] and [6] proved that $32 \leq g^{(4)}(18)$. In this paper, we show that $g^{(4)}(17) = 31$ by showing that there does not exist a PBD$(17, \{2, 3, 4\})$ with exactly 30 blocks, and that $g^{(4)}(18) = 33$ by showing that there does not exist a PBD$(18, \{2, 3, 4\})$ with exactly 32 blocks. The proofs involve showing that no partial design can be completed to be a PBD$(v, \{2, 3, 4\})$ with exactly b blocks for $(v, b) = (17, 30)$ or $(18, 32)$.

This has been achieved by case analysis and computational techniques improved by some preliminary analysis. The reason for the reliance on computer searches is that the analytic arguments became quite detailed and long as the number of subcases increased. The results in this paper, combined with other papers referenced, means that $g_\lambda^{(4)}(v)$ is now known for all cases.

1 Preliminaries

We begin by introducing some terminology and notation. In this section the pair $(v, b) = (17, 30)$ or $(18, 32)$. Let g_i be the number of blocks of size i for $i = 2, 3, 4$. Then counting pairs of points in two ways gives

$$g_2 + 3g_3 + 6g_4 = \binom{v}{2}.$$

Also, $g_2 + g_3 + g_4 = b$. There are three integer solutions to these two equations for $(v, b) = (17, 30)$, namely $(g_2, g_3, g_4) = (1, 13, 16), (4, 8, 18)$ and $(7, 3, 20)$. For $(v, b) = (18, 32)$ the integer solutions are $(g_2, g_3, g_4) = (0, 13, 19)$, $(3, 8, 21)$ and $(6, 3, 23)$.

Let $\mathcal{B}'$ be a subset of the block set $\mathcal{B}$. The *volume* of $\mathcal{B}'$ is $V(\mathcal{B}') = \sum_{B \in B'} |B|$. If two distinct points occur in the same block then it is said that they *are a pair* or *are paired*.

A point x has *point type* $P(x) = 2^{\alpha_2}3^{\alpha_3}4^{\alpha_4}$ or $(\alpha_2, \alpha_3, \alpha_4)$ if x is contained in exactly α_2 blocks of size 2 (*doubles*), α_3 blocks of size 3 (*triples*) and α_4 blocks of size 4 (*quads*). Each each point type must satisfy

$$\alpha_2 + 2\alpha_3 + 3\alpha_4 = v - 1 \quad \text{and} \quad \alpha_k \le g_k. \tag{1}$$

The *point type distribution* of a collection of points is a collection of values expressed in the form $d \times 2^{\alpha_2}3^{\alpha_3}4^{\alpha_4}$ which indicates that there are d points which each occur in α_2 blocks of size 2, α_3 blocks of size 3, and α_4 blocks of size 4.

Let $d_j \times 2^{\alpha_{2,j}}3^{\alpha_{3,j}}4^{\alpha_{4,j}}$ with $1 \le j \le t$ be the point type distribution of some PBD. Then the following equations are used implicitly throughout this paper.

$$\sum_{j=1}^{t} d_j = v \quad \text{and} \quad \sum_{j=1}^{t} d_j \alpha_{k,j} = k g_k \quad \text{for each } k \in \{2, 3, 4\}. \tag{2}$$

2 Case $(v, b) = (17, 30)$

Given the relations in (1) with $v = 17$ it is easily checked that in a PBD with $(g_2, g_3, g_4) = (1, 13, 16)$ then the only possible point types are

$$2^1 4^5, 3^2 4^4, 2^1 3^3 4^3, 3^5 4^2, 2^1 3^6 4^1, 3^8;$$

with $(g_2, g_3, g_4) = (4, 8, 18)$ then the only possible point types are

$$2^1 4^5, 2^2 3^1 4^4, 3^2 4^4, 2^4 4^4, 2^1 3^3 4^3, 2^3 3^2 4^3, 3^5 4^2, 2^2 3^4 4^2, 2^4 3^3 4^2, 2^1 3^6 4^1, 2^3 3^5 4^1, 3^8, 2^2 3^7, 2^4 3^6;$$

and with $(g_2, g_3, g_4) = (7, 3, 20)$ then the only possible point types are

$$2^1 4^5, 2^2 3^1 4^4, 3^2 4^4, 2^4 4^4, 2^1 3^3 4^3, 2^3 3^2 4^3, 2^5 3^1 4^3, 2^7 4^3, 2^4 3^3 4^2, 2^6 3^2 4^2, 2^7 3^3 4^1.$$

2.1 Subcase $(g_2, g_3, g_4) = (1, 13, 16)$

Assume that there is a point x with point type 3^8. Then 16 points occur once each in the triples which include x. By consideration of the possible point types it can be seen that there is no point which occurs in exactly one triple. So each of the 16 points occur in a triple without x. This is a contradiction as there are 5 remaining triples. Note that as $g_2 = 1$ there are at most 2 points of type $2^1 4^5$.

Beginning with a given number of points of point type $2^1 4^5$ it is easily checked using (2) that the only possible point type distributions are:

17.1: $14 \times 3^2 4^4, 2 \times 2^1 3^3 4^3, 1 \times 3^5 4^2$;

17.2: $15 \times 3^2 4^4, 1 \times 2^1 3^3 4^3, 1 \times 2^1 3^6 4^1$;

17.3: $1 \times 2^1 4^5, 13 \times 3^2 4^4, 1 \times 2^1 3^3 4^3, 2 \times 3^5 4^2$;

17.4: $1 \times 2^1 4^5, 14 \times 3^2 4^4, 1 \times 3^5 4^2, 1 \times 2^1 3^6 4^1$; and

17.5: $2 \times 2^1 4^5, 12 \times 3^2 4^4, 3 \times 3^5 4^2$.

The only feasible point types are: $2^1 4^5, 3^2 4^4, 2^1 3^3 4^3, 3^5 4^2, 2^1 3^6 4^1$.

2.2 Subcase $(g_2, g_3, g_4) = (4, 8, 18)$

Given that there is only one possible point type involving 5 quads, namely a point of type $2^1 4^5$, and given that $g_2 = 4$, $\alpha_4 \leq 5$ and the volume of the quads is 72, there are between 4 and 8 occurrences of the point type $2^1 4^5$. This means that a point x of point type 3^8 is not possible as if there is more than one point of type $2^1 4^5$ then there are not 16 distinct points available to complete the triples containing x.

If there are 8 points of type $2^1 4^5$ then there are 9 points which have point types $3^2 4^4$ or $3^5 4^2$, which are the only point types which do not include doubles. The only possible choice for the 9 points is that they have point type distribution $7 \times 3^2 4^4, 2 \times 3^5 4^2$. Then the five triples containing one of the points with point type $3^5 4^2$ cannot be completed using the remaining eight points.

If there are 7 points of type $2^1 4^5$ then there is a point of type $2^1 3^3 4^3$ or $2^1 3^6 4^1$ to fill the doubles and there are 9 points which have point types $3^2 4^4$ or $3^5 4^2$. The point type $2^1 3^6 4^1$ cannot occur as the 6 triples cannot be completed with 9 points. If the point type $2^1 3^3 4^3$ occurs then the remaining 9 points have point type distribution $8 \times 3^2 4^4, 1 \times 3^5 4^2$. Again, if there are five triples containing the point type $3^5 4^2$, then these cannot be completed using the remaining eight points.

With at most 6 points of type $2^1 4^5$ and the volume of the quads being 72, each of the remaining points occur in at least two quads, so the point types $2^1 3^6 4^1, 2^3 3^5 4^1, 2^2 3^7, 2^4 3^6$ cannot occur.

Beginning with a given number of points of point type $2^1 4^5$ it is easily checked that the only possible point type distributions are:

17.6: $4 \times 2^1 4^5, 11 \times 3^2 4^4, 2 \times 2^2 3^1 4^4$;

17.7: $4 \times 2^1 4^5, 12 \times 3^2 4^4, 1 \times 2^4 4^4$;

17.8: $5 \times 2^1 4^5, 10 \times 3^2 4^4, 1 \times 2^2 3^1 4^4, 1 \times 2^1 3^3 4^3$;

17.9: $5 \times 2^1 4^5, 11 \times 3^2 4^4, 1 \times 2^3 3^2 4^3$;

17.10: $6 \times 2^1 4^5, 9 \times 3^2 4^4, 2 \times 2^1 3^3 4^3$;

17.11: $6 \times 2^1 4^5, 9 \times 3^2 4^4, 1 \times 2^2 3^1 4^4, 1 \times 3^5 4^2$; and

17.12: $6 \times 2^1 4^5, 10 \times 3^2 4^4, 1 \times 2^2 3^4 4^2$.

The only feasible point types are: $2^1 4^5, 2^2 3^1 4^4, 3^2 4^4, 2^4 4^4, 2^1 3^3 4^3, 2^3 3^2 4^3, 3^5 4^2, 2^2 3^4 4^2$.

Point type distribution 17.11 can be excluded by considering pairings of points of type $2^1 4^5$. At least two pairs of the six points of type $2^1 4^5$ occur in doubles. So that all points are paired, there are six pairs of points of type $2^1 4^5$ in quads which also contain a point of type $3^5 4^2$. Therefore the remaining 16 quads contain 24 occurrences of points of type $2^1 4^5$, resulting in at least 8 pairs of this point type in these quads. So there are at least 16 pairings of points of type $2^1 4^5$ and this exceeds the required 15 pairs of this point type.

2.3 Subcase $(g_2, g_3, g_4) = (7, 3, 20)$

Given that $g_4 = 20$ and $g_3 = 3$ there is one possible point type distribution: $12 \times 2^1 4^5, 4 \times 3^2 4^4, 1 \times 2^2 3^1 4^4$. This leads to a contradiction as each of 4 points occur in two of the three triples, and thus one pair of the points must occur in two triples. So there is no PBD$(17, \{2, 3, 4\})$ with the configuration $(g_2, g_3, g_4) = (7, 3, 20)$.

3 Case $(v, b) = (18, 32)$

Given the relations in (1) with $v = 18$ it is easily checked that in a PBD with $(g_2, g_3, g_4) = (0, 13, 19)$ then the only possible point types are

$$3^1 4^5, 3^4 4^3, 3^7 4^1;$$

and with $(g_2, g_3, g_4) = (3, 8, 21)$ then the only possible point types are

$$3^1 4^5, 2^2 4^5, 2^1 3^2 4^4, 2^3 3^1 4^4, 3^4 4^3, 2^2 3^3 4^3, 2^1 3^5 4^2, 2^3 3^4 4^2, 3^7 4^1, 2^2 3^6 4^1, 2^1 3^8 4^0, 2^3 3^7 4^0.$$

A PBD with $(g_2, g_3, g_4) = (6, 3, 23)$ is not possible as the volume of the quads is 92, requiring a point to occur in at least 6 quads, which means that it is paired with another point at least twice in the quads.

3.1 Subcase $(g_2, g_3, g_4) = (0, 13, 19)$

Since $\alpha_4 \le 5$ for all point types, as there are no partial point types 4^2 or 4^4, and as the volume of the quads is 76, there are at least 11 occurrences of the point type $3^1 4^5$.

The following two point type distributions are possible:

18.1: $11 \times 3^1 4^5, 7 \times 3^4 4^3$ or
18.2: $12 \times 3^1 4^5, 5 \times 3^4 4^3, 1 \times 3^7 4^1$.

Here, point type distribution 18.2 can immediately be eliminated as follows. Suppose that there are two points x, y of type $3^4 4^3$ which are not paired in a triple, then there are eight distinct triples containing x, y. There are at least five more triples containing the point z of type $3^7 4^1$. Each other point of type $3^4 4^3$ is contained in at least one triple which contains none of the three points x, y, z. So all together we need at least 14 distinct triples for 18.2, a contradiction to $g_3 = 13$.

Thus all points of type $3^4 4^3$ are paired within the triples. All of these points are paired in a triple with the $3^7 4^1$ point as otherwise we would need 14 triples to include all of the points of type $3^4 4^3$ without a duplicate pair. So the remaining 6 triples contain 15 occurrences of points of type $3^4 4^3$. This means that there are at least 12 pairs of points of type $3^4 4^3$, which exceeds the required number of 10 pairs.

3.2 Subcase $(g_2, g_3, g_4) = (3, 8, 21)$

To reduce the number of combinations to be considered it is useful to reduce the number of occurrences of a given point type and the number of occurrences of different point types.

Since $\alpha_4 \le 5$ for all point types and the volume of the quads is 84, the point types $3^1 4^5$ and $2^2 4^5$ occur between 12 and 16 times in each possible PBD and there can be at most one point type including the partial point types 4^0 or 4^1. The point type $2^2 4^5$ can occur at most 3 times as $g_2 = 3$. Therefore the point type $3^1 4^5$ occurs at least 9 times.

If the point type $2^1 3^8 4^0$ or $2^3 3^7 4^0$ occurs then there must be 16 of the point types $3^1 4^5$ or $2^2 4^5$, and one of the point types $2^1 3^2 4^4$ or $2^3 3^1 4^4$. It is easy to check that there is a duplicated pair in either the doubles or triples for each combination of the point types which satisfy the volume constraints on the collection of blocks of each given size.

If the partial point type 4^1 occurs exactly once then there must be 16 points with the partial point type 4^5 and one with partial point type 4^3, or 15 points with the partial point type 4^5 and two points with the partial point type 4^4. Hence, if the point type $3^7 4^1$ or $2^2 3^6 4^1$ occurs exactly once and there are 16 of the point types $3^1 4^5$ or $2^2 4^5$, then there must be one of the point type $2^2 3^3 4^3$ as the point of type $3^4 4^3$ would cause a duplicate pair in the triples. There is now a duplicate pair in the doubles or triples for each volume-feasible combination of the point types. In the case of a point of type $3^7 4^1$ or $2^2 3^6 4^1$ and 15 points with partial point type 4^5 the only possible point type distributions are $13 \times 3^1 4^5, 2 \times 2^2 4^5, 2 \times 2^1 3^2 4^4, 1 \times 3^7 4^1$ or $14 \times 3^1 4^5, 1 \times 2^2 4^5, 2 \times 2^1 3^2 4^4, 1 \times 2^2 3^6 4^1$.

The point type $3^1 4^5$ cannot occur exactly 16 times as the two remaining points cannot fill the doubles. If the point type $3^1 4^5$ occurs exactly 15 times then the remaining three points must have point type $2^2 3^3 4^3$. Then there must be two points which are paired in both a double and a triple, so this is not possible.

By considering the number of times that the point type $3^1 4^5$ can occur (between 12 and 14 inclusive), it is easy to check that the following 16 point type distributions are possible (including the 2 reproduced from above):

18.3: $12 \times 3^1 4^5, 6 \times 2^1 3^2 4^4$;
18.4: $12 \times 3^1 4^5, 1 \times 2^2 4^5, 4 \times 2^1 3^2 4^4, 1 \times 3^4 4^3$;
18.5: $12 \times 3^1 4^5, 2 \times 2^2 4^5, 2 \times 2^1 3^2 4^4, 2 \times 3^4 4^3$;
18.6: $13 \times 3^1 4^5, 3 \times 2^1 3^2 4^4, 1 \times 2^3 3^1 4^4, 1 \times 3^4 4^3$;
18.7: $13 \times 3^1 4^5, 4 \times 2^1 3^2 4^4, 1 \times 2^2 3^3 4^3$;
18.8: $13 \times 3^1 4^5, 1 \times 2^2 4^5, 2 \times 2^1 3^2 4^4, 1 \times 3^4 4^3, 1 \times 2^1 3^3 4^3$;
18.9: $13 \times 3^1 4^5, 1 \times 2^2 4^5, 3 \times 2^1 3^2 4^4, 1 \times 2^1 3^5 4^2$;
18.10: $13 \times 3^1 4^5, 2 \times 2^2 4^5, 2 \times 3^4 4^3, 1 \times 2^2 3^3 4^3$;
18.11: $13 \times 3^1 4^5, 2 \times 2^2 4^5, 1 \times 2^1 3^2 4^4, 1 \times 3^4 4^3, 1 \times 2^1 3^5 4^2$;
18.12: $13 \times 3^1 4^5, 2 \times 2^2 4^5, 2 \times 2^1 3^2 4^4, 1 \times 3^7 4^1$;
18.13: $14 \times 3^1 4^5, 2 \times 2^1 3^2 4^4, 2 \times 2^2 3^3 4^3$;
18.14: $14 \times 3^1 4^5, 2 \times 2^1 3^2 4^4, 1 \times 2^3 3^1 4^4, 1 \times 2^1 3^5 4^2$;
18.15: $14 \times 3^1 4^5, 3 \times 2^1 3^2 4^4, 1 \times 2^3 3^4 4^2$;
18.16: $14 \times 3^1 4^5, 1 \times 2^2 4^5, 1 \times 3^4 4^3, 2 \times 2^2 3^3 4^3$;
18.17: $14 \times 3^1 4^5, 1 \times 2^2 4^5, 1 \times 2^1 3^2 4^4, 1 \times 2^2 3^3 4^3, 1 \times 2^1 3^5 4^2$;
18.18: $14 \times 3^1 4^5, 1 \times 2^2 4^5, 2 \times 2^1 3^2 4^4, 1 \times 2^2 3^6 4^1$.

The only feasible point types are: $3^1 4^5, 2^2 4^5, 2^1 3^2 4^4, 3^4 4^3, 2^2 3^3 4^3, 2^1 3^5 4^2, 2^3 3^4 4^2, 2^2 3^6 4^1$.

Again, one of these point type distributions, 18.12, can be eliminated. The two points x, y of type $2^2 4^5$ need to be paired with the $3^7 4^1$ point in a common quad. This leads to a contradiction as x, y must also be paired in a double.

4 Search for the Desired PBDs

In this section, we describe the method used to search by computer for a PBD$(17, \{2, 3, 4\})$ with 30 blocks or a PBD$(18, \{2, 3, 4\})$ with 32 blocks. There are three basic steps. In the first step we determine for each point type distribution from the previous section the numerical conditions which need to be satisfied when distributing the point types to the blocks. Then, we construct with respect to these distributions all suitable partial PBDs containing only blocks of size 2 and 3 – the so called *prestructure*. In a third step it is attempted to complete the prestructures obtained with blocks of size 4.

4.1 Distributing the Point Types to the Blocks

Given a possible point type distribution $d_j \times 2^{\alpha_{2,j}} 3^{\alpha_{3,j}} 4^{\alpha_{4,j}}$ where $1 \leq j \leq t$ we have for each single block to answer the question how many points of each type occur in that block. This leads to the following definition. The *single block type* of a block is a collection of values expressed in the form $s \times 2^{\alpha_2} 3^{\alpha_3} 4^{\alpha_4}$ which indicates that there are exactly s points in the block of point type $2^{\alpha_2} 3^{\alpha_3} 4^{\alpha_4}$. If it is clear from the context that we refer to a particular point type distribution we simply write $\mathbf{s} = (s_1, \ldots, s_t)$ instead of $s_1 \times 2^{\alpha_{2,1}} 3^{\alpha_{3,1}} 4^{\alpha_{4,1}}, \ldots, s_t \times 2^{\alpha_{2,t}} 3^{\alpha_{3,t}} 4^{\alpha_{4,t}}$. The *single block type distribution* of a collection of blocks is a collection of values expressed in the form $c \times (s_1, \ldots, s_t)$ which indicates that there are exactly c blocks of single block type $(s_1, \ldots, s_t)$. Again, if it is clear that we refer to single block types $\mathbf{s}_1, \ldots, \mathbf{s}_r$ we briefly write $\mathbf{c} = (c_1, \ldots, c_r)$.

In the following we study the numerical conditions which need to be satisfied by the single block types and single block type distribution of a partial PBD(v, K) with point type distribution $d_j \times 2^{\alpha_{2,j}} 3^{\alpha_{3,j}} 4^{\alpha_{4,j}}$ where $1 \leq j \leq t$. Let $\mathbf{s} = (s_1, \ldots, s_t)$ be the single block type of some block in the partial PBD. Then clearly

$$k(\mathbf{s}) := \sum_{j=1}^{t} s_j \text{ is a size from } K; \tag{3}$$

$$s_j \leq d_j \quad \text{for } j = 1, \ldots, t; \text{ and} \tag{4}$$

$$\alpha_{k(S),j} = 0 \quad \text{implies } s_j = 0 \text{ for } j = 1, \ldots, t. \tag{5}$$

Furthermore, let $\mathbf{c} = (c_1, \ldots, c_r)$ be the single block type distribution of the partial PBD with respect to the set of distinct single block types $\mathbf{s}_1, \ldots, \mathbf{s}_h = (s_{h,1}, \ldots, s_{h,t}), \ldots, \mathbf{s}_r$. If we define $I_k = \{i \in \{1, \ldots, r\} : k(\mathbf{s}_i) = k\}$ to be the set of indices i such that $\mathbf{s}_i$ belongs to a block of size k, then

$$\sum_{h \in I_k} c_h s_{h,j} = d_j \alpha_{k,j} \quad \text{for all } j \in \{1, \ldots, t\} \text{ and all } k \in K, \tag{6}$$

$$\sum_{h=1}^{r} c_h s_{h,j}(s_{h,j}-1) \le d_j(d_j-1) \quad \text{for all } j \in \{1,\ldots,t\}, \tag{7}$$

and

$$\sum_{h=1}^{r} c_h s_{h,j} s_{h,\ell} \le d_j d_\ell \quad \text{for all } j,\ell \in \{1,\ldots,t\}, i \ne \ell. \tag{8}$$

In (6) points of the j-th point type which are contained in blocks of size k are counted in two ways, (7) counts pairs of points of the same point type, and in (8) pairs of points of distinct point types are counted. Equality holds in (7) and (8) for all $j,\ell \in \{1,\ldots,t\}$ simultaneously if and only if the partial PBD is a PBD.

The task is now to determine all possible single block types for each possible point type distribution $(d_1,\ldots,d_t)$ in Section 2 and Section 3. That is, determine all vectors $(s_1,\ldots,s_t)$ which satisfy (3), (4), and (5). Once this accomplished, we compute for the r possible single block types $\mathbf{s}_1,\ldots,\mathbf{s}_r$ obtained all possible single block type distributions. That is, we complete all vectors $(c_1,\ldots,c_r)$ which satisfy (6), (7), and (8).

The results for $(v,b) = (17,30)$ and $(18,32)$ are quite lengthy, therefore, we decided to include the cases 17.1–17.3 as an example in Appendix A and to put the rest of the cases into an extra file which we provide at the web page `http://www.math.uni-rostock.de/~mgruttm/g4v17v18/index.html`. All material from that web page is also available from authors upon request.

4.2 Search for Suitable Prestructures

Our aim is to find all non-isomorphic prestructures with blocks of size 2 and 3 whose point type distribution and single block type distribution is the restriction to blocks of size 2 and 3 of one of the possible point type distributions and single block type distributions listed in Appendix A or our web page. This is achieved with a backtracking algorithm. Details of the algorithm are already described in [6]. To avoid unnecessary repetitions we refer the interested reader to that paper and just mention that backtracking searches for the dual of the prestructure are used rather than for the prestructure itself to increase the speed of the isomorphism testing. The number of prestructures found by the backtrack algorithm are also given in Appendix A or at the web page. In some cases several thousand non-isomorphic prestructures are found which, of course, can not be listed here. Therefore we provide all prestructures constructed also on our web page `http://www.math.uni-rostock.de/~mgruttm/g4v17v18/index.html`.

Note that it might well be that although two single block type distributions are distinct there restrictions to blocks of size 2 and 3 are the same. Hence, the corresponding prestructures are exactly the same, but they are treated differently in the next step so they are counted repeatedly.

4.3 Search for a Completion of a Given Prestructure

By another backtracking algorithm we try to find a completion of each of the prestructures constructed using blocks of size 4. This approach takes into account that we know the single block types of the missing quads and also considers the automorphisms of the prestructure to speed up the search. More details are explained in [6]. We found that none of the prestructures were completable to a PBD.

5 Conclusion

We have computed in the previous sections all non-isomorphic prestructures for a PBD$(17, \{2, 3, 4\})$ with exactly 30 blocks or a PBD$(18, \{2, 3, 4\})$ with exactly 32 blocks and have shown that none of these prestructures is completable. Therefore, we have established:

Theorem 5.1. *There does not exist a PBD on 17 points with 30 blocks of size at most* 4 *nor a PBD on 18 points with 32 blocks of size at most* 4.

Corollary 5.1. $g^{(4)}(17) = 31$ *and* $g^{(4)}(18) = 33$.

A Single Block Type Distributions

Below is a list of all possible single block type distributions for each of the point type distributions 17.1 – 17.3. All possible single block type distributions for each of the point type distributions 17.1 – 17.10, 17.12, 18.1, 18.3 – 18.11, 18.13 – 18.18 are provided on the web page `http://www.math.uni-rostock.de/~mgruttm/g4v17v18/index.html`. Each parameter set is presented in the form:

number: point type distribution
possible single block types
possible single block type distribution : number of non-isomorphic prestructures

17.1: $1 \times 2^0 3^5 4^2, 2 \times 2^1 3^3 4^3, 14 \times 2^0 3^2 4^4$

(0,2,0), (1,2,0), (1,1,1), (1,0,2), (0,2,1), (0,1,2), (0,0,3), (1,2,1), (1,1,2), (1,0,3), (0,2,2), (0,1,3), (0,0,4)

(1,0,2,3,0,4,4,0,0,2,0,6,8) : 51970
(1,0,1,4,0,5,3,0,1,1,0,5,9) : 35016
(1,0,0,5,0,6,2,0,2,0,0,4,10) : 3612

17.2: $1 \times 2^1 3^6 4^1, 1 \times 2^1 3^3 4^3, 15 \times 2^0 3^2 4^4$

(1,1,0), (1,1,1), (1,0,2), (0,1,2), (0,0,3), (1,1,2), (1,0,3), (0,1,3), (0,0,4)

(1,0,6,3,4,0,1,3,12) : 925

17.3: $2 \times 2^0 3^5 4^2, 1 \times 2^1 3^3 4^3, 13 \times 2^0 3^2 4^4, 1 \times 2^1 3^0 4^5$

(0,1,0,1), (2,1,0,0), (2,0,1,0), (1,1,1,0), (1,0,2,0), (0,1,2,0), (0,0,3,0), (2,1,1,0), (2,1,0,1), (2,0,2,0), (2,0,1,1), (1,1,2,0), (1,1,1,1), (1,0,3,0), (1,0,2,1), (0,1,3,0), (0,1,2,1), (0,0,4,0), (0,0,3,1)

(1,1,0,0,8,2,2,0,0,0,0,0,0,2,2,3,0,6,3) : 1632
(1,0,1,2,6,1,3,0,0,0,0,0,0,2,2,3,0,6,3) : 12921
(1,0,1,1,7,2,2,0,0,0,0,1,0,1,2,2,0,7,3) : 9567
(1,0,0,2,8,1,2,0,0,1,0,0,0,0,2,3,0,7,3) : 2518
(1,0,1,0,8,3,1,0,0,0,0,2,0,0,2,1,0,8,3) : 874
(1,0,0,0,10,3,0,1,0,0,0,0,0,0,2,2,0,8,3) : 57
(1,0,0,2,8,1,2,0,0,0,1,0,0,2,0,3,0,6,4) : 2518
(1,0,0,1,9,2,1,0,0,0,1,1,0,1,0,2,0,7,4) : 1028
(1,0,0,0,10,3,0,0,0,0,1,2,0,0,0,1,0,8,4) : 57

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