On Frobenius and tensor induction

Reinhard Knörr*

Institut für Mathematik, Universität Rostock

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Abstract: Tensor inducing a Frobenius induced character leads to a sum of characters where each summand is given by first tensor induction, then Frobenius induction. This qualitative statement is made precise. As an example, tensor induction of monomial characters is considered. *Key Words: group representation, character theory, tensor induction*
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The starting point for this paper was the question how to describe the character which results from applying tensor induction to a sum of characters or to a character induced from a subgroup; this is a problem very similar in spirit to working out a power of a sum but it poses a notational problem. The description offered here uses the concept of inductible maps and their induction. The precise definition of inductible maps and a description how they can be induced are given in Section 2; it uses the language of G-sets (where G is a group, mostly assumed finite) and G-maps. Consequently, these form the topic of Section 1, which contains some basic lemmas and the definition of the tensor product of two G-sets. Inducing an inductible map leads from (generalized) characters to (generalized) characters and is flexible enough to include addition, multiplication, ordinary (Frobenius) induction and tensor induction of class functions; it also describes Mackey decomposition at no extra cost. The answer to the question above is then given in Section 3. As a corollary, one finds that tensor inducing a sum of monomial characters leads again to a sum of monomial characters. In Section 4, transitivity of this 'new' (it isn't) induction process is proved by giving an explicit description of the sets and maps involved.

*Correspondence: Reinhard Knörr, Institut für Mathematik, Universität Rostock, D-18051 Rostock, Germany; E-mail: reinhard.knoerr@uni-rostock.de

1 G-Sets

We need a few facts about G-sets. They are all very elementary and probably well known. We state them to get the generalities out of the way. Notation is standard; in particular, stabilizers in G are denoted by subscripts. Maps are mostly written on the right, as is the G-action. In the following, X and Y are G-sets.

1.1 Lemma: *Orbits*

Let $\alpha : X \to Y$ be a G-map and let $Y = \dot{\bigcup\limits_{j\in J}} jG$. For every j, let $X_j = \{x \in X \mid x\alpha = j\}$. Then X_j is a G_j-set (possibly empty). If $X_j = \dot{\bigcup\limits_{i\in I_j}} iG_j$, then $X = \dot{\bigcup\limits_{i\in I}} iG$, where $I = \dot{\bigcup\limits_{j\in J}} I_j$.

Proof: Easy.

1.2 Lemma: *G-Maps*

Let X and Y be G-sets and let $X = \dot{\bigcup\limits_{i\in I}} iG$. Denote $Y^i = \mathrm{Fix}_Y(G_i)$, the set of fixed points, and let $[X, Y]_G$ be the set of G-maps $X \to Y$. Then $\alpha \mapsto (i\alpha)_{i\in I}$ is a bijection from $[X, Y]_G$ onto the direct product $\prod\limits_{i\in I} Y^i$.

Proof: The inverse sends $(y_i)_{i\in I}$ to the G-map α well-defined by $ig\alpha = y_i g$.

In the following, as in the next sections, R is a commutative ring with 1.

1.3 Lemma: *Distributivity*

Keep the notation of the previous lemma and assume that X and Y are finite. Further let $T : Y \to R$ be any map. Then the following holds:

$$\prod_{i\in I} \sum_{y\in Y^i} T(y) = \sum_{\alpha\in[X,Y]_G} \prod_{i\in I} T(i\alpha).$$

Let in addition $\mu : Y \to X$ be a G-map, and denote $Y(i) = Y(i, \mu) = \{y \in Y^i \mid y\mu = i\}$. Then

$$\prod_{i\in I} \sum_{y\in Y(i)} T(y) = \sum_{\substack{\alpha\in[X,Y]_G \\ \alpha\mu=\mathrm{id}_X}} \prod_{i\in I} T(i\alpha) \ .$$

Proof: The first statement is just distributivity in R combined with 1.2. The second follows from this by observing that for $\alpha \in [X, Y]_G$, one has $\alpha\mu = \mathrm{id}_X$ if and only if $i\alpha \in Y(i)\ \forall i \in I$.

1.4 Definition: *Tensor products of G-sets*

Only part of the following construction is needed in this paper. Still, it may clarify the concept to state it in more generality. So let X and Y be G-sets. Then their tensor product $X \otimes_G Y = (X \times Y)/G$ is by definition simply the set of G-orbits on $X \times Y$; one then denotes by $x \otimes y$ the orbit containing (x, y) . This will look more familiar if Y is considered as a *left* G-set by defining $gy = yg^{-1}$ because then $xg \otimes y = x \otimes gy$.

1.5 Remark:

So far, $X \otimes_G Y$ is just a set, but it will be an H-set in the obvious way if Y is a G-H-biset, i.e. a $G \times H$-set. Similarly, the H-maps $[Y, Z]_H$ form a G-set, if Z is an H-set and Y a G-H-biset.
The name 'tensor product' is justified by the fact that the tensor functor is adjoint to the Hom functor: if X is a G-set, Y is G-H-biset and Z is an H-set, then

$$[X \otimes_G Y, Z]_H \cong [X, [Y, Z]_H]_G \quad \text{(naturally)} \ .$$

This is easily seen by observing that the natural isomorphism of sets

$$[X \times Y, Z] \cong [X, [Y, Z]]$$

is an $G \times H$-map, and then taking fixed points. We concentrate here on two special instances:
First, let $G \leq H$ and $Y = {}_G H_H$. Then $X \otimes_G H_H$ is often written as X^H and called the induced H-set (this will be used below). Since $[H, Z]_H \cong Z_{|G}$ as G-sets, the isomorphism above becomes $[X^H, Z]_H \cong [X, Z_{|G}]_G$. This is a form of Frobenius reciprocity.
Second, let $H \leq G$. With $Y = {}_G G_H$ then, $X \otimes_G G_H \cong X_{|H}$, so the above isomorphism reads $[X_H, Z]_H \cong [X, [G, Z]_H]_G$. A certain amount of confusion is perhaps generated by the fact that the functor $Z \mapsto [G, Z]_H$ is known as tensor induction (instead of 'Hom induction' or some such) and written as $Z^{\otimes G}$. Then the isomorphism can be rephrased again as $[X_{|H}, Z]_H \cong [X, Z^{\otimes G}]_G$. This may be called Dress reciprocity.
The reason for the name 'tensor induction' will become clear in 2.2 below. The reader is referred to [2], §80 for more information. In particular, Theorems 80.26 and 80.37 in [2] (minus the finiteness assumptions made there) are precisely Frobenius and Dress reciprocity.

1.6 Remarks:

Let $H \leq G$.

(i) If the H-set X contains just one element, then X^G is sometimes also written as $[G : H]$ and is isomorphic to the G-set consisting of the (right) cosets of H in G. In this case, $[X, Z_{|H}]_H$ is naturally isomorphic to $\mathrm{Fix}_Z(H)$, so $[[G : H], Z]_G \cong \mathrm{Fix}_Z(H)$. With different notation, this is a special case of Lemma 1.2 (assume that X is transitive).

(ii) Let $N \triangleleft G$ and $\overline{G} = G/N$. If X is a G-set, we can form the $\overline{G}$-set $\overline{X} = X/N$ consisting of the N-orbits on X; clearly $\overline{X}$ is isomorphic to $X \otimes_G \overline{G}_{\overline{G}}$.
A natural example of this kind is $X = N$ with G acting by conjugation. Then X/N consists of the conjugacy classes of N, permuted by G (or $\overline{G}$).

(iii) A word of warning is in order: If M is an RH-module, then it is also an H-set. One can therefore use it to construct the induced G-set $M \otimes_H G$ as just described. However, it is *not* a good idea to denote the result by M^G, because this notation is routinely used for the Frobenius induced of M, i.e. for $M \otimes_{RH} RG$, which is a G-module (and therefore also a G-set). If $H \neq G$, the induced set and the induced module are not isomorphic as G-sets: the induced module contains a point (namely 0) fixed by G, whereas all stabilizers of elements of the induced set are conjugate to subgroups of H.

(iv) On the positive side, the notation suggests the correct result when inducing permutation modules, i.e. $R[X]^G \cong R[X^G]$ for any H-set X (module induction on the left, set induction on the right).

(v) To describe the induced module M^G as G-set (for an H-module M), one can use tensor induction of sets:

$$[G, M]_H \cong M^G$$

if the index $|G : H|$ is finite. If $\{r_1, \ldots, r_n\}$ is a set of right coset representatives for H in G, then

$$[G, M]_H \ni a \mapsto \sum_i r_i^{-1} a \otimes r_i \in M^G$$

is an isomorphism of G-sets, in fact of G-modules (note that $[G, M]_H$ has a natural G-module structure). Moreover, the map is natural and independent of the choice of the r_i's.
For RH-modules, the notation $M^{\otimes G}$ (which was used bevor as synonymous with $[G, M]_H$) will be given *another* meaning in 2.1.

(vi) For any H-set, $X \ni x \mapsto x \otimes 1$ defines an injective H-map $X \to X^G$, so we view X as a subset of X^G. Inclusion then maps the H-orbits of X bijectively to the G-orbits of X^G. In particular, X^G is transitive if and only if X is.

(vii) This is not true for tensor induction: The transitivity of X is a necessary, but not sufficient condition for the transitivity of $X^{\otimes G}$.

To see necessity, observe that for $x, y \in X$ and $G = \dot{\bigcup}_{i \in I} g_i H$, we can define $a, b \in X^{\otimes G} = [G, X]_H$ by $(g_i h)a = xh$ and $(g_i h)b = yh$. If a and b belong to the same G-orbit, then x and y belong to the same H-orbit, as is easily seen.
For finite G, let $n = |G : H|$ and let X be a regular H-set. Then X is transitive. Since $|X^{\otimes G}| = |X|^n = |H|^n$ is usually much larger than $|G| = |H|n$, the G-set $X^{\otimes G}$ cannot be transitive. In fact, for $1 < H < G$, there is only one exception, namely $G = C_4$.

1.7 Remark: *Piecemeal G-sets*

If $Y = \dot{\bigcup}_{j \in J} jG$ is a G-set and for every $j \in J$ a G_j-set X_j is given, then one can construct a G-set $X = \dot{\bigcup}_{j \in J} X_j^G = \dot{\bigcup}_{j \in J} X_j \otimes_{G_j} G$ and define a canonical G-map $\alpha : X \to Y$ by $(x \otimes g)\alpha = jg$ if $x \in X_j$. Again, we will use the notation $X_y = \{x \in X \mid x\alpha = y\}$. There is a slight ambiguity here for the X_j's. However, X_j is canonically embedded into X, and

its image is just the inverse image of j under α.
This construction can be extended to maps: if for any $j \in J$ a G_j-map $\varphi_j : X_j \to Z$ is given, where Z is some G-set, then a G-map $\varphi = \dot{\bigcup\limits_{j \in J}} \varphi_j^G : X \to Z$ is defined by $(x \otimes g)\varphi = x\varphi_j g$ if $x \in X_j$.
Similarly, if for any $j \in J$ a G_j-map $\varphi_j : X_j \to Z_j$ is given, where Z_j is some G_j-set, then a G-map $\varphi = \dot{\bigcup\limits_{j \in J}} \varphi_j^G : X \to Z = \dot{\bigcup\limits_{j \in J}} Z_j^G = \dot{\bigcup\limits_{j \in J}} Z_j \otimes_{G_j} G$ is defined by $(x \otimes g)\varphi = x\varphi_j \otimes g$ if $x \in X_j$.
These constructions will be repeatedly used in the Section 4.

2 Inductible Maps

Everybody working in group representation theory is familiar with Frobenius induction, which constructs to a class function φ of a subgroup $H \leq G$ a class function φ^G of G by

$$\varphi^G(g) = \sum_{\substack{Hx \in [G:H] \\ Hxg = Hx}} \varphi^x(g).$$

Of course, this may not make sense if $|G : H|$ is infinite. So we assume from now on for the rest of this paper that *all groups G considered and all occurring G-sets are finite, unless otherwise stated.*
Somewhat less popular then Frobenius induction, but occasionally useful, is tensor induction, introduced by Berger in [1] and also used by Dade and Isaacs. For the readers convenience, we review the construction briefly.

2.1 Definition: *Tensor induction*

Let X be an R-module on which the group H acts; for every $n \in \mathbb{N}$ then, H^n acts in the obvious way on the n-fold tensor product $X \otimes_R \ldots \otimes_R X$. We have also a natural action of the symmetric group S_n on $X \otimes \ldots \otimes X$ by $(x_1 \otimes \ldots \otimes x_n)\sigma = x_{1\sigma^{-1}} \otimes \ldots \otimes x_{n\sigma^{-1}}$ for $\sigma \in S_n$. Similarly, S_n acts on H^n ; these actions are compatible. Therefore, $X \otimes \ldots \otimes X$ is a module for the semidirect product $S_n H^n$; this group is also known as the wreath product $H \wr S_n$.
Now let G be a group containing H as a subgroup of index n, say, and let $\{r_1, \ldots, r_n\}$ be a set of right coset representatives. Then for any $c \in G$, we have $r_i c = h_i r_{i\sigma}$ for uniquely defined elements $h_i = h_i(c) \in H,\ i = 1, \ldots, n$ and a unique permutation $\sigma = \sigma(c) \in S_n$. Therefore, $c\varphi = \sigma h_{1\sigma^{-1}} \ldots h_{n\sigma^{-1}} \in H \wr S_n$. Since $\varphi : G \to H \wr S_n$ turns out to be a homomorphism, we can view $X \otimes \ldots \otimes X$ as G-module via φ. This module is then called the tensor induced module and written as $X^{\otimes G}$. The notation is justified by the (easily checked) fact that a different choice of representatives $r_i' = a_i r_{i\tau}$ (with $a_i \in H$ and a permutation τ) will lead to a map φ' which differs from φ only by an inner automorphism of $H \wr S_n$ (more precisely, φ is φ' followed by conjugation with $\tau a_{1\tau^{-1}} \ldots a_{n\tau^{-1}} \in H \wr S_n$), and will therefore define an isomorphic G-module.
If X has a finite R-basis, then so has $X^{\otimes G}$ and one can calculate the trace of $c \in G$ on

$X^{\otimes G}$: Let $C =< c >$and let $G = \dot{\bigcup}_{j=1,\ldots,s} Hg_jC$ be the double coset decomposition (i.e. the decomposition of $[G : H]$ into C-orbits). If $n_j := |Hg_jC : H|$ is the orbit length, then a set of coset representatives for H in Hg_jC is $\{g_jc^t \,|\, t = 0,\ldots,n_j-1\}$. Using these and writing $k_j = g_jc^{n_j}g_j^{-1} (\in H)$, the action of c on $X^{\otimes G}$ is given by

$$\begin{array}{ccccc} [\,(x_{1,1} \otimes x_{1,2} \otimes \ldots \otimes x_{1,n_1}) & \otimes \ldots \otimes & (x_{s,1} \otimes x_{s,2} \otimes \ldots \otimes x_{s,n_s})\,] \cdot c & = \\ (x_{1,n_1}k_1 \otimes x_{1,1} \otimes \ldots \otimes x_{1,n_1-1}) & \otimes \ldots \otimes & (x_{s,n_s}k_s \otimes x_{s,1} \otimes \ldots \otimes x_{s,n_s-1}) \ . \end{array}$$

Now let $\{b_1,\ldots,b_d\}$ be a basis of X. Then the elements

$$B_f := b_{1f} \otimes \ldots \otimes b_{nf} \ , \ f : \{1,\ldots,n\} \to \{1,\ldots,d\}$$

form a basis of $X^{\otimes G}$, as is well known. It is clear from the above that the contribution of B_f to the trace of c on $X^{\otimes G}$ is 0 unless f is constant on the orbits of σ. If this is the case, say $if = jt$ for i in the j-th orbit of σ, then the contribution of B_f is the product for $j = 1,\ldots,s$ of the contributions of b_{jt} to the trace of k_j on X . Summation over all f's which are constant on σ-orbits gives

$$tr_{X^{\otimes G}}(c) = \prod_{j=1}^{s} tr_X(k_j) \ .$$

If one replaces tr_X on the right hand side of this formula by an arbitrary class function γ of H, then

$$\gamma^{\otimes G}(c) = \prod_{j=1}^{s} \gamma(k_j) \ .$$

defines a class function $\gamma^{\otimes G}$ of G. The map $\gamma \mapsto \gamma^{\otimes G}$ is called <u>tensor induction</u>. The connection with the module construction just described motivates the name and shows that the tensor induced of a character is a character. (For more details on tensor induction, see [2], § 13, and [1].)

2.2 Remarks:

(i) At this point, we have defined tensor induction to construct a G-module from an H-module ($H \leq G$), but also to construct a G-set from an H-set (in 1.4). As for Frobenius induction, one has to be careful in case M is an RH-module (compare 1.6), because then set tensor induction and module tensor induction can both be applied to M, but will in general produce non-isomorphic G-sets. In this case, we use the notation $M^{\otimes G}$ only for module tensor induction.

(ii) Again, permutation modules are well-behaved: If X is an H-set, then the permutation G-module over the tensor induced set $X^{\otimes G}$ is isomorphic to the module obtained by tensor inducing the permutation H-module $R[X]$, i.e.

$$R[X^{\otimes G}] \cong R[X]^{\otimes G}$$

(see [2], Prop. 80.38). In fact, if $G = \dot{\bigcup}_{i=1,\ldots,n} Hr_i$ and $a \in X^{\otimes G} = [G,X]_H$, then define $a\tau = r_1^{-1}a \otimes \cdots \otimes r_n^{-1}a$. It turns out that τ is compatible with the action of G and maps a basis of $R[X^{\otimes G}]$ bijectively onto a basis of $R[X]^{\otimes G}$.

(iii) When calculating determinants of Frobenius induced modules, tensor induction (of linear characters) introduces itself naturally:

$$\det(M^G) = (\mathrm{sign}_{[G:H]})^{\dim\, M} \cdot (\det\, M)^{\otimes G}\ ,$$

if M is a module (with finite basis) for the subgroup H (see [2], Prop. 13.15).

(iv) For an H-module M, there is a canonical map $\kappa : M^G \to M^{\otimes G}$: if again $G = \dot{\bigcup\limits_{i=1,\dots,n}} Hr_i$, then $(\sum\limits_i m_i \otimes r_i)\kappa = m_1 \otimes \cdots \otimes m_n$. It turns out that κ is a G-map, but if $H < G$ and $M \neq 0$, then κ is definitely not R-linear and not injective; it is surjective if and only if $\dim_R M = 1$. It may well happen that the only G-map $M^{\otimes G} \to M^G$ is the 0-map.

(v) There are at least two good reasons for the general preference of Frobenius induction over tensor induction: First, the degrees of characters remain more manageable (with the exception of linear characters, but these tend to become trivial under tensor induction). Second, there is no analog of Frobenius reciprocity (which is one of the most useful tools in character theory), not even if one starts dreaming of 'tensor restriction'.

We wish to study a more general procedure which generates a class function of a group from class functions of subgroups and still call it 'induction'. To justify the name, such a procedure should certainly have some good properties (respecting characters and invariance under field (or ring) automorphisms come to mind), but I am unable to state these; in other words, I cannot define 'induction'. Instead, I will give an example.

2.3 Notation:

Let R be a commutative ring with 1. For any subgroup U of G, let U^* be the set of all maps from U to R, written on the left (against the convention above). Let $\widetilde{G} = \dot{\bigcup\limits_{U \leq G}} U^*$. If $a \in U^*$ and $g \in G$, define (as usual) $a^g \in (U^g)^*$ by $a^g(v) = a(v^{g^{-1}})$ for $v \in U^g$. Clearly, this defines a G-action on $\widetilde{G}$; this G-set of course is infinite if R is.

2.4 Definition: *Inductible maps*

Let M be a G-set. An <u>inductible map</u> (for G on M) is a G-map $\theta : M \to \widetilde{G}$ with $m\theta \in (G_m)^*$ for all $m \in M$.

2.5 Examples:

(i) Let $H \leq G$ be a subgroup and φ a class function of H. Let $M = [G : H]$ and define $\theta : M \to \widetilde{G}$ by $(Hg)\theta = \varphi^g$. Then θ is inductible.

(ii) Let γ be a complex valued class function of G and $N \triangleleft G$ a normal subgroup. Then $M = \mathrm{Irr}\,(N)$ is a G-set. For any $\tau \in M$, let $e_\tau \in \mathbb{C}N$ be the corresponding central idempotent. Define $\theta : M \to \widetilde{G}$ by $(\tau\theta)(g) = \gamma(ge_\tau)$ for $g \in G_\tau$. Then θ is inductible.

(iii) Let $K \leq N$ both be normal subgroups of G and κ a G-stable linear character of K. Then $M := N/K$ is a G-set under conjugation. If $m = \overline{n} \in M$ and $c \in G_m$, then the commutator $[n, c] \in K$ and $(m\theta)(c) = \kappa[n, c]$ defines a linear character $m\theta$ of G_m. As before, $\theta : M \to \widetilde{G}$ is inductible. In fact, M can be viewed as a $\overline{G} = G/K$-set and $m\theta$ as character of $\overline{G}_m = G_m/K$, thereby defining an inductible map for $\overline{G}$.

2.6 Remarks:

(i) If θ is inductible and $g, h \in G_m$ for some $m \in M$, then $(m\theta)(hgh^{-1}) = (m\theta)^h(g) = (mh\theta)(g) = (m\theta)(g)$, so $m\theta$ is not arbitrary $G_m \to R$, but a *class function*. We may therefore think of θ as providing class functions of certain subgroups of G (the G_m's), and of doing so in a reasonably coherent way.

(ii) Let $\theta : M \to \widetilde{G}$ be inductible and let N be a normal subgroup of G. We wish to define an inductible map $\overline{\theta} : \overline{M} \to \widetilde{\overline{G}}$, where $\overline{G} = G/N$ and $\overline{M} = M/N$ is the $\overline{G}$-set of N-orbits on M. To make the construction more suggestive, we assume that $|N|$ is a unit in R. For any $m \in M$ then, $m\theta$ is a class function of G_m, which can be used to construct a class function $\overline{m\theta}$ of G_m/N_m by taking the mean

$$(\overline{m\theta})(gN_m) = \frac{1}{|N_m|} \sum_{x \in gN_m} (m\theta)(x)$$

for $g \in G_m$; careful, the 'bar' has nothing to do with complex conjugation, even if $R = \mathbb{C}$. If $\overline{g} = gN \in \overline{G}_{\overline{m}} = G_m N/N \cong G_m/N_m$, then $gn \in G_m$ for a suitable $n \in N$. We may now define $(\overline{m}\overline{\theta})(\overline{g}) = (\overline{m\theta})(gnN_m)$. The reader can easily check that $\overline{\theta} : \overline{M} \to \widetilde{\overline{G}}$ is a well-defined inductible map for $\overline{G}$. We can call $(M, \theta)/N := (\overline{M}, \overline{\theta})$ the <u>factor</u> inductible map (modulo N).

(iii) If $\eta_j : P_j \to \widetilde{H}_j \subseteq \widetilde{G}$ are inductible maps for subgroups $H_j \leq G$, then according to 1.7, one can construct

$$\eta = \dot{\bigcup_j} \eta_j^G : P = \dot{\bigcup_j} P_j^G \to \widetilde{G} \ ,$$

which is an inductible map.
It is tempting to call this process 'induction' (of inductible maps), but we reserve this name for another concept. Instead, just the notation will have to do. Since it is also used for Frobenius induction of class functions of subgroups, this may be dangerous. Still, it should always be clear from context if one is dealing with class functions or with inductible maps.

(iv) Let H be a subgroup of G. An inductible map for G on M can be 'restricted' to an inductible map for H on M since restriction maps G_m^* to H_m^*. More generaly: if $\varphi : H \to G$ is a group homomorphism and $\theta : M \to \widetilde{G}$ an inductible map for G, then M is an H-set via φ and $(m\eta)(h) = (m\theta)(h^\varphi)$ for $h \in H_m$ defines an inductible map η for H on M.

(v) An inductible map for G can be 'induced', i.e. used to construct from it a class function of G. In fact, there are usually several ways to do so. The next definition describes how.

2.7 Definition: *Inducing an inductible map*

Let $\theta : M \to \widetilde{G}$ be an inductible map and let $\alpha : M \to A$ be a G-map into a G-set A. We will define a class function $\theta^\alpha : G \to R$, called the $\underline{\alpha\text{-induced of } \theta}$, or $\underline{\theta \text{ induced with respect to } \alpha}$; this is done in two steps.
First, fix $a \in A$ and let $M_a = \{m \in M \mid m\alpha = a\}$, so this is a G_a-subset of M. For $c \in G_a$, denote $C =< c >$ and let $M_a = \dot{\bigcup\limits_{i \in I}} iC$. For every $i \in I$, let $n_i =\mid iC \mid$ and $c_i = c^{n_i}$, so $C_i =< c_i >$; in particular, $c_i \in G_i$. Then define

$$\theta_a^\alpha(c) = \prod_{i \in I} (i\theta)(c_i) \ ;$$

note that this does not depend on the choice of I. So far, we have a map $\theta_a^\alpha : G_a \to R$.
Now the second step is simple: $\theta^\alpha(c)$ for any $c \in G$ is defined by

$$\theta^\alpha(c) = \sum_{\substack{a \in A \\ ac=a}} \theta_a^\alpha(c) \ .$$

As usual, an empty sum is 0, an empty product 1 (both in R) per definition.

2.8 Examples:

(i) The simplest and most important situation is $A = M$ and $\alpha = id$. Then $M_m = \{m\}$ and $\theta_m^{id} = m\theta$. Therefore

$$\theta^{id}(c) = \sum_{\substack{m \in M \\ mc=m}} (m\theta)(c) \ .$$

(ii) Let $\theta_1, \ldots, \theta_m$ be class functions of G and let $M = \{1, \ldots, m\}$ with trivial G-action. Define $\theta : M \to \widetilde{G}$ by $i\theta = \theta_i$. If $A = \{a\}$ has only one element, so $i\alpha = a$ for all $i \in M$, then $\theta^\alpha = \theta_a^\alpha = \prod\limits_i \theta_i$ is precisely the product.

(iii) Let M and θ be as in (ii). Then $\theta^{id} = \sum\limits_i \theta_i$ is simply the sum.

(iv) Let M and θ be as in 2.5, (i). Then $\theta^{id} = \varphi^G$ is just Frobenius induction.

(v) With M and θ as in (iv), let $A = \{a\}$. Then $\theta^\alpha = \theta_a^\alpha = \varphi^{\otimes G}$ is tensor induction.

(vi) Let M and θ be as in 2.5, (ii). Then $\theta^{id} = \gamma$. To see this, note that

$$\gamma(c) = \gamma(c \cdot \sum_{\tau \in M} e_\tau) = \sum_{\tau \in M} \gamma(ce_\tau) = \sum_{\substack{\tau \in M \\ \tau^c = \tau}} \gamma(ce_\tau) = \sum_{\substack{\tau \in M \\ \tau^c = \tau}} (\tau\theta)(c) \ ,$$

because

$$\gamma(ce_\tau) = \gamma(ce_\tau^2) = \gamma(e_\tau ce_\tau) = \gamma(ce_\tau^c e_\tau) = \gamma(ce_{\tau^c} e_\tau) = 0 \ ,$$

if $\tau^c \neq \tau$.
This example can be taken as a starting point for Clifford theory.

(vii) Assume that G acts faithfully on M as a Frobenius group with Frobenius kernel K and complement G_m for some fixed $m \in M$. Let $\alpha : M \to A$ be a surjective G-map and $a := m\alpha$. Then $G_a = G_m \cdot H$ for $H = G_a \cap K$, a G_m-stable subgroup of K. Let φ be a class function of G_m; as bevor, define $\theta : M \to \widetilde{G}$ by $(mg)\theta = \varphi^g$ and let $C :=< c >$ for $c \in G$. We leave it to the reader to check

$$\theta^\alpha(c) = \begin{cases} \mathbf{1}_H^K(c)\varphi(1)^{\,|H|\,/|C|} & \text{if } c \in K \\ \varphi(c)\varphi(1)^{\,(|H|-1)\,/|C|} & \text{if } c \in G_m^\times \,. \end{cases}$$

Note that this describes θ^α, as each c belongs either to K or – up to conjugation – to $G_m^\times$. Note also that $|C|$ divides $|H|$ if $c \in K$ and $\mathbf{1}_H^K(c) \neq 0$, while $|C|$ divides $|H| - 1$ if $c \in G_m$.

Of course, to use this formula may be difficult. It is greatly simplified in two cases:

(α) $A = M$, $\alpha = id$, $\varphi(1) = 0$. Then $H = 1$, so θ^α is an extension of φ to G which is constantly 0 on K.

(β) $A = \{a\}$, $\varphi(1) = 1$. Then $H = K$, so again θ^α is an extension of φ to G, this time constantly 1 on K.

Either of these cases can be used to prove the existence of the Frobenius kernel; if χ is a character of G_m, subtract a suitable multiple of the trivial character to get φ. Together, this means that $(\varphi + \mathbf{1}_{G_m})^{\otimes G} = \varphi^G + \mathbf{1}_G$ if $\varphi(1) = 0$. This is a special instance of a general – and more complicated – formula, as we will see in the next section.

2.9 Remarks:

(i) The reader should check that θ^α as defined in 2.7 is indeed a class function!

(ii) If $a \in A \setminus \mathrm{Im}\,(\alpha)$, then $\theta_a^\alpha = \mathbf{1}_{G_a}$, since $M_a = \emptyset$. Therefore the contribution of $A \setminus \mathrm{Im}\,(\alpha)$ to θ^α is just a permutation character.

(iii) The first of the two induction steps in 2.7 is clearly multiplicative: if θ and γ are two inductible maps on M, then so is $\theta\gamma$ (or $\theta + \gamma, \dots$) and $(\theta_a^\alpha)(\gamma_a^\alpha) = (\theta\gamma)_a^\alpha$ for every $a \in A$.

(iv) The reader will have guessed by now that 'A' stands for 'addition' and 'M' for 'multiplication'. The above examples show that these two operations as well as Frobenius induction and tensor induction can be described by special inductible maps and special α's. The next result will show that inducing an inductible map can always be expressed as a combination of these four operations. So inductible maps introduce no new concepts. One should rather look at them as a notational device which allows to describe uniformly such messy things as 'tensor induction of sums of (Frobenius) induced characters' by storing all the information in two G-sets (M and A) and two G-maps (θ and α).

(v) To think of addition or multiplication of class functions in terms of inductible maps may seem strange, and indeed it is. For Frobenius induction, it takes perhaps some getting used to the G-set $[G : H]$ and the corresponding inductible map. The second example in 2.5 as continued in 2.8, (vi) seems quite natural in view of Clifford theory. The third example there will be taken up in a forthcoming paper. There are other examples where the relevant G-sets and G-maps are naturally given.

(vi) Why not interchange the order of addition and multiplication in 2.7? Couldn't one equally well define a class function the other way round? Yes, one could, but not equally well. The reason is the asymmetry between addition and multiplication introduced by distributivity: every product of sums can be expanded into a sum of products, but rare is the sum of products which is a product of sums. More on the expansion in the next section.

(vii) Let $g \in G \geq H$ and let $\theta : M \to \widetilde{H}$ be an inductible map on the H-set M. Further let $\alpha : M \to A$ be an H-map into some H-set A. Then M^g and A^g are H^g-sets, $\theta^g : M^g \to \widetilde{H^g}$ is inductible and $\alpha^g : M^g \to A^g$ is an H^g-map. It is straightforward to check that $(\theta^g)^{\alpha^g} = (\theta^\alpha)^g$.

2.10 Proposition:

Let an inductible map θ and α as in 2.7 be given.

(1) Fix $a \in A$ and let $M_a = \dot{\bigcup\limits_{j\in J}} jG_a$. Then $\theta^\alpha_a = \prod\limits_{j\in J} (j\theta)^{\otimes G_a}$ (a product of tensor induced class functions).

(2) Let $A = \dot{\bigcup\limits_{b\in B}} bG$. Then $\theta^\alpha = \sum\limits_{b\in B} (\theta^\alpha_b)^G$ (a sum of Frobenius induced class functions).

(3) If $m\theta$ is a (generalized) character of G_m for every m, then θ^α is a (generalized) character of G.

(Of course, 'character of G' means 'trace of G on some RG-module with finite R-basis', and a generalized character is a difference of two characters.)

Proof: The first statement follows directly from the definition of tensor induction, the second from the definition of Frobenius induction. It is well known that Frobenius induction takes (generalized) characters to (generalized) characters. The same holds for tensor induction: for characters, we have written down the module in 2.1. For generalized characters, the proof is more complicated. For the case $R = \mathbb{C}$, i.e. ordinary generalized characters, an argument was given in [5], Prop. 1.8., using Brauer's characterization of characters. A simpler proof for this fact (and also for generalized permutation characters) was given by Gluck and Isaacs in [3]; their proof argues with the Galois group and algebraic integers. An elementary proof for general R will be given in 3.5 at the end of the next section.

2.11 Remark:

The last result cries out for a shorthand notation for a set of representatives of the G-orbits on some G-set M. We denote these simply by M/G and can then write $\theta^\alpha_a = \prod\limits_{j\in M_a/G_a} (j\theta)^{\otimes G_a}$ and $\theta^\alpha = \sum\limits_{b\in A/G} (\theta^\alpha_b)^G$. Strictly speaking of course, M/G is the set of G-orbits on M, but all the constructions are independent of the choice of representatives. Using this notation, the simplest special case of the proposition is $\theta^{id} = \sum\limits_{m\in M/G} (m\theta)^G$.

2.12 Lemma: *Mean and Induction*

Let $\theta : M \to \widetilde{G}$ be inductible and $N \lhd G$; assume that $|N|$ is a unit in R. Let $(\overline{M}, \overline{\theta})$ be the factor inductible map as in 2.6, (ii). Then

$$\overline{\theta}^{id} = \overline{\theta^{id}} \ .$$

Proof: We have to show that

$$|N| \cdot \overline{\theta}^{id}(gN) = \sum_{x \in gN} \theta^{id}(x)$$

for every $g \in G$. For given $m \in M$, there is an $x \in gN = \overline{g}$ with $mx = m$ if and only if $\overline{m}\,\overline{g} = \overline{m}$. If so, then the elements y in gN with this property form precisely one coset xN_m and

$$\sum_{\substack{y \in gN \\ my = m}} (m\theta)(y) = \sum_{y \in xN_m} (m\theta)(y) = |N_m| \cdot (\overline{m\theta})(xN_m) = |N_m| \cdot (\overline{m}\overline{\theta})(\overline{g})$$

by definition of $\overline{\theta}$. Summing over the $|N : N_m|$ elements in the N-orbit $\overline{m}$ of m gives therefore precisely

$$\sum_{\substack{y \in gN \\ mu' \in \overline{m} \\ mu'y = m'}} (m'\theta)(y) = |N| \cdot (\overline{m}\overline{\theta})(\overline{g}) \ ,$$

still under the assumption that $\overline{m}\,\overline{g} = \overline{m}$. Hence

$$\begin{aligned}
\sum_{y \in gN} \theta^{id}(y) &= \sum_{\substack{y \in gN \\ mu \in M \\ my = m}} (m\theta)(y) \\
&= \sum_{\substack{\overline{m} \in \overline{M} \\ \overline{m}\,\overline{g} = \overline{m}}} \ \sum_{\substack{y \in gN \\ mu' \in \overline{m} \\ mu'y = m'}} (m'\theta)(y) \\
&= |N| \sum_{\substack{\overline{m} \in \overline{M} \\ \overline{m}\,\overline{g} = \overline{m}}} (\overline{m}\overline{\theta})(\overline{g}) \\
&= |N| \cdot \overline{\theta}^{id}(\overline{g})
\end{aligned}$$

2.13 Remark:

To rephrase the last result in more familiar terms, consider the situation where a normal subgroup N of G and a class function φ of a subgroup H are given; we wish to construct a class function of $\overline{G} = G/N$ from these data. Two ways of doing so come to mind:
First, use Frobenius induction to obtain the class function φ^G of G, then take the mean over cosets of N to get $\overline{\varphi^G}$, a class function of $\overline{G}$.
Second, take the mean of φ over cosets of the normal subgroup $H \cap N$ of H to obtain the class function $\overline{\varphi}$ of $H/(H \cap N)$. Since $H/(H \cap N) \cong HN/N = \overline{H}$, we can view $\overline{\varphi}$ as a class function of $\overline{H}$, using the canonical isomorphism. But $\overline{H}$ is a subgroup of $\overline{G}$, so

Frobenius induction produces the class function $\overline{\varphi}^{\overline{G}}$ of $\overline{G}$.
The content of the lemma is that these two methods lead to the same result. This is no surprise; however, the corresponding statement with Frobenius induction replaced by tensor induction is false in general.

3 Distributivity

We now reverse the order of tensor induction and Frobenius induction in 2.7. So let P and S be G-sets, $\varphi : P \to \widetilde{G}$ an inductible map, and $\pi : P \to S$ a G-map. For $s \in S$, consider the G_s-set $P_s = \{p \in P \mid p\pi = s\}$. Denote

$$ {}^{\pi}_{s}\varphi = \sum_{p \in P_s/G_s} (p\varphi)^{G_s} \ , $$

so this is a class function of G_s, and

$$ {}^{\pi}\varphi = \prod_{s \in S/G} ({}^{\pi}_{s}\varphi)^{\otimes G} \ . $$

Note the analogy and the difference to 2.10. In a good sense, ${}^{\pi}\varphi$ is a product of sums. We wish to expand this product. The obvious approach works.

3.1 Proposition: *Distributivity of induction*

Let $A = \{a : S \to P \mid a\pi = \mathrm{id}_S\}$. This is a G-set under conjugation. Also, $M = S \times A$ is a G-set, and $(s,a)\alpha = a$ defines a G-map $\alpha : M \to A$. Further, let $(s,a)\vartheta = (sa\varphi)_{|G_{(s,a)}}$. Then $\vartheta : M \to \widetilde{G}$ is an inductible map and

$$ {}^{\pi}\varphi = \vartheta^{\alpha} \ . \tag{1} $$

Proof: Clearly, $a^g\pi = (a\pi)^g = \mathrm{id}_S$ for $a \in A$ and $g \in G$, so $a^g \in A$ and A is a G-set. The statements about M and α are then trivial. For $s \in S$, we have $(sa)g = sga^g$, hence $G_{(s,a)} \leq G_{sa}$. Since $sa\varphi \in (G_{sa})^*$ by assumption on φ, we can restrict it to get $(s,a)\vartheta \in (G_{(s,a)})^*$. Moreover,

$$ \begin{aligned} (s,a)g\vartheta &= (sga^g\varphi)_{|G_{(sg,a^g)}} \\ &= (sag\varphi)_{|G^g_{(s,a)}} \\ &= (sa\varphi)^g_{|G^g_{(s,a)}} \qquad \text{since } \varphi \text{ is a } G\text{-map} \\ &= [(sa\varphi)_{|G_{(s,a)}}]^g \\ &= [(s,a)\vartheta]^g \ , \end{aligned} $$

so ϑ is a G-map and therefore inductible. It remains to proof (1).
So let $C =< c >$ for $c \in G$ and let $S = \dot{\bigcup\limits_{i \in I}} iC$ (compare 2.7). Then

$$ {}^{\pi}\varphi(c) = \prod_i {}^{\pi}_{i}\varphi(c_i) \ . $$

For any i, by definition,

$$ {}^{\pi}_{i}\varphi(c_i) = \sum_{\substack{p\in P_i\\ pc_i=p}} (p\varphi)(c_i) \ , $$

so

$$ {}^{\pi}\varphi(c) = \prod_{i\in I} \sum_{\substack{p\in P_i\\ pc_i=p}} (p\varphi)(c_i) \ . $$

Now use 1.3 with $G = C,\ X = S,\ Y = P,\ \mu = \pi$ and $T(p) = (p\varphi)(c_i)$ for $p \in P_i$. Note that

$$ \begin{aligned} Y(i) &= \{p \in P \mid p \text{ fixed by } C_i \text{ and } p\pi = i\} \\ &= \{p \in P_i \mid pc_i = p\} \ . \end{aligned} $$

Therefore

$$ \begin{aligned} {}^{\pi}\varphi(c) &= \sum_{\substack{a\in[S,P]_C\\ a\pi=id_S}} \prod_{i\in I} (ia\varphi)(c_i) \\ &= \sum_{\substack{a\in A\\ a^c=a}} \prod_{i\in I} (i,a)\vartheta(c_i) \\ &= \sum_{\substack{a\in A\\ a^c=a}} \vartheta^{\alpha}_{a}(c) \\ &= \vartheta^{\alpha}(c) \ . \end{aligned} $$

(For the third equality, note that $M_a = S \times \{a\} = \dot{\bigcup\limits_{i\in I}} (i,a)C$ if $a^c = a$.)

3.2 Remarks:

(i) Explicitly, (1) can be rewritten as

$$ \prod_{s\in S/G} \big[\sum_{p\in P_s/G_s} (p\varphi)^{G_s} \big]^{\otimes G} = \sum_{a\in A/G} \big[\prod_{s\in S/G_a} (sa\varphi_{|G_{(s,a)}})^{\otimes G_a} \big]^{G} \tag{2} $$

This is the expansion mentioned above.
Note that on the right of this formula, tensor induction is only applied to restrictions of class functions $p\varphi$, *not* to sums of these nor to class functions obtained from them by Frobenius induction. As an immediate consequence, one gets the first corollary below.

(ii) We look more closely at the case that G_s acts trivially on P_s for all $s \in S$. For a later application, we give first a different description of this situation.
Assume that for all $s \in S$, there is an index set I_s, depending only on the G-orbit of s, i.e. $I_s = I_{sg}$, and class functions $\varphi_{s,i}$ of G_s such that $\varphi^g_{s,i} = \varphi_{sg,i}$ for all g, s and $i \in I_s$. Define the G-set $P := \{(s,i) \mid s \in S,\ i \in I_s\}$ with G-action only on the first component; also define $\varphi : P \to \widetilde{G}$ by $(s,i)\varphi = \varphi_{s,i}$. Then φ is inductible; the projection $\pi : P \to S$ is clearly a G-map. With the notation introduced at the beginning of this section, we have

$$ {}^{\pi}_{s}\varphi = \sum_{i\in I_s} \varphi_{s,i} \ . $$

The set A then also has a simple description, namely $A = \prod_{s \in S} I_s$. (Strictly speaking, we use the canonical isomorphism $\prod_{s \in S} I_s \ni f \mapsto a_f \in A$, where a_f is defined by $sa_f = (s, sf)$, to identify the two sets.) Using (2), we get

$$\prod_{s \in S/G} \Big[\sum_{i \in I_s} \varphi_{s,i}\Big]^{\otimes G} = \sum_{a \in A/G} \varphi_a^G \;, \tag{3}$$

where

$$\varphi_a = \prod_{s \in S/G_a} (\varphi_{s,sa})_{|G_{s,a}}^{\otimes G_a} \;.$$

This shows how to change the order of summation and tensor induction.

(iii) In case S is a transitive G-set, one may think of (3) as the character theoretical analog of the multinomial formula $(\sum_{i=1}^{t} x_i)^n = \sum_{\nu} \binom{n}{\nu} x^\nu$, where the second sum runs over all tupels $\nu = (\nu_1, \ldots, \nu_t) \in \mathbb{N}_0^t$ with $n = \sum_i \nu_i$ and where x^ν is shorthand for $\prod_i x_i^{\nu_i}$. Indeed, this formula follows with $t = |I_s|$ by specializing to $G = \mathcal{S}_n$, the symmetric group, acting naturally on $S = \{1, \ldots, n\}$ and then evaluating at 1. Here, of course, $x_i = \varphi_i(1)$; the G_a's are then Young subgroups.

(iv) The statement and the proof of the last proposition come naturally if one looks at induction in terms of G-sets and inductible maps. This may justify the definition.

3.3 Corollary: *Tensor induction of monomial characters*

Tensor-inducing a sum of monomial characters gives again a sum of monomial characters.

Proof: Restricting, conjugating or tensor-inducing a linear character gives a linear character. Also, products of linear characters are linear. Now use formula (2) with all $p\varphi$ linear.

3.4 Remark:

It is clear from the argument that one may take other class functions instead of the linear characters, provided they are closed under these four operations. For instance, one could take all linear characters λ with order$(\lambda) \in T$ if $T \subseteq \mathbb{N}$ is closed under divisors and least common multiples, i.e. $d|t \in T \Rightarrow d \in T$ and $s, t \in T \Rightarrow \mathrm{lcm}(s,t) \in T$ (e.g. all divisors of some fixed $n \in \mathbb{N}$, all powers of some fixed prime,...). The case $T = \{1\}$ leads to the permutation characters considered in 2.2. Also, the statement remains true if, in addition, one allows taking products.

3.5 Corollary: *Tensor induction of generalized characters*

Let γ be a generalized character of some subgroup H of G. Then $\gamma^{\otimes G}$ is a generalized character of G. More precisely, let α and β be class function of H and write $S = [G : H]$

(so S is a G-set); then

$$(\alpha - \beta)^{\otimes G} = \sum_{\{X \subseteq S\}/G} \varphi_X^G \ ,$$

where

$$\varphi_X = (-1)^{|X|} \cdot \mathrm{sign}_X \prod_{s \in (S \setminus X)/G_X} (\alpha^s)^{\otimes G_X}_{|G_{s,X}} \cdot \prod_{s \in X/G_X} (\beta^s)^{\otimes G_X}_{|G_{s,X}}$$

and sign_X is the sign character of G_X acting on X. So if α and β are characters, then $(\alpha - \beta)^{\otimes G}$ is an alternating sum of characters.

Proof: Use (3) to expand $(\alpha - \beta)^{\otimes G}$ and identify the maps $a : S \mapsto \{\alpha, -\beta\}$ with the subsets X of S by $x \in X \Leftrightarrow xa = -\beta$. This gives

$$(\alpha - \beta)^{\otimes G} = \sum_{\{X \subseteq S\}/G} \varphi_X^G$$

with

$$\varphi_X = \prod_{s \in (S \setminus X)/G_X} (\alpha^s)^{\otimes G_X}_{|G_{s,X}} \cdot \prod_{s \in X/G_X} (-\beta^s)^{\otimes G_X}_{|G_{s,X}} \ .$$

Since tensor induction is multiplicative, it is enough to show that

$$\prod_{s \in X/G_X} (-\mathbf{1}_{G_{s,X}})^{\otimes G_X} = (-1)^{|X|} \cdot \mathrm{sign}_X$$

and this follows easily from observing that

$$\prod_{s \in X/G_X} (-\mathbf{1}_{G_{s,X}})^{\otimes G_X}(c) = (-1)^{o(c)}$$

where $o(c)$ is the number of orbits of $c \in G_X$ on X.

3.6 Remark:

The last two results can be combined: If $\mu = \sum_i z_i \lambda_i^H$ is a generalized monomial character of a subgroup $H \leq G$, where the the linear characters λ_i belong to a subset of all linear characters as in the last remark, then $\mu^{\otimes G}$ is of the same type; note that the sign character is $\mathbf{1}_{A_n}^{S_n} - \mathbf{1}_{S_n}$, so is a generalized permutation character. In particular, tensor inducing a generalized permutation character gives a generalized permutation character (for $R = \mathbb{C}$, this is the result of Gluck and Isaacs mentioned above; recall that all results in this section hold for any commutative ring R.) Of course, taking *all* linear characters makes the result trivial, since every generalized character is a generalized monomial character by Brauer's theorem on induced characters (again for $R = \mathbb{C}$).

4 Transitivity

We observed in 2.10 that inducing an inductible map means essentially – apart from products and sums – first tensor induction (Ti), then Frobenius induction (Fi). Repeating the process gives therefore Ti · Fi · Ti · Fi . As shown in the last section, the second and third operation can be rewritten as Ti · Fi. Since both tensor induction and Frobenius induction are transitive, it comes as no surprise that induction of inductible maps is also transitive. For the sake of completeness, we give an explicit description in this section. Unfortunately, the result is rather technical.

So let an inductible map $\vartheta : M \to \widetilde{G}$ be given and assume that the $m\vartheta$'s are defined by inducing suitable inductible maps for the subgroups G_m with respect to some G_m-maps. We wish to calculate ϑ^α (for some $\alpha : M \to A$) directly from these maps.

Before describing the result, we have to deal with the difficulty that the relevant sets for the subgroups G_m may not be compatible, even though the induced class functions are conjugate. This may happen even for Frobenius induction. For example, if $U \leq H \leq G$ and $V \leq H^g \leq G$ for some $g \in G$ and if further λ and μ are characters of U and V, respectively, such that $(\lambda^H)^g = \mu^{(H^g)}$, then this does not imply that $\mu = \lambda^g$ nor even $V = U^g$.

However, if we are only interested in the induced characters, we can replace V by U^g and μ by λ^g (or vice versa), that is, we can assume that the underlying structures are conjugate (but we do make a choice). This is what we will do in general for inductible maps:

So let $M = \dot{\bigcup\limits_{j\in J}} jG$ be the orbit decompositions (here, we *choose* the representatives). For every j then, $j\vartheta$ is a class function of G_j which by assumption is given by inducing an inductible map, say $j\vartheta = \eta_j^{\tau_j}$, where $\eta_j : P_j \to \widetilde{G}_j$ is an inductible map on some G_j-set P_j and $\tau_j : P_j \to S_j$ is some G_j-map. Denote $S = \dot{\bigcup\limits_{j}} S_j^G$ and let $\sigma : S \to M$ be the canonical G-map defined by $(s \otimes g)\sigma = jg$ if $s \in S_j$ (compare 1.7), so $S = \dot{\bigcup\limits_{m\in M}} S_m$, where $S_m = \{s \in S \mid s\sigma = m\}$. In the same way, define $\pi : P = \dot{\bigcup\limits_{j}} P_j^G \to M$ and the corresponding subsets P_m. Define $\tau : P \to S$ by $(p \otimes g)\tau = p\tau_j \otimes g$ if $p \in P_j$. Finally, let $\eta : P \to \widetilde{G}$ be the inductible map defined by $(p \otimes g)\eta = (p\eta_j)^g$ if $p \in P_j$ (compare 2.6, (iii)); note that P and η depend on the P_j's and the η_j's (so on the choice of J), but not on the σ_j's or α. Then we have a commutative diagram of G-maps:

$$\begin{tikzcd}
S \arrow[dr, "\sigma"'] & P \arrow[l, "\tau"'] \arrow[r, "\eta"] \arrow[d, "\pi"] & \widetilde{G} \\
 & M \arrow[r, "\alpha"] & A
\end{tikzcd}$$

Let $\tau_m = \tau_{|P_m}$ the restriction, viewed as G_m-map into S_m, and $\eta_m = \eta_{|P_m}$, viewed as G_m-map into $\widetilde{G}_m$. If $M \ni m = jg$, then it is easy to check that $\eta_m = \eta_j^g$ and $\tau_m = \tau_j^g$. Therefore $\eta_m^{\tau_m} = (\eta_j^g)^{\tau_j^g} = (\eta_j^{\tau_j})^g = (j\theta)^g = (jg)\theta = m\theta$ (compare 2.9, (vii)).

So far, we have just argued that not only the class functions $m\vartheta$ and $(mg)\vartheta$ are conjugate, but that they can be obtained by inducing conjugate inductible maps (i.e. $P_{mg} = P_m g$ and $\eta_{mg} = \eta_m^g$) with respect to conjugate maps (i.e. also $S_{mg} = S_m g$ and $\tau_{mg} = \tau_m^g$). We are now ready for transitivity:

4.1 Notation:

As before, let $M_a = \{m \in M \mid m\alpha = a\}$ for every $a \in A$. Define

$$F_a = \{f : M_a \to S \mid mf\sigma = m \ \forall m \in M_a\} \ .$$

Note that $f^g \in F_{ag}$ if $f \in F_a$ and that $F = \dot{\bigcup_a} F_a$ is a G-set with this action. It is easy to check that

$$Q := \{(p, f) \mid p \in P, \ f \in F_{p\pi\alpha}, \ p\tau = p\pi f\}$$

is a G-subset of $P \times F$. Define $\varphi : Q \to \widetilde{G}$ by $(p, f)\varphi = (p\eta)_{|G_{p,f}}$, the restriction of the class function $p\eta$ of G_p to the subgroup $G_{p,f}$. Then φ is an inductible map for G on Q. We let $\beta : Q \to F$ be the projection, i.e. $(p, f)\beta = f$; this is clearly a G-map.

4.2 Theorem: *Transitivity*

Keep the above notation. Then $\vartheta^\alpha = \varphi^\beta$.

Proof: Take $c \in G$. By definition,

$$\vartheta^\alpha(c) = \sum_{\substack{a \in A \\ ac = a}} \vartheta^\alpha_a(c) \ ,$$

and

$$\varphi^\beta(c) = \sum_{\substack{f \in F \\ f^c = f}} \varphi^\beta_f(c) = \sum_{\substack{a \in A \\ ac = a}} \sum_{\substack{f \in F_a \\ f^c = f}} \varphi^\beta_f(c) \ ,$$

so it is clearly enough to show that

$$\vartheta^\alpha_a(c) = \sum_{\substack{f \in F_a \\ f^c = f}} \varphi^\beta_f(c) \tag{4}$$

for every $a \in A$ and $c \in G_a$. So fix such a and c.
Now, by definition,

$$\vartheta^\alpha_a(c) = \prod_{i \in I} (i\vartheta)(c_i) \ , \tag{5}$$

where $M_a = \dot{\bigcup_{i \in I}} iC$. But $i\vartheta = \eta_i^{\tau_i}$, so

$$(i\vartheta)(c_i) = \sum_{\substack{s \in S_i \\ sc_i = s}} \eta^{\tau_i}_{i,s}(c_i) \ . \tag{6}$$

Now use the definition of $\eta^{\tau_i}_{i,s}$:

$$\eta^{\tau_i}_{i,s}(c_i) = \prod_{j \in J_s} (j\eta_i)(c_j) \ ,$$

where

$$P_{i,s} = \{p \in P_i \mid p\tau_i = s\} = \dot{\bigcup_{j \in J_s}} jC_i \ . \tag{7}$$

Write

$$T(s) = \prod_{j \in J_s} (j\eta_i)(c_j) = \eta_{i,s}^{\tau_i}(c_i) \tag{8}$$

for short (still $s \in S_i$). Then

$$(i\vartheta)(c_i) = \sum_{\substack{s \in S_i \\ sc_i = s}} T(s) \tag{9}$$

by (6) and therefore

$$\vartheta_a^\alpha(c) = \prod_{i \in I} \sum_{\substack{s \in S_i \\ sc_i = s}} T(s) \tag{10}$$

from (5) and (9).
So much for the left hand side of (4). Now for the right: To calculate $\varphi_f^\beta(c)$ for $f \in F_a$ with $f^c = f$, we need the decomposition of

$$Q_f = \{q \in Q \mid q\beta = f\} = \{(p, f) \mid p \in P,\ p\pi\alpha = a,\ p\tau = p\pi f\}$$

into orbits under C. Let $P_f = \{p \in P \mid (p, f) \in Q_f\}$. Then $P_f \subseteq P_a := \{p \in P \mid p\pi \in M_a\}$, so $\pi : P_f \to M_a$ is a G_f-map, in particular a C-map. According to 1.1, we get the orbit decomposition of P_f (hence of Q_f) by first decomposing $M_a = \dot{\bigcup_{i \in I}} iC$ (see above under (5)), and then further decomposing the inverse image of i under π in P_f into orbits under C_i. But

$$\{p \in P_f \mid p\pi = i\} = \{p \in P_i \mid p\tau_i = if\} = P_{i,if}$$

(see (7) and recall that τ_i is just the restriction of τ to P_i). Therefore the contribution to $\varphi_f^\beta(c)$ for some fixed $i \in I$ is just $T(if)$ (compare (8) and recall that $(p, f)\varphi$ is obtained by restricting $p\eta = p\eta_i$). It follows that

$$\varphi_f^\beta(c) = \prod_{i \in I} T(if) \ . \tag{11}$$

Therefore, using (10) and (11), to prove (4), we need

$$\prod_{i \in I} \sum_{\substack{s \in S_i \\ sc_i = s}} T(s) = \sum_{\substack{f \in F_a \\ f^c = f}} \prod_{i \in I} T(if) \ .$$

This follows from the second part of 1.3 with $G = C$, $X = M_a$, $Y = S_a := \dot{\bigcup_{m \in M_a}} S_m$ and $\mu = \sigma_{|S_a}$. Note that then

$$\begin{aligned} Y(i) &= \{y \in Y \mid y\mu = i \text{ and } yG_i = y\} \\ &= \{s \in S_i \mid sc_i = s\} \end{aligned}$$

and

$$\begin{aligned} \{f \in [X, Y]_G \mid f\mu = id_X\} &= \{f : M_a \to S_a \mid f^c = f \text{ and } mf\sigma = m\ \forall m \in M_a\} \\ &= \{f \in F_a \mid f^c = f\} \ . \end{aligned}$$

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