Partial inner products

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Abstract

A 'partial' inner product with respect to a given subset of $\mathrm{Irr}\,(G)$ is introduced and its basic properties are studied. As applications, a weak second orthogonality is shown in a Clifford setting and a version of the Burnside-Brauer theorem is given.
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Throughout this paper, G is a finite group. Characters and class functions are over $\mathbb{C}$ and $\chi \mapsto \overline{\chi}$ denotes complex conjugation (of numbers or class functions).

Definition *Partial inner product*
Let $S \subseteq \mathrm{Irr}\,(G)$ be any subset.

(1) For class functions α and β of G, call

$$(\alpha, \beta)_G^S := \sum_{\sigma \in S} (\alpha, \sigma)_G (\sigma, \beta)_G$$

the S-partial inner product of α and β.
Closely related is the projection

$$\alpha_S := \sum_{\sigma \in S} (\alpha, \sigma)_G \, \sigma \ ,$$

because clearly $(\alpha, \beta)_G^S = (\alpha_S, \beta)_G = (\alpha, \beta_S)_G = (\alpha_S, \beta_S)_G$ for any α and β.

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(2) For $g \in G$, denote

$$\rho_S^g = \sum_{\sigma \in S} \overline{\sigma(g)}\, \sigma \quad .$$

Remark 1

(i) Clearly, ρ_S^g depends only on the conjugacy class of g.

(ii) There is a symmetry: $\rho_S^g(h) = \overline{\rho_S^h(g)}$ for all $g, h \in G$.

(iii) By its very definition, $\rho_S^g \in < S >$, the subspace of all class functions spanned by S. In fact, $< S >$ is generated by the ρ_S^g's, because for $\tau \in S$, we have

$$\sum_{g \in G} \frac{\tau(g)}{|G|} \rho_S^g = \sum_{g \in G} \frac{\tau(g)}{|G|} \sum_{\sigma \in S} \overline{\sigma(g)} \sigma = \sum_{\sigma \in S} \left(\frac{1}{|G|} \sum_{g \in G} \tau(g) \overline{\sigma(g)} \right) \sigma = \sum_{\sigma \in S} (\tau, \sigma)_G \sigma = \tau \quad .$$

(iv) The most prominent member of the family $\{\rho_S^g\}$ is of course the regular character $\rho = \rho^1_{\mathrm{Irr}(G)}$, and as with the regular character, one can calculate inner products as values: $(\alpha,\, \rho_S^g)_G = \alpha_S(g)$ for every class function α.

Example
If $S = \mathrm{Irr}\,(G/N)$ for some $N \lhd G$, then $\alpha_S(g) = \frac{1}{|N|} \sum_{n \in N} \alpha(gn)$ for any $g \in G$.

Proposition 1 *Associativity for partial inner products*
Let $S \subseteq \mathrm{Irr}\,(G)$ and a class function η of G be given and denote $T = \mathrm{Irr}\,(G) \setminus S$. For $c \in \mathbb{C}$, let $G(c) = G(c, \eta) = \{g \in G \mid \eta(g) = c\}$. Then the following are equivalent:

(i) For all class functions α and β of G, one has $(\alpha\eta, \beta)_G^S = (\alpha, \overline{\eta}\beta)_G^S$.

(ii) If $0 \neq \sum_{\sigma \in S} \sigma(g)\overline{\sigma(h)}$ for elements $g, h \in G$, then $\eta(g) = \eta(h)$.

(iii) Every $0 \neq \rho_S^g$ is an eigenvector for η.

(iv) $< S >$ is closed under multiplication with η.

(v) $\sum_{g \in G(c)} \sigma(g)\overline{\tau(g)} = 0$ for every $c \in \mathbb{C}$, $\sigma \in S$, $\tau \in T$

(vi) $< S >$ is closed under multiplication with $\overline{\eta}$.

(vii) $< S >$ and $< T >$ are closed under multiplication with η.

(viii) $(\alpha\eta)_S = \alpha_S \eta$ for every α.

Proof:
(i)$\Rightarrow$(ii) Let $K = g^G$ and $L = h^G$, and let χ_K and χ_L be the characteristic functions for these conjugacy classes. Then it is easily seen that

$$(\chi_K \eta, \chi_L)_G^S = \eta(g) \frac{|K||L|}{|G|^2} \sum_{\sigma \in S} \overline{\sigma(g)} \sigma(h) \ ,$$

while

$$(\chi_K, \overline{\eta}\chi_L)_G^S = \eta(h) \frac{|K||L|}{|G|^2} \sum_{\sigma \in S} \overline{\sigma(g)} \sigma(h) \ .$$

Since these two values are equal by assumption, (ii) follows.
(ii)⇒(iii) If $0 \neq \rho_S^g(h) = \sum_{\sigma \in S} \overline{\sigma(g)}\sigma(h)$, then $\eta(g) = \eta(h)$, so $\eta(h)\rho_S^g(h) = \eta(g)\rho_S^g(h)$ for every h. This means $\eta\rho_S^g = \eta(g)\rho_S^g$, so ρ_S^g is an eigenvector for η with eigenvalue $\eta(g)$.
(iii)⇒(iv) This clear since $< S >$ is spanned by the ρ_S^g's.
(iv)⇒(v) Take $\sigma \in S$ and $\tau \in T$; let $c_1, \dots, c_m \in \mathbb{C}$ be all the different values of η. Define $a_i(\sigma, \tau) = \sum_{g \in G(c_i)} \sigma(g)\overline{\tau(g)}$. For fixed i, let $\pi = \pi(i) = \prod_{j \neq i} (\eta - c_j\mathbf{1})$. Then π vanishes on $G \setminus G(c_i)$ and is a non-zero constant d, say, on $G(c_i)$. Since π is a polynomial in η, it leaves $< S >$ invariant, so

$$0 = (\pi\sigma, \tau)_G = \frac{d}{|G|}\, a_i(\sigma, \tau) \ ,$$

which implies $a_i(\sigma, \tau) = 0$. This proves the statement if c is one of the c_i's. If not, $G(c)$ is empty and the assertion holds trivially.
(v)⇒(vi) Keep the notation and let γ be any class function which is constant, say equal to γ_i, on every $G(c_i)$; then

$$(\gamma\sigma, \tau)_G = \frac{1}{|G|}\sum_{i=1}^{m} \gamma_i a_i(\sigma, \tau) \ = \ 0$$

for every $\sigma \in S,\ \tau \in T$. This implies $\gamma\sigma \in < S >$. The argument applies in particular to $\gamma = \overline{\eta}$, proving (vi).
(vi)⇒(vii) Using (iv)⇒(vi) with $\overline{\eta}$ instead of η, one finds that $< S >$ is closed under multiplication with η. Let $\tau \in T$. If $\eta\tau \notin < T >$, there is some $\sigma \in S$ with $0 \neq (\sigma, \eta\tau)_G = (\sigma\overline{\eta}, \tau)_G$, a contradiction, since $\sigma\overline{\eta} \in < S >$ by assumption.
(vii)⇒(viii) Let $\alpha = \alpha_S + \alpha_T$ be given. Then in $\alpha\eta = \alpha_S\eta + \alpha_T\eta$, the first summand belongs to $< S >$, the second to $< T >$, by hypothesis. Therefore $(\alpha\eta)_S = \alpha_S\eta$.
(viii)⇒(i)

$$(\alpha\eta, \beta)_G^S = ((\alpha\eta)_S, \beta)_G = (\alpha_S\eta, \beta)_G = (\alpha_S, \overline{\eta}\beta)_G = (\alpha, \overline{\eta}\beta)_G^S \ ,$$

where the second equality holds by assumption.

Remark 2

(i) The values $c_1, \dots, c_m$ of η come in no particular order. However, it is reasonable to agree that $c_1 = \eta(1)$. Note that this means $G(c_1) = \mathrm{Ker}\,(\eta)$ if η is a character; also, all the $G(c_i)$'s are then unions of cosets of $G(c_1)$.

(ii) The class function $\pi(i)$ constructed in the above proof will in fact be a generalized character if η is and if c_i is rational (for instance $i = 1$). This follows by observing that then the c_j's are algebraic integers permuted by the Galois group.

(iii) The functions constant on all $G(c_i)$ are precisely the polynomials in η (of degree $< m$), as the usual interpolation argument shows.

(iv) Statements (ii) and (v) are versions of the orthogonality relations: Condition (iv) is trivially satisfied for every S if one takes $\eta = \mathbf{1}$. Then $G(1) = G$, so (v) reduces to $(\sigma, \tau)_G = 0$ for $\sigma \neq \tau \in \mathrm{Irr}\,(G)$ (taking $S = \{\sigma\}$).

Again, condition (iv) is trivially satisfied for every η if one takes $S = \mathrm{Irr}\,(G)$. Since we may choose η such that $\eta(g) \neq \eta(h)$ if g, h are not conjugate, we conclude from (ii) that then $0 = \sum_{\sigma \in \mathrm{Irr}(G)} \sigma(g)\overline{\sigma(h)}$. This can be generalized:

Corollary 1
Let φ be an irreducible character of some normal subgroup N of G and let $g, h \in G$. If $\overline{g}$ and $\overline{h}$ are not conjugate in $\overline{G} = G/N$ then

$$0 = \sum_{\chi \in \mathrm{Irr}(G|\varphi)} \chi(g)\overline{\chi(h)} \quad .$$

Proof: It is clear that $< \mathrm{Irr}\,(G|\varphi) >$ is invariant under multiplication with any class function of $\overline{G}$. Since there is such a class function η with $\eta(g) \neq \eta(h)$, the statement follows from (iv)$\Rightarrow$(ii) above.

Remark 3

(i) Trivially, there are class functions which take different values on all conjugacy classes of G. It is not hard to prove that there is a *character* (in general reducible) with that property.

(ii) The question naturally arises what the value of $\sum_{\chi \in \mathrm{Irr}(G|\varphi)} \chi(g)\overline{\chi(h)}$ is if $\overline{g}$ and $\overline{h}$ are conjugate in $\overline{G}$. This will be investigated in a separate paper [2].

Corollary 2
Assume that η takes only non–negative real values. Then the conditions of the proposition are equivalent to $(\alpha\eta,\, \alpha)_G^S \geq 0$ for every α.
Proof: Assume that condition (viii) holds. Then

$$(\alpha\eta, \alpha)_G^S = ((\alpha\eta)_S, \alpha_S)_G = (\alpha_S\eta, \alpha_S)_G \geq 0 \quad .$$

Conversely, assume that $(\alpha\eta, \alpha)_G^S \geq 0$ for every α. It is enough to show that $< S >$ is closed under multiplication with $\overline{\eta}$ $(= \eta)$. If not, there is $\sigma \in S$ and $\tau \in T = \mathrm{Irr}\,(G) \setminus S$ such that $(\tau, \overline{\eta}\sigma)_G \neq 0$. Choose $c \in \mathbb{C}$ such that $c \cdot (\tau, \overline{\eta}\sigma)_G = -(\sigma, \overline{\eta}\sigma)_G - 1$ and let $\alpha = \sigma + c\tau$, so $\alpha_S = \sigma$. It follows that

$$0 \leq (\alpha\eta, \alpha)_G^S = (\alpha\eta, \alpha_S)_G = (\alpha\eta, \sigma)_G = (\sigma + c\tau, \overline{\eta}\sigma)_G = (\sigma, \overline{\eta}\sigma)_G + c \cdot (\tau, \overline{\eta}\sigma)_G = -1 \quad ,$$

a contradiction.

Corollary 3
Let $S \subseteq \mathrm{Irr}\,(G)$. Define a graph $\Delta = \Delta(S)$ with vertices all the conjugacy classes of G and an edge between two classes K and L if and only if $0 \neq \sum_{\sigma \in S} \sigma(g)\overline{\sigma(h)}$, where $g \in K$ and $h \in L$. The class functions η which are constant on the connected components of Δ are then precisely the ones satisfying the conditions of the proposition.

Proof: Clear from condition (ii) of the proposition.

Corollary 4
Let η be a class function of G taking exactly m values $c_1, \ldots, c_m$. Define a graph $\Gamma = \Gamma(\eta)$ with vertices all the irreducible characters of G and an edge between σ and τ if and only if $0 \neq \sum\limits_{x \in G(c_i)} \sigma(x)\overline{\tau(x)}$ for some i; here again $G(c_i) = \{g \in G \mid \eta(g) = c_i\}$. The unions of connected components of Γ are then precisely the sets S satisfying the conditions of the proposition.

Proof: Clear from condition (v) of the proposition.

Corollary 5
Let η be a faithful class function of G, i.e. $\eta(g) \neq \eta(1)$ for $g \neq 1$. Then $S = \mathrm{Irr}\,(G)$ and $S = \emptyset$ are the only subsets of $\mathrm{Irr}\,(G)$ such that the conditions of the proposition are satisfied.

Proof: Taking $c_1 = \eta(1)$, we have $G(c_1) = \{1\}$, so $\sum\limits_{x \in G(c_1)} \sigma(x)\overline{\tau(x)} = \sigma(1)\tau(1) \neq 0$ for every σ, $\tau \in \mathrm{Irr}\,(G)$. Therefore the graph Γ of the last corollary is connected (even complete).

Corollary 5 can be read as a version of the Burnside-Brauer theorem (compare [1], Theorem 4.3), because $\eta < S > \leq < S >$ is certainly true if η is a character and $S = \{\sigma \in \mathrm{Irr}\,(G) \mid (\sigma, \eta^i)_G \neq 0 \text{ for some } i\}$.
Even if η is not faithful, an idea in the proof of Proposition 1 can be used to predict, for a given character γ, which irreducibles of G will show up as constituents of some $\gamma\eta^i$:

Poposition 2 *Burnside–Brauer*
Let γ and η be characters of G and let $K = \mathrm{Ker}\,(\eta)$; further let m be the number of different values of η. Then for any $\chi \in \mathrm{Irr}\,(G)$, the following are equivalent:

(i) $(\chi, \gamma)_K \neq 0$
(ii) $(\chi, \gamma\eta^i)_G \neq 0$ for some $m > i \geq 0$
(iii) $(\chi, \gamma\eta^i)_G \neq 0$ for some $i \geq 0$.

Proof:
(i)⇒(ii) Let $\eta(1) = c_1, \ldots, c_m$ be the values of η and let $\pi = \prod\limits_{i>1} (\eta - c_i\mathbf{1})$. Then π vanishes off K and is a non-zero constant on K, so $\pi = c \cdot \mathbf{1}_K^G$ for some $c \neq 0$. Therefore $(\chi, \gamma\,\pi)_G = \bar{c} \cdot (\chi, \gamma\,\mathbf{1}_K^G)_G = \bar{c} \cdot (\chi, \gamma)_K \neq 0$. Since π is a polynomial of degree $m-1$ in η, it follows that χ is a constituent of $\gamma\eta^i$ for some $i < m$.
(ii)⇒(iii) Trivial.
(iii)⇒(i) Since $(\chi, \gamma\eta^i)_G \neq 0$ for some i, a fortiori $0 \neq (\chi, \gamma\eta^i)_K = \eta^i(1)(\chi, \gamma)_K$.

Remark 4

(i) Note that the first condition depends on η only via its kernel.
(ii) Clearly, one gets the classical result for $\gamma = \mathbf{1}$ and η faithful, since then $K = 1$.

(iii) As the proof of (i)$\Rightarrow$(ii) shows, the assumption that γ and η are characters can be weakened for this implication: γ may be any class function; for η, we need only assume that $K = \{g \in G \mid \eta(g) = \eta(1)\}$ is a (normal) subgroup; there are many class functions besides the characters satisfying this condition.

(iv) However, here are two simple examples showing that the condition that γ and η are characters is essential for the implication (iii)$\Rightarrow$(i). In both examples, one of γ, η is a character, the other a generalized character. The group is C_4 and λ is a faithful linear character.
First, take $\gamma = \mathbf{1} - \lambda^2$ and $\eta = \mathbf{1} + \lambda^2$, so $K = C_2$. Then $\mathbf{1}$ is a constituent of $\gamma\eta^0$ but $(\mathbf{1}, \gamma)_K = 0$.
Second, take $\gamma = \mathbf{1}$ and $\eta = \lambda(\mathbf{1} - \lambda^2)$, so again $K = C_2$ (even though η is not a character). Then λ is a constituent of $\gamma\eta^1$ but $(\lambda, \gamma)_K = 0$.

References

[1] I.M. Isaacs, Character theory of finite groups, Academic Press, New York 1976

[2] R. Knörr, Isaacs' bilinear form and the second orthogonality relation. In preparation.